Girraween High School

2019

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time: 5 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple choice questions by completely colouring in the appropriate circle on your multiple choice answer sheet on the front page of your answer booklet.
- In questions 11-16 start all questions on a separate page in your answer booklet and show all relevant mathematical reasoning and/or calculations.

Total Marks: 100

<u>**Section 1**</u> (Pages 2 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

<u>**Section 2**</u> (Pages 5-13) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

Section 1 (10 marks)

Attempt Questions 1-10

Allow about 15 minutes for this section

Question 1

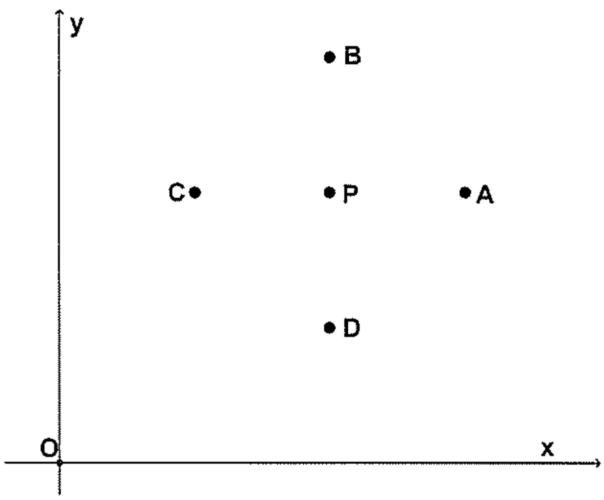
$$\left(\frac{1+i\sqrt{3}}{2}\right)^{2019} =$$

(A) i (B) $-i$ (C) 1 (D) -1

Question 2

If $\overrightarrow{OP} = z,\ z - i$ could be

(A) $\overrightarrow{OA}$ (B) $\overrightarrow{OB}$ (C) $\overrightarrow{OC}$ (D) $\overrightarrow{OD}$



Question 3

The foci of the conic $25x^2 + 9y^2 = 225$ are

(A) $(\pm 4, 0)$ (B) $(0, \pm 4)$ (C) $(\pm\sqrt{34}, 0)$ (D) $(0, \pm\sqrt{34})$

Question 4

A conic has foci $(\pm 8, 0)$ and directrices $x = \pm 5$. Its intercepts with the coordinate axes are

(A) $(\pm\sqrt{40}, 0)$ (B) $(0, \pm\sqrt{40})$ (C) $\left(\pm\frac{4\sqrt{5}}{5}, 0\right)$ (D) $\left(0, \pm\frac{4\sqrt{5}}{5}\right)$

Question 5

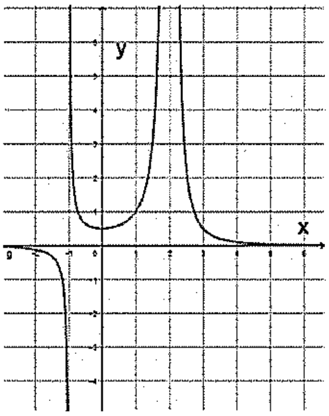
If α, β, γ and δ are the roots of $3x^4 - 2x + 1 = 0$, then $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 =$

(A) -1 (B) -4 (C) $-\frac{1}{3}$ (D) $-\frac{4}{3}$

Examination continues on the following page

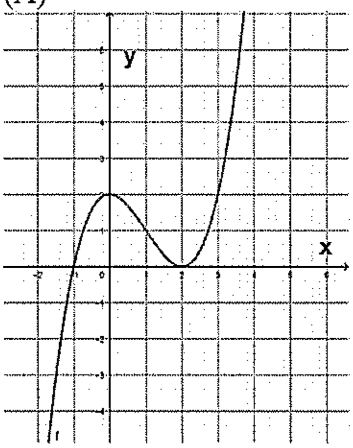
Question 6

The graph of $y = \frac{1}{f(x)}$ is sketched below:

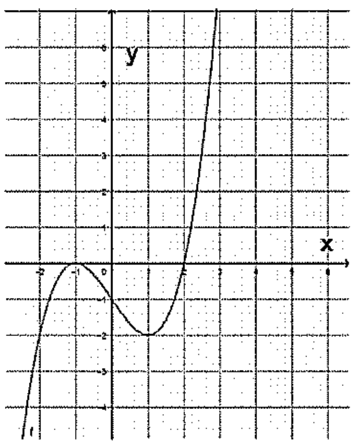


$y = f(x)$ could be:

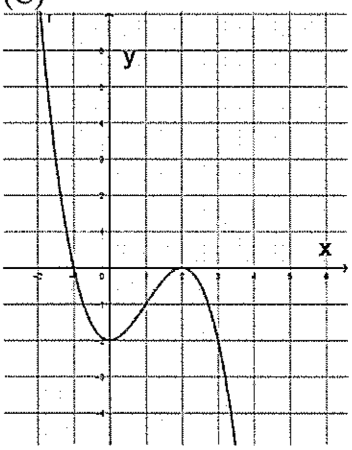
(A)



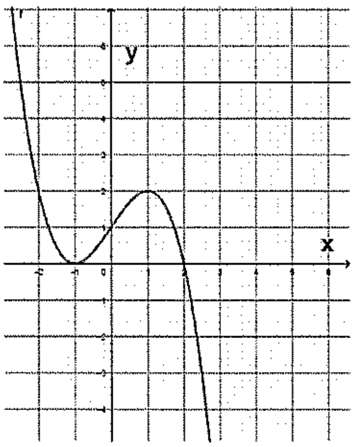
(B)



(C)



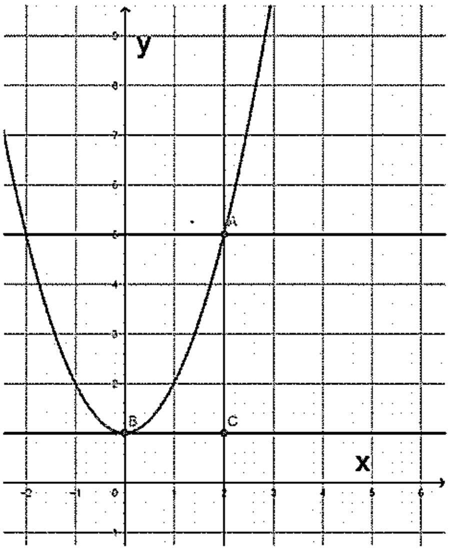
(D)



Examination continues on the following page

Question 7

The area bounded by $y = x^2 + 1, x = 2$ and $y = 1$ is to be rotated around the line $y = 5$.



The volume of the resulting solid using the method of cylindrical shells would be given by

(A) $V = 2\pi \int_0^2 x(x^2 + 1)\,.dx$ (B) $V = 2\pi \int_1^5 y\sqrt{y-1}\,.dy$

(C) $V = 2\pi \int_0^2 (2 - x)(4 - x^2)\,.dx$ (D) $V = 2\pi \int_1^5 (2 - \sqrt{y-1})(5 - y).dy$

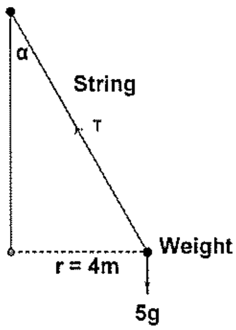
Question 8

If $f(x)$ is even and $g(x)$ is odd, then $\int_{-a}^{a} f(x) - g(x).dx =$

(A) 0 (B) $2\int_0^a f(x).dx$ (C) $2\int_0^a g(x).dx$ (D) $2\int_{-a}^{a} f(x).dx$

Question 9

A weight with a mass of 5 kilograms is attached by a taut string to the point P as shown.



If $g = 9.8m/s^2$ and the weight is rotating in a horizontal circle at a rate of 5 metres per second, the angle α that the taut string makes with the vertical is

(A) $32^o32'$ (B) $7^o16'$ (C) $57^o28'$ (D) $82^o44'$

Examination continues on the following page

Question 10

A rock is sinking in water. It experiences gravity, as well as resistance from the water proportional to the *cube* of its velocity. Given that down is POSITIVE, its acceleration is given by

(A) $\ddot{x} = kv^3 - g$ (B) $\ddot{x} = kv^3 + g$ (C) $\ddot{x} = -kv^3 - g$ (D) $\ddot{x} = -kv^3 + g$

Section II (90 marks)

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations.

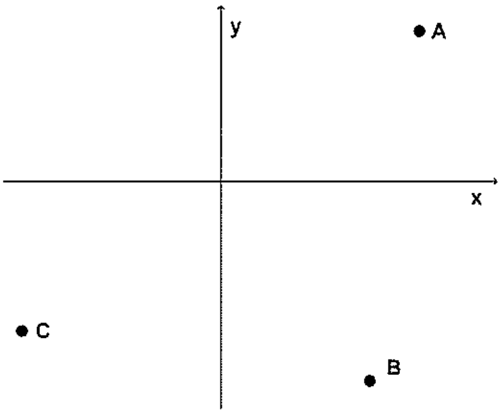
Question 11 *(15 Marks)*	**Marks**
(a) If $z_1 = -1 + i$ and $z_2 = \sqrt{3} + i$	
(i) Find $\frac{z_1}{z_2}$ in Cartesian form.	**2**
(ii) Express z_1 and z_2 in modulus/argument form.	**2**
(iii) Hence find the exact value of $sin\frac{7\pi}{12}$.	**1**
(b) Sketch the region in the complex plane where $\lvert z - 2i + 2\rvert \leq 2$ and $\frac{\pi}{2} \leq Arg\, z \leq \frac{3\pi}{4}$.	**3**

Examination continues on the following page

Question 11 (continued) **Marks**

(c) z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3|$.

If on the diagram below, $\overrightarrow{OA} = z_1, \overrightarrow{OB} = z_2$ and $\overrightarrow{OC} = z_3$,



(i) Find expressions for $\overrightarrow{CA}$ *and* $\overrightarrow{CB}$ in terms of $\boldsymbol{z_1, z_2}$ ***and*** $\boldsymbol{z_3}$. **2**

(ii) Show that $Arg\left(\frac{z_1}{z_2}\right) = 2\,Arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right)$. **1**

(d) The polynomial equation $8x^3 + ax^2 + bx + 3 = 0$ has a double root at $x = \frac{1}{2}$ and a single root at $x = r$. Find the values of a, b *and* r. **4**

Question 12 ***(15 Marks)***

(a) If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and γ are the roots of $3x^3 + 11x^2 + 11x - 5 = 0$, form the polynomial equation with roots α^2, β^2 and γ^2. **2**

(b) Find

(i) $\int \frac{1}{x^2\sqrt{1+x^2}} . dx$ **3**

(ii) $\int \frac{1}{3+cosx} . dx$ **3**

(iii) $\int \quad x\, sinx. dx$ **2**

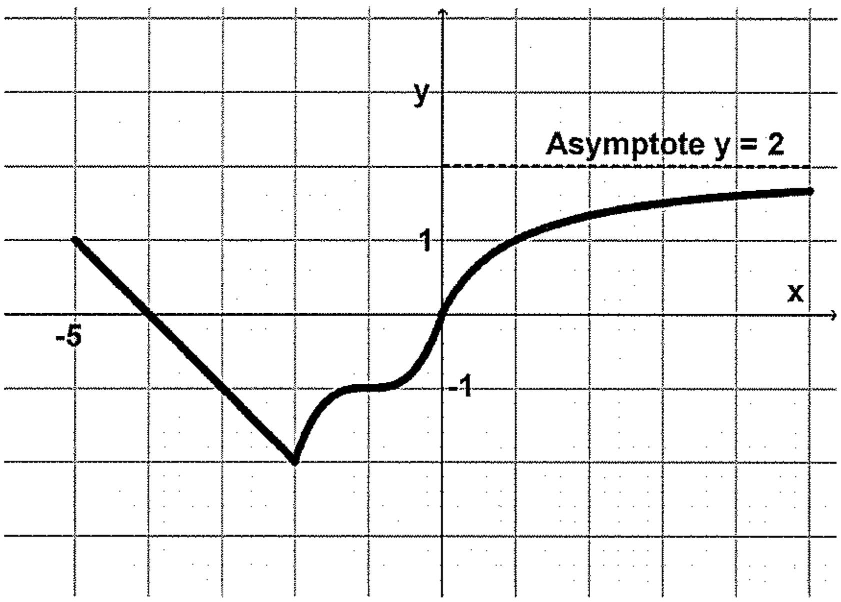
(c) (i) If $\frac{-8x-11}{(x-2)^2(x+1)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \frac{C}{(x+1)}$, find the values of A, B and C. **3**

(ii) Hence find $\int \frac{-8x-11}{(x-2^2(x+1)} . dx$ **2**

Examination continues on the following page

Question 13 *(15 Marks)* **Marks**

(a) The graph of $y = h(x)$ is sketched below:



Sketch on separate number planes from $x = -5$ *to* $x = 5$ in your answer booklet:

(i) $y = h|x|$ **1**

(ii) $y = (h(x))^2$ **3**

(iii) $y = e^{h(x)}$ **3**

(b) For the relation $3x^2 - 2xy + 3y^2 = 12$

(i) Find all intercepts with the coordinate axes. **3**

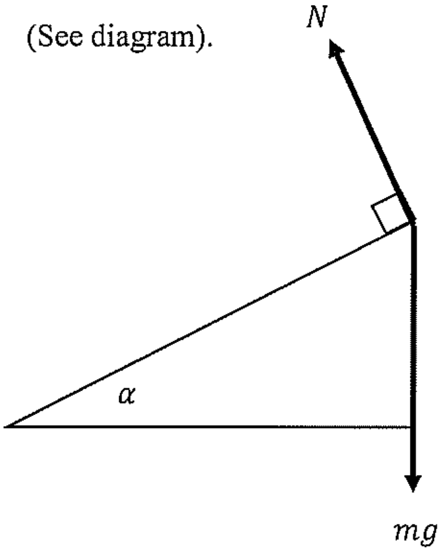
(ii) Find all turning points and points where the tangent to the graph is vertical. **3**

(iii) Sketch the graph of $3x^2 - 2xy + 3y^2 = 12$ showing all of these features. **2**

Examination continues on the following page

Question 14 *(15 Marks)* **Marks**

(a) A bend with horizontal radius 500 metres on a racetrack is banked at an angle of α to the horizontal so that a motorbike travelling at 144 km/h experiences no sideways friction. The motorbike does experience gravity and a normal force. (See diagram).



(i) By resolving forces in the horizontal and vertical directions, show that the track is banked at an angle of 18^o to the horizontal to the nearest degree. **3**

(Use $g = 9.8\ m/s^2$). (Note: Use $\alpha = 18^o$ for the rest of this question).

(ii) If the combined weight of the motorbike and rider is 200 kilograms, find the amount of lateral friction that the motorbike experiences when travelling around the bend at 135 km/h. **3**

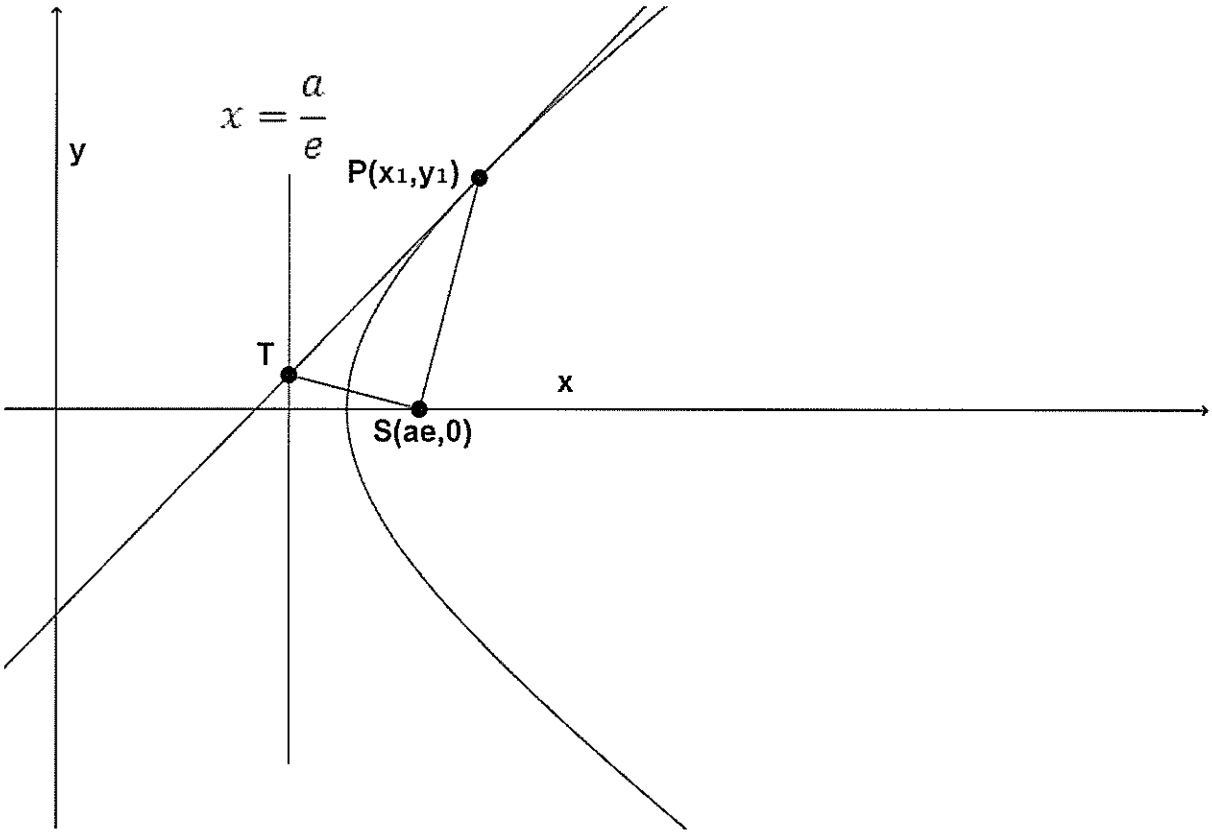
(iii) The bend is engineered so that the coefficient of friction is 0.1 (that is, friction can be up to 0.1 times the normal force before the motorbike starts to slip). What is the fastest speed that the motorbike can travel before it starts skidding outwards on this bend? **3**

Examination continues on the following page

Question 14 (continued) **Marks**

(b) The tangent at the point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

intersects with the directrix at T *(see diagram).*



If $S(ae, 0)$ is the focus

(i) Show that the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2**

(ii) Show that the point T is $\left(\frac{a}{e}, \frac{b^2(x_1 - ae)}{aey_1}\right)$ **1**

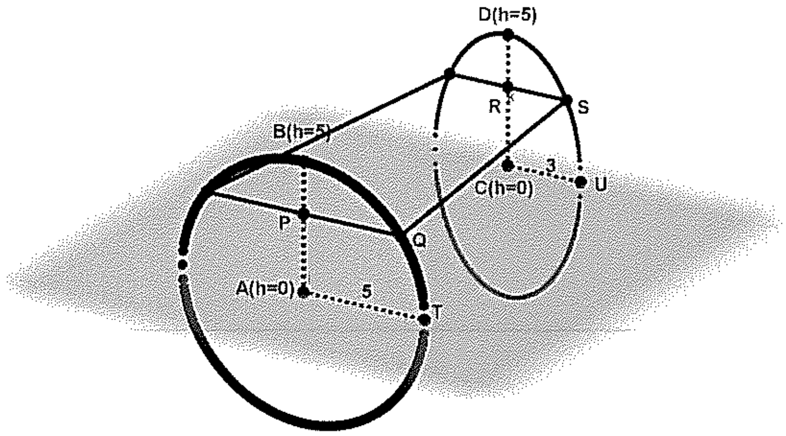
(iii) Show that $\angle PST = 90^o$. **3**

Examination continues on the following page

Question 15 *(15 Marks)* **Marks**

(a) By taking slices perpendicular to the axis of rotation, find the volume of the solid of revolution formed when the area enclosed by $y = x^2$ and $y = 4x$ is rotated around the line $y = 16$. **3**

(b) A piece of wood is shaped so that it is circular at one end with a radius of $5cm$ and elliptical at the other end with a semi major axis of $CD = 5cm$ and a semi minor axis of $CU = 3cm$ $(see\ diagram)$. The length of the piece of wood AC is $10cm$.



(i) If h is the distance AP or CR, explain why $PQ = \sqrt{25 - h^2}$ and find a similar expression for the distance RS. **1**

(ii) By considering the volume of each trapezoidal slice, find the volume of the piece of wood. **3**

(c) If $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cot^n x.dx$

(i) Show that $I_n = \frac{1}{n-1} - I_{n-2}$ **2**

(ii) Find I_4 **1**

(iii) Using $\lim_{n\to\infty} I_n = 0$, find $\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{(2k-1)}$ **1**

(i.e. $\lim_{k\to\infty}\ 1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{k-1}\left(\frac{1}{2k-1}\right)$.)

Examination continues on the following page

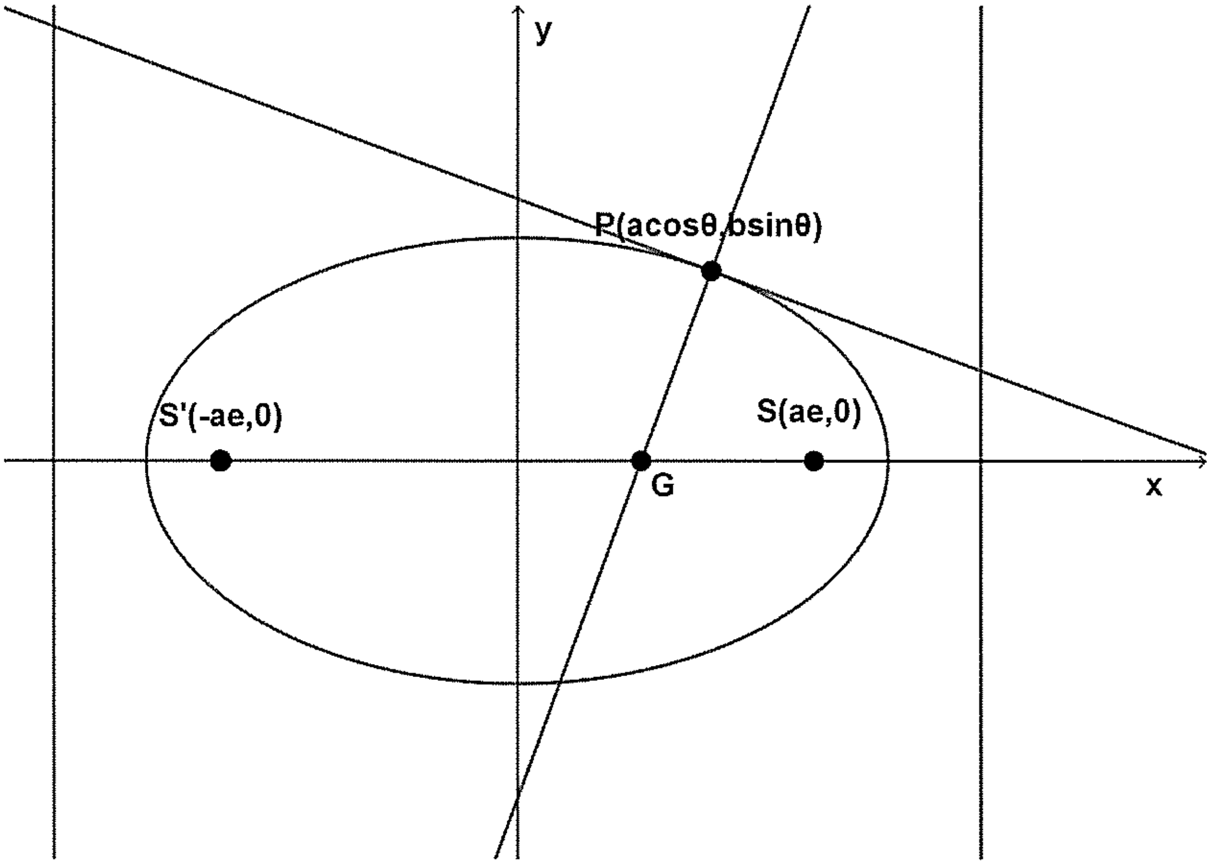
Question 15 (continued) **Marks**

(d) The point $P\ (acos\Theta, bsin\Theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The normal at P is $axsin\Theta - bycos\Theta = (a^2 - b^2)sin\Theta cos\Theta$.

(Do NOT prove this!!!)

The normal to the ellipse at P cuts the x axis at G. If S and S' are the foci of the ellipse *(see diagram below)*



(i) Show that the coordinates of G are $(ae^2cos\Theta, 0)$ 1

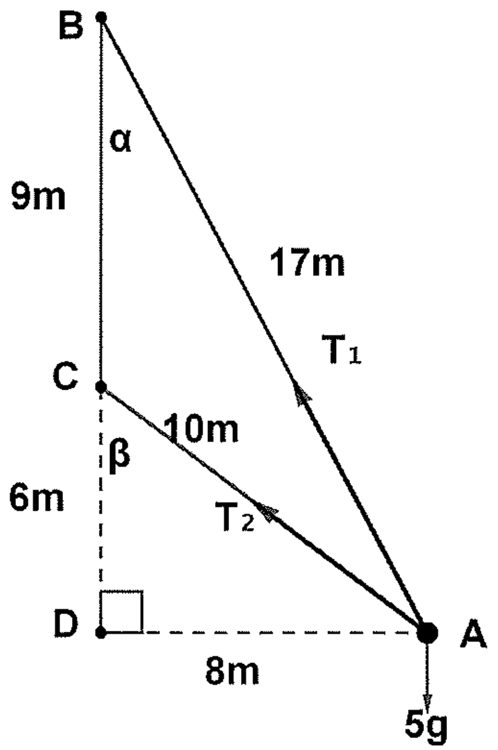
(ii) Show that $(PG)^2 = (1 - e^2) \times PS \times PS'$ 3

Examination continues on the following page

Question 16 *(15 marks)* **Marks**

(a) A particle with mass 5 kilograms at A is attached to two strings of length 17 metres and 10 metres, which are both attached to a vertical pole at points B and C respectively. The particle is rotating in a horizontal circle with radius 8 metres around point D on the pole. C is 9 metres directly below B and D is 6 metres directly below C. α and β are the semi-vertical angles of the two strings *(see diagram.)*

(In this question use $g = 9.8\, m/s^2$*)*



(i) The only forces on the particle at A are gravity ($g = 9.8m/s^2$) and the tensions in the two strings. By resolving the forces on the particle at A in the vertical and horizontal directions, show that **2**

$75T_1 + 51T_2 = 425g$ and

$$40T_1 + 68T_2 = \frac{425v^2}{8}$$

(ii) Find the speed below which the lower string (AC) will no longer be taut. **2**

(iii) Find the speed above which the upper string (AB) will no longer be taut. **2**

Examination continues on the following page

Question 16 (continued) **Marks**

(b) A 10 kilogram cannonball is launched at a speed of 40 m/s at an angle of 30^o up from the horizontal. It experiences gravity of $10g$ in the vertical (y) direction and air resistance of $\frac{v^2}{32}$ in both the horizontal (x) and vertical (y) directions. *(Note: In this question use $g = 9.8m/s^2$)*

(i) By resolving forces in the vertical direction, show that **3**

$y = -160ln\left(\frac{v^2+320g}{400+320g}\right)$ where $v = \dot{y}$ and find how high the cannonball is when it reaches its maximum height.

(ii) Find the TIME taken for the cannonball to reach its maximum height. **2**

(iii) By resolving forces in the horizontal direction, show that **3**

$v = \frac{960}{3t+16\sqrt{3}}$ where $v = \dot{x}$ and $x = 320ln\left(\frac{3t}{16\sqrt{3}} + 1\right)$.

(iv) Find the distance the cannonball has travelled horizontally before it reaches its maximum height. **1**

END OF EXAMINATION!!!

Solutions

Multiple choices

Q.(1) D

$$\left(\frac{1+i\sqrt{3}}{2}\right)^{2019}$$

$$= \left[\text{cis}\frac{\pi}{3}\right]^{2019}$$

$$= \text{cis}\,\frac{2019\pi}{3}$$

$$= \text{cis}\,673\pi$$

$$= -1.$$

(2) $z-i$ will be 1 unit BELOW P if $\overrightarrow{OP} = z$.

D

(3) $25x^2 + 9y^2 = 225$.

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

$e^2 = 1 - \frac{a^2}{b^2}$ as major axis of ellipse $= y$ axis

$$= 1 - \frac{9}{25}.$$

$$e = \frac{4}{5}.$$

Foci $= (0, \pm 4)$

B

(4) $ae = 8$

$$\left(\frac{a}{e}\right) = 5.$$

$$e^2 = \frac{8}{5}$$

$$e =$$

As $ae = 8$,

$$\frac{2a\sqrt{10}}{5} = 8$$

$$a = 2\sqrt{10}$$

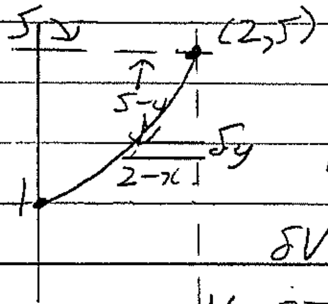
$$= \sqrt{40}$$

A

(5)
$$3\alpha^4 - 2\alpha + 1 = 0$$
$$3\beta^4 - 2\beta + 1 = 0 \quad +$$
$$3\gamma^4 - 2\gamma + 1 = 0$$
$$3\delta^4 - 2\delta + 1 = 0$$
$$3(\alpha^4+\beta^4+\gamma^4+\delta^4) - 2(\alpha+\beta+\gamma+\delta) + 4 = 0$$
$$3(\alpha^4+\beta^4+\gamma^4+\delta^4) - 2\times 0 + 4 = 0$$
$$\alpha^4+\beta^4+\gamma^4+\delta^4 = -\frac{4}{3} \rightarrow \text{D}$$

(6) A $\rightarrow$ Original must go from negative to positive at $x=-1$.

(7)



Inner $r = 5-y$ | $y = x^2+1$

$H = 2-x$ | $y-1 = x^2$

$= 2-\sqrt{y-1}$ | $\sqrt{y-1} = x.$

$$\delta V = 2\pi(5-y)(2-\sqrt{y-1})$$

$$V = 2\pi\int (2-\sqrt{y-1})(5-y)\,dy$$

D

(8)
$$\int_{-a}^{a} f(x) - g(x)\,dx$$
$$= \int_{-a}^{a} f(x)\,dx - \int_{-a}^{a} g(x)\,dx$$
$$= 2\int_0^a f(x)\,dx - 0 \text{ (as } f(x) \text{ even, } g(x) \text{ odd)}$$
$$= 2\int_0^a f(x)\,dx \quad \text{B}$$

(9) Resolving horizontally: $T\sin\alpha = \frac{5\times 5^2}{4}$ (1)

vertically: $T\cos\alpha = 5g$ (2)

(1) ÷ (2) $\tan\alpha = \frac{125}{196}$

A $\alpha = 32^\circ 32'$

(10) $F = ma = mg - mkv^3$.

mg $\quad a = \ddot{x} = g - kv^3$

mkv^3 $\quad = -kv^3 + g$ D

p. 2 Year 12 Extension 2 2019 Trial Examination Solutions.

10 mins.

Multiple Choice: (1) D (2) D (3) B (4) A (5) D (6) A (7) D (8) B (9) A (10) D

Q. (11) (a) (i) $\frac{z_1}{z_2} = \frac{-1+i}{\sqrt{3}+i} \times \frac{(\sqrt{3}-i)}{(\sqrt{3}-i)}$

$= \frac{1}{4}\left[(1-\sqrt{3}) + i(\sqrt{3}+1)\right]$.

(ii) $z_1 = \sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ $\quad z_2 = 2\,\text{cis}\,\frac{\pi}{6}$.

(iii) $\frac{z_1}{z_2} = \frac{\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}}{2\ \text{cis}\ \frac{\pi}{6}}$

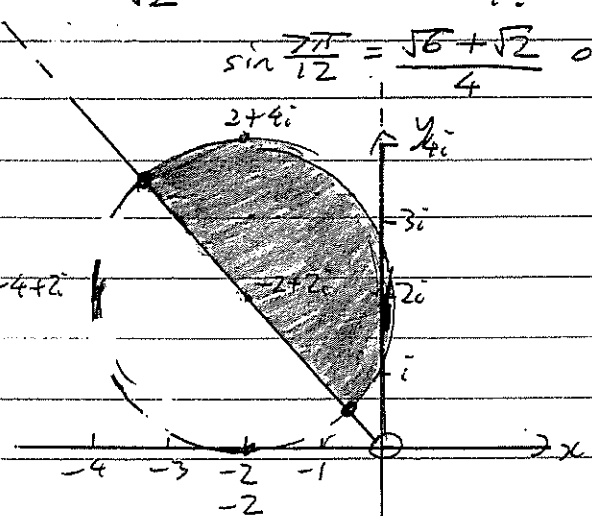
$= \frac{1}{\sqrt{2}}\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]$.

Equating with imaginary parts of (a)(i),

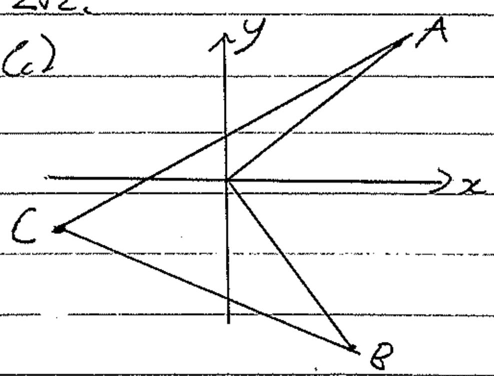
$\frac{1}{\sqrt{2}}\sin\frac{7\pi}{12} = \frac{(\sqrt{3}+1)}{4}$.

$\sin\frac{7\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$ or $\frac{\sqrt{3}+1}{2\sqrt{2}}$.

(b)



(c)



(i) $\vec{CA} = \vec{CO} + \vec{OA}$
$= -z_3 + z_1$
$= z_1 - z_3$.

$\vec{CB} = \vec{CO} + \vec{OB}$
$= -z_3 + z_2$
$= z_2 - z_3$.

(ii) As $|A| = |B| = |C|$, A, B & C are on the circumference of a circle, centre O.

$\angle AOB$ $\left[\text{Arg}\left(\frac{z_1}{z_2}\right)\right] = 2 \times \angle ACB$ $\left[= \text{Arg}\left(\frac{z_1 - z_3}{z_2 - z_3}\right)\right]$.

[$\angle$ at centre of circle ABC $= 2 \times \angle$ at circumference subtended by arc AB].

Q.(11)(d) $P(x) = 8x^3 + ax^2 + bx + 3$.

Double root at $x = \frac{1}{2}$. $8(\frac{1}{2})^3 + a(\frac{1}{2})^2 + b(\frac{1}{2}) + 3 = 0$

$1 + \frac{1}{4}a + \frac{1}{2}b + 3 = 0$ ×4 & simplify.

$a + 2b = -16$ (1)

As $x = \frac{1}{2}$ is a DOUBLE root, $P'(\frac{1}{2}) = 0$ too.

$P'(x) = 24x^2 + 2ax + b$.

$P'(\frac{1}{2}) = 24(\frac{1}{2})^2 + 2a(\frac{1}{2}) + b = 0$

$a + b = -6$ (2)

(1) − (2): $b = -10$

$\therefore a = 4$.

$\therefore P(x) = 8x^3 + 4x^2 - 10x + 3$.

Single root at $x = r$. Roots $= \frac{1}{2}, \frac{1}{2}, r$.

By sum of roots, $\frac{1}{2} + \frac{1}{2} + r = -\frac{4}{8} = -\frac{1}{2}$.

$r = -\frac{3}{2}$.

$\therefore a = 4, b = -10, r = -\frac{3}{2}$.

35 mins.

Q.(12)(a) Let $y = x^2$

$\therefore x = \sqrt{y}$

$3x^3 + 11x^2 + 11x - 5 = 0$ becomes

$3y\sqrt{y} + 11y + 11\sqrt{y} - 5 = 0$

$\sqrt{y}\ (3y + 11) = 5 - 11y$

Squaring BS

$y(9y^2 + 66y + 121) = 25 - 110y + 121y^2$.

$9y^3 - 55y^2 + 231y - 25 = 0$.

The equation with roots α^2, β^2 & γ^2

is $9x^3 - 55x^2 + 231x - 25 = 0$.

Q.(12)(b)(i) $\int \frac{1}{x^2\sqrt{1+x^2}}\cdot dx$ Let ~~a~~ ~~z~~ $x=\tan\theta$.

$dx = \sec^2\theta . d\theta$

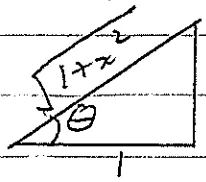
$$= \int \frac{1}{\tan^2\theta\sec\theta}\cdot\sec^2\theta . d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\, . d\theta$$

$$= \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}. d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\, . d\theta \quad \left[\text{By } \int [f(x)]^n \cdot f'(x).dx = \frac{1}{n+1}[f(x)]^{n+1}\right.$$

$$= -\frac{1}{\sin\theta} + C$$



$x = \tan\theta$

$\frac{1}{\sin\theta} = \frac{\sqrt{1+x^2}}{x}$.

$$= -\frac{\sqrt{1+x^2}}{x} + C.$$

(ii) $\int \frac{1}{3+\cos x} . dx$ Let $t = \tan(\frac{x}{2})$

$\frac{dt}{dx} = \frac{1}{2}\sec^2(\frac{x}{2})$

$= \frac{1+t^2}{2}.$

$\therefore dx = \frac{dx}{dt}. dt$

$= \frac{2}{1+t^2}. dt$

$$= \int \frac{1}{3+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}. dt$$

$$= \int \frac{2}{2t^2+4} dt$$

$$= \int \frac{1}{t^2+2} .. dt$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C.$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\quad}{\sqrt{2}}\right) + C.$$

(iii) $\int x\cdot\sin x . dx$ $u = x$ $v = -\cos x$.

$u' = 1$. $v' = \sin x$

By $\int uv' . dx = uv - \int vu' . dx$

$$\therefore \int x\sin x . dx = -x\cos x + \int \cos x . dx$$

$$= \underline{-x\cos x + \sin x + C.}$$

Q.(12)(c) (i) $\dfrac{-8x-11}{(x-2)^2(x+1)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)} + \dfrac{C}{(x+1)}$

$\times BS$ by $(x-2)^2(x+1)$

$-8x-11 = A(x+1)+B(x-2)(x+1)+C(x-2)^2 \text{ - } (1)$

Sub. in $x=2$ to (1)

$-8\times 2-11 = 3A$

$-9 = A \quad (2)$

Sub. in $x=-1$ to (1):

$-8\times -1-11 = C\times(-1-2)^2$

$-\frac{1}{3} = C \quad (3)$

Sub. in $A=-9$, $C=-\frac{1}{3}$, $x=0$ to (1):

$-11 = -9\times 1 + -2B + 4\times -\frac{1}{3}$

$B = \frac{1}{3}$

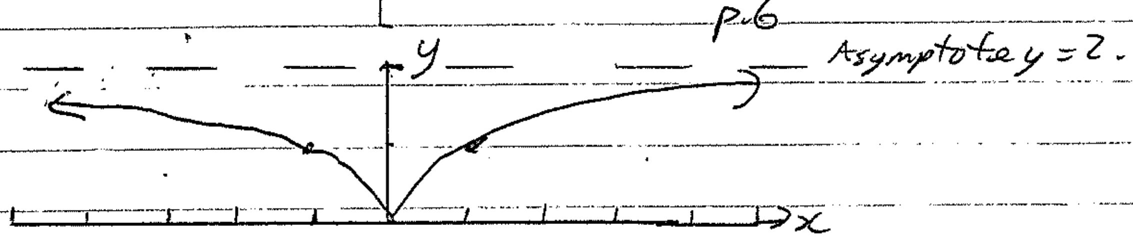
$\therefore A=-9,\ B=\frac{1}{3},\ C=-\frac{1}{3}$

(ii) $\displaystyle\int \frac{-8x-11}{(x-2)^2(x+1)}\,dx$

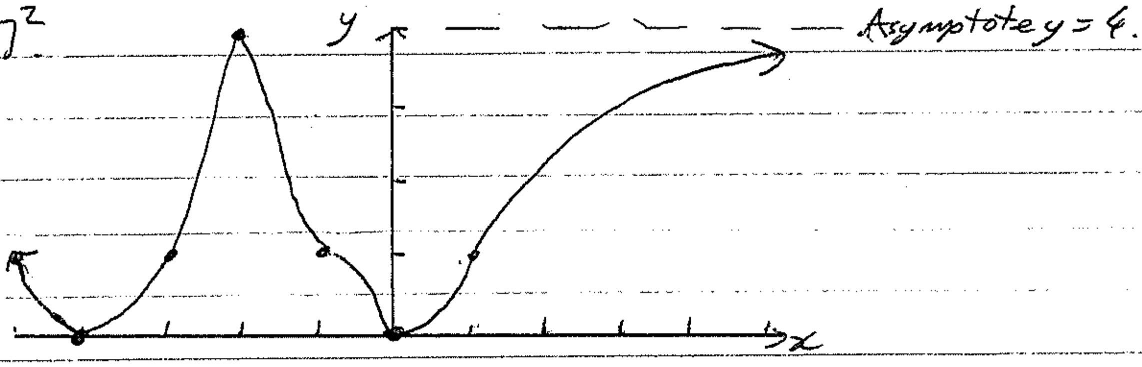
$\displaystyle = \int \frac{-9}{(x-2)^2} + \frac{1}{3(x-2)} - \frac{1}{3(x+1)}\,dx$

$\displaystyle = \frac{9}{(x-2)} + \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x+1) + c$

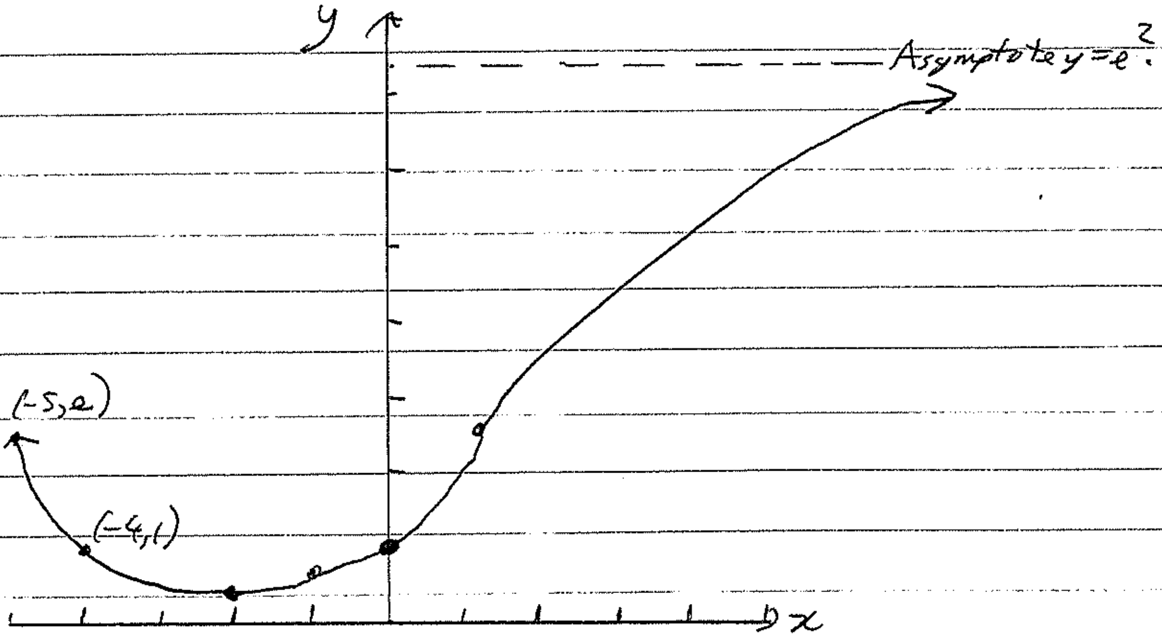
Q.(13)(a)(i) $y=h(x)$



(ii) $y=(h(x))^2$



(iii) $y=e^{h(x)}$



Q. [13](b)(i) $3x^2 - 2xy + 3y^2 = 12$.

x intercepts [y=0]. y intercepts [x=0].

$3x^2 = 12$. $3y^2 = 12$

$x = \pm 2$. $y = \pm 2$.

(ii) $3x^2 - 2xy + 3y^2 = 12$ (1)

Differentiating w.r. to x:

$$6x - \left[2y + 2x\frac{dy}{dx}\right] + 6y\frac{dy}{dx} = 0$$

$$(6y - 2x)\frac{dy}{dx} = 2y - 6x.$$

$$\frac{dy}{dx} = \frac{y - 3x}{3y - x}.$$

$\frac{dy}{dx} = 0$: $y - 3x = 0$.

$y = 3x$.

Sub. $y = 3x$ in (1):

$3x^2 - 2x \times 3x + 3 \times (3x)^2 = 12$

$24x^2 = 12$

$x = \pm\frac{1}{\sqrt{2}}$

& $y = 3x = \pm\frac{3}{\sqrt{2}}$.

Turning points $= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{3}{\sqrt{2}}\right)$.

$\frac{dy}{dx}$ is undefined:

$3y - x = 0$

$x = 3y$.

Sub. $x = 3y$ in (1):

$3(3y)^2 - 2 \times 3y \times y + 3y^2 = 12$.

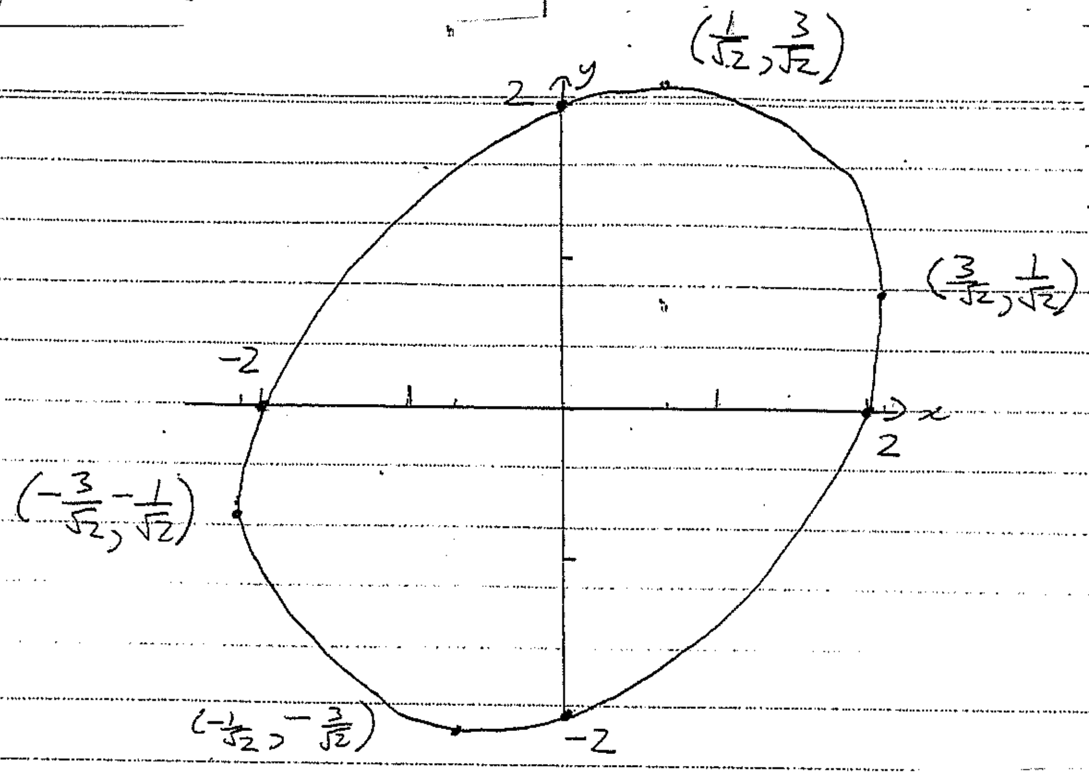
$24y^2 = 12$

$y = \pm\frac{1}{\sqrt{2}}$, $x = \pm\frac{3}{\sqrt{2}}$

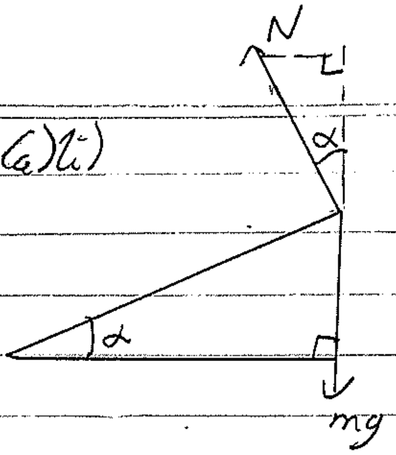
Points where tangent is vertical

$= \left(\pm\frac{3}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$.

(iii)



Q. (14)(a)(i)



Resolving horizontally: $N\sin\alpha = \frac{mv^2}{r}$ (1)

Resolving vertically: $N\cos\alpha = mg$ (2)

Note: 144km/h = 40m/s.

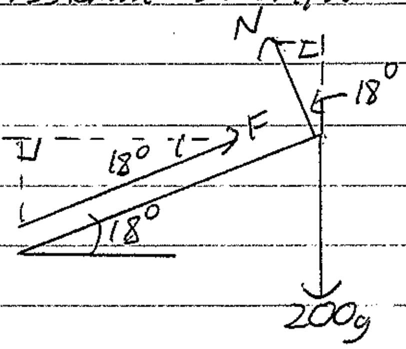
(1) ÷ (2) $\tan\alpha = \frac{v^2}{rg}$

$= \frac{40^2}{500 \times 9.8}$

$\alpha = 18°5'$

$\doteqdot 18°$

(ii) 135km/h = 37.5m/s. → Slower than optimum → Friction will go UP the plane.



Resolving vertically:

$N\cos 18° + F\sin 18° = 200g$ (1)

Resolving horizontally:

$N\sin 18° - F\cos 18° = \frac{200 \times 37.5^2}{500}$ (2)

$= 562.5$.

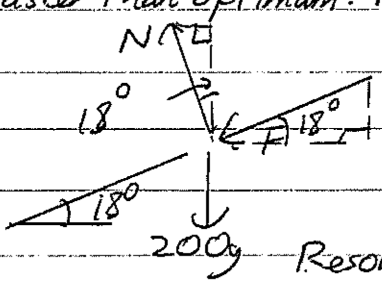
(1) × sin18° = (3) $N\cos 18° \sin 18° + F\sin^2 18° = 200g \sin 18°$

(2) × cos 18° = (4) $N\cos 18° \sin 18° - F\cos^2 18° = 562.5 \cos 18°$

$F(\sin^2 18° + \cos^2 18°) = 70.70401..$

The motorbike experiences friction of 70.7 Newtons UP the slope.

(iii) Faster than optimum: Friction will go DOWN the plane. F = 0.1N.



Resolving vertically:

$N\cos 18° - 0.1N\sin 18° = 200g$.

$N = \frac{200g}{\cos 18° - 0.1\sin 18°}$ (1)

Resolving horizontally:

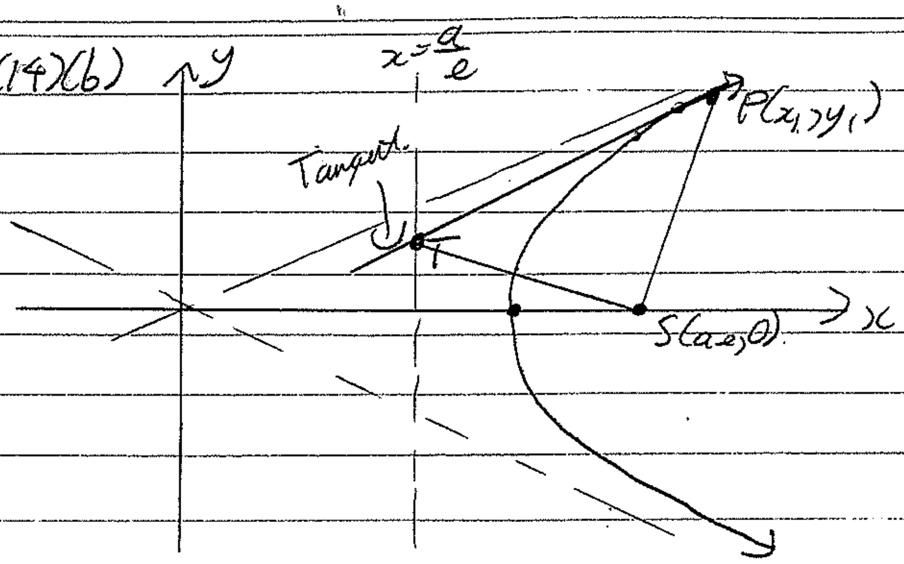
$N\sin 18° + 0.1N\cos 18° = \frac{200 \times v^2}{500}$

$\frac{500N(\sin 18° + 0.1\cos 18°)}{200} = v^2$.

Sub. in N from (1): $v = 46.3899$ m/s $= 167$ km/h.

Maximum speed before skidding = 167 km/h.

Q.(14)(b)



(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{2x}{a^2} = \frac{2y}{b^2}\cdot\frac{dy}{dx}$$

$$\frac{b^2x}{a^2y} = \frac{dy}{dx}.$$

Sub. in (x_1, y_1) m of tangent $= \frac{b^2x_1}{a^2y_1}$.

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$$

$$+ b^2x_1^2 - a^2yy_1$$

$$b^2x_1^2 - a^2y_1^2 = b^2xx_1 - a^2yy_1$$

$$\div a^2b^2.$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{yy_1}{b^2}$$

As $P(x_1, y_1)$ is on the hyperbola, $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$

$\therefore$ Tangent is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$

(ii) T is where tangent hits $x = \frac{a}{e}$

$$\therefore \frac{a}{e}\frac{x_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

$$\frac{x_1}{ae} = 1 + \frac{yy_1}{b^2}$$

$$\frac{x_1 - ae}{ae} = \frac{yy_1}{b^2}$$

$$\frac{b^2(x_1 - ae)}{aey_1} = y.$$

T is $\left(\frac{a}{e}, \frac{b^2(x_1 - ae)}{aey_1}\right)$

PTO →

Q.(14)[continued] (b)(iii)

Proving $PS \perp ST$:

$$mPS = \frac{y_1}{x_1 - ae} \quad (1)$$

$$mST = \frac{\frac{-b^2(x_1 - ae)}{aey_1}}{ae - \frac{a}{e}} \quad \times aey_1$$

$$= \frac{-b^2(x_1 - ae)}{a^2e^2y_1 - a^2y_1}$$

Note: As $e^2 = 1 + \frac{b^2}{a^2}$

$$a^2e^2 = a^2 + b^2$$

$$a^2(e^2 - 1) = b^2. \quad (3)$$

$$= \frac{-b^2(x_1 - ae)}{a^2(e^2 - 1)y_1} \quad (2)$$

$$= \frac{-b^2(x_1 - ae)}{b^2 y_1} \quad \text{[Sub. (3) in (2)]}.$$

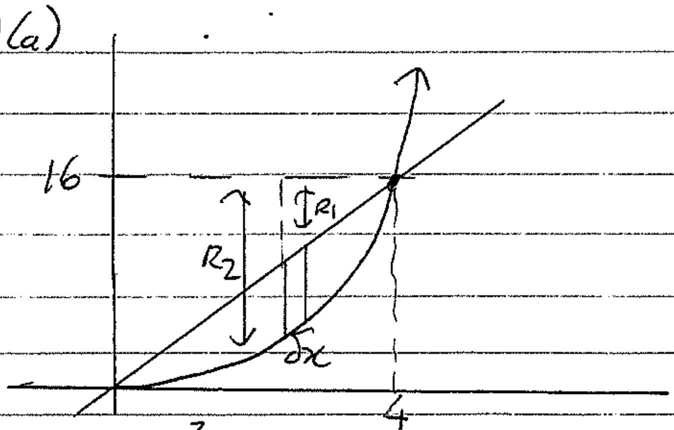
$$= -\frac{(x_1 - ae)}{y_1}.$$

$$\therefore mPS \times mST = \frac{y_1}{x_1 - ae} \times \frac{-(x_1 - ae)}{y_1}$$

$$= -1.$$

$PS \perp ST$.

Q.(15)(a)



$R_2 = 16 - x^2$.

$R_1 = 16 - 4x$.

$$\delta V = \pi\left[R_2^2 - R_1^2\right].\delta x$$

$$= \pi\left[(16 - x^2)^2 - (16 - 4x)^2\right].\delta x$$

$$= \pi(x^4 - 48x^2 + 128x).\delta x$$

$$V = \pi \lim_{\delta x \to 0} \sum_{x=0}^{x=4} (x^4 - 48x^2 + 128x).\delta x$$

Letting $\delta x \to 0$

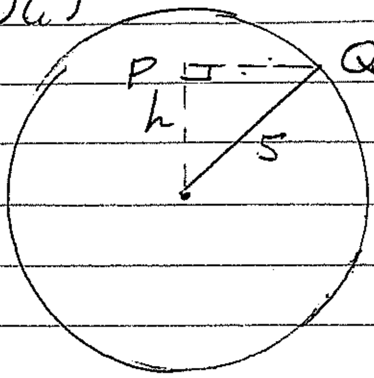
$$V = \pi \int_0^4 (x^4 - 48x^2 + 128x).dx$$

$$= \pi\left[\frac{1}{5}x^5 - 16x^3 + 64x^2\right]_0^4$$

$$= \pi\left[\frac{1}{5} \times 4^5 - 16 \times 4^3 + 64 \times 4^2\right] - 0$$

$$= \frac{1024\pi}{5} u^3.$$

Q.(15)(b)(i)



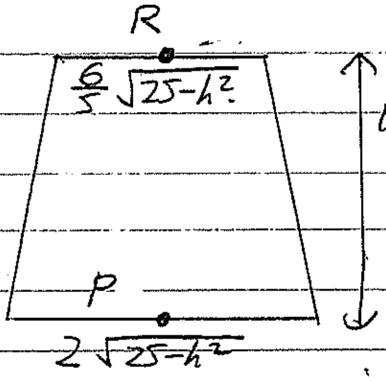
By Pythagoras,

$PQ = \sqrt{5^2 - h^2}$

$= \sqrt{25 - h^2}$.

$RS = \frac{3}{5} PQ = \frac{3}{5}\sqrt{25-h^2}$.

(ii)



$A = \frac{1}{2} \times 10 \times \left(2\sqrt{25-h^2} + \frac{6}{5} \times \sqrt{25-h^2}\right)$

$= 16\sqrt{25-h^2}$.

$\delta V = 16\sqrt{25-h^2} \cdot \delta h$.

Max. $h = 5$. Min. $h \Rightarrow$ take 0 & double.

$\therefore V = 2 \times 16 \lim_{\delta h \to 0} \sum_{h=0}^{h=5} \sqrt{25-h^2} \cdot \delta h$

Letting $\delta h \to 0$

$V = 32 \int_0^5 \sqrt{25-h^2} \cdot dh$. This is the area of a $\frac{1}{4}$ circle $= \frac{\pi}{4} \times 5^2 = \frac{25\pi}{4}$.

$\therefore V = 32 \times \frac{25\pi}{4}$

$= 200\pi \text{ cm}^3$.

(c)(i) $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \cdot dx$

$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \cdot dx$

$I_n = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \cdot (-\operatorname{cosec}^2 x) \cdot dx - I_{n-2}$.

$I_n = -\left[\frac{1}{n-1} \cot^{n-1} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} {}^{*} - I_{n-2}$ [By ${}^{*} f'(x) \cdot [f(x)]^n \cdot dx = \frac{1}{n+1}[f(x)]^{n+1}$.]

$= -\left[\frac{1}{n-1} \times 0 - \frac{1}{n-1} \times 1\right] - I_{n-2}$.

$\underline{I_n = \frac{1}{n-1} - I_{n-2}}$.

Q.(15)(c)(ii) $I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1\,dx$

$= \frac{\pi}{4}$.

$I_2 = 1 - \frac{\pi}{4}$.

$I_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$

$= \frac{\pi}{4} - \frac{2}{3}$.

(iii) Using $I_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$.

$I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4}$..

$I_{10} = \frac{1}{9} - \frac{1}{7} + \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$.

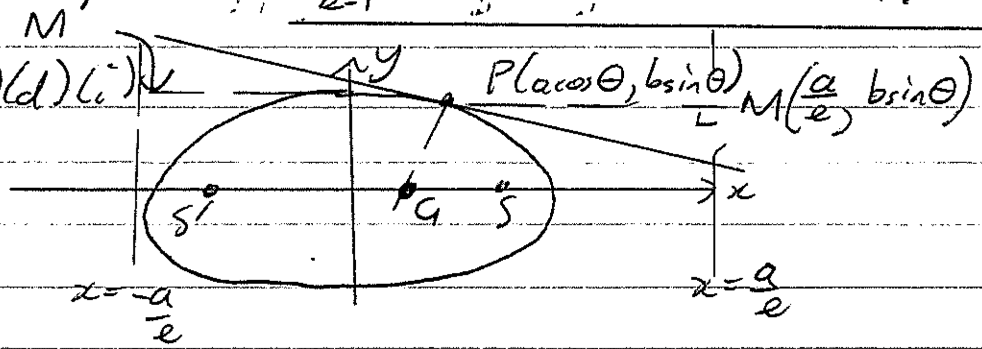
As $I_n \to 0$ as $n \to \infty$,

$\therefore 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots - \frac{\pi}{4} = 0$.

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

i.e. $\sum_{k=1}^{\infty} \frac{1}{2k-1} \dots = \frac{\pi}{4}$.

(15)(d)(i)



At G, $y = 0$

$\therefore ax \sin\theta = (a^2 - b^2)\sin\theta\cos\theta$.

$x = \frac{a(a^2-b^2)\cos\theta}{a}$.

$= a\left(\frac{a^2}{a^2} - \frac{b^2}{a^2}\right)\cos\theta$

$= a\left(1 - \frac{b^2}{a^2}\right)\cos\theta$

$x = ae^2\cos\theta$.

$G = (ae^2\cos\theta, 0)$.

PTO →

Q.(15)(d)(ii) $PS = ePM$
$= e\left(\frac{a}{e} - a\cos\theta\right)$
$= a - ae\cos\theta.$

$PS' = ePM'$
$= e\left(\frac{a}{e} + a\cos\theta\right)$
$= a + ae\cos\theta.$

$\therefore PS \times PS'$
$= a(1-e\cos\theta) \times a(1+e\cos\theta)$
$= a^2(1-e^2\cos^2\theta)$ (1)

Note for PG: $1-e^2$
$= 1-\left(1-\frac{b^2}{a^2}\right)$
$= \frac{b^2}{a^2}$ (2)

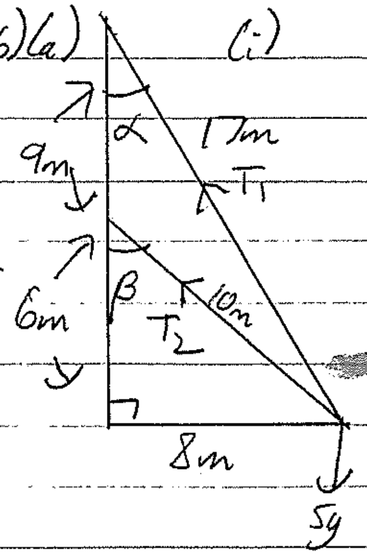
Also: $a^2(1-e^2) = b^2$

By distance formula, $(PG)^2 = (a\cos\theta - ae^2\cos\theta)^2 + b^2\sin^2\theta.$
$= a^2\cos^2\theta(1-e^2)^2 + b^2\sin^2\theta.$
$= a^2(1-e^2)\cos^2\theta(1-e^2) + b^2\sin^2\theta$
$= a^2(1-e^2)\cos^2\theta(1-e^2) + a^2(1-e^2)\sin^2\theta$

$= (1-e^2)\left[a^2(1-e^2)\cos^2\theta + a^2(1-\cos^2\theta)\right]$
$= (1-e^2)a^2\left[\cancel{\cos^2\theta} - e^2\cos^2\theta + 1 - \cancel{\cos^2\theta}\right]$
$= (1-e^2)a^2\left[1-e^2\cos^2\theta\right]$
$= (1-e^2) \times PS \times PS'$ [By (1)]

QED.

Q.(16)(a)(i)



Vertical: $T_1\cos\alpha + T_2\cos\beta = 5g$
Note: $\cos\alpha = \frac{15}{17}$ $\cos\beta = \frac{6}{10} = \frac{3}{5}$

$\therefore \frac{15T_1}{17} + \frac{3T_2}{5} = 5g$ $\times 85$

$75T_1 + 51T_2 = 425g$. (1)

Horizontal: $T_1\sin\alpha + T_2\sin\beta = \frac{mv^2}{r}$ $\sin\alpha = \frac{8}{17}$

$\frac{8T_1}{17} + \frac{4T_2}{5} = \frac{5v^2}{8}$ (2) $\times 85$ $\sin\beta = \frac{8}{10} = \frac{4}{5}$

$40T_1 + 68T_2 = \frac{425v^2}{8}$

Q.(16)(a) (continued):

(ii) Range of speeds → Slowest when $T_2 = 0$ & all tension is in T_1.

Using (1): $75T_1 = 425g$

$T_1 = \frac{17g}{3}$ (3)

Sub. (3) in (2):

$40T_1 = \frac{425v^2}{8}$

$\frac{40 \times 17g}{3} = \frac{425v^2}{8}$

$41.81\ldots = v^2$

$6.466\ldots = v$

→ Slowest speed is 6.5 m/s (1DP).

(iii) → Fastest when $T_1 = 0$ & all tension is in T_2.

Using (1): $51T_2 = 425g$

$T_2 = \frac{25g}{3}$ (4)

Sub. (4) in (2): $68T_2 = \frac{425v^2}{8}$

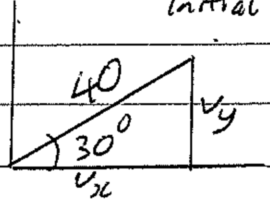
$\frac{68 \times 25g}{3} = \frac{425v^2}{8}$

$104.53\ldots = v^2$

$10.22\ldots = v$

→ Fastest speed is 10.2 m/s (1DP).

Q. (16)(b)(i)



Initial $v_y = 40\sin 30^0$ [Note: Initial $v_x = 40\cos 30^0$
$= 20 m/s$. $= 20\sqrt{3}\ m/s$].

Resolving forces vertically: $\downarrow 10g$ $\downarrow \frac{v^2}{32}$.

$F = ma = 10\ddot{y} = 10g + \frac{v^2}{32}$ [DOWN].

$\ddot{y} = -g - \frac{v^2}{320}$.

$v.\frac{dv}{dy} = -g - \frac{v^2}{320}$.

$= \frac{-320g - v^2}{320}$.

$\frac{dv}{dy} = \frac{-320g - v^2}{320v}$

$\frac{dy}{dv} = \frac{-320v}{v^2 + 320g}$

$y = -160 \int \frac{2v}{v^2 + 320g} . dv$

$= -160 \ln [v^2 + 320g] + c$.

As $y = 0$ when $v = 20$, $0 = -160 \ln [20^2 + 320g] + c$

$160 \ln [20^2 + 320g] = c$.

$y = -160 \ln [v^2 + 320g] + 160 \ln [400 + 320g]$.

$y = -160 \ln \left[\frac{v^2 + 320g}{400 + 320g}\right]$.

Max. height: $\dot{y} = v = 0 \therefore y = -160 \ln \left[\frac{320g}{400 + 320g}\right]$

Max height $\doteqdot 19.207..\ m$. [$= 19.2m$ (1DP)].

(ii) Time taken: $\ddot{y} = \frac{dv}{dt} = -g - \frac{v^2}{320}$.

$= \frac{-320g - v^2}{320}$.

$\frac{dt}{dv} = \frac{-320}{v^2 + 320g}$.

PTO →

Q.(16)(b)(ii)(continued): $t = -320\int \frac{1}{v^2+320g}\,dv$

$$= -320\int \frac{1}{v^2+3136}\,dv$$

$$t = -320 \times \frac{1}{56}\tan^{-1}\left(\frac{v}{56}\right) + C$$

As $v = 20$ when $t = 0$,

$$0 = -320 \times \frac{1}{56}\tan^{-1}\left(\frac{20}{56}\right) + C$$

$$\frac{40}{7}\tan^{-1}\left(\frac{5}{14}\right) = C$$

$$\therefore t = -\frac{40}{7}\tan^{-1}\left(\frac{v}{56}\right) + \frac{40}{7}\tan^{-1}\left(\frac{5}{14}\right).$$

When $v = 0$, $t = -\frac{40}{7}\tan^{-1}\left(\frac{0}{56}\right) + \frac{40}{7}\tan^{-1}\left(\frac{5}{14}\right)$

$$\doteqdot 1.9601\ldots \text{ seconds.}$$

(iii)

$$\leftarrow \frac{v^2}{32}.$$

$F = ma = 10\ddot{x} = \frac{v^2}{32}$ (in negative direction).

$$\ddot{x} = \frac{-v^2}{320}.$$

$$\frac{dv}{dt} = \frac{-v^2}{320} \quad [v = \dot{x}].$$

$$\frac{dt}{dv} = -\frac{320}{v^2}$$

$$t = \int \frac{-320}{v^2}\,dv$$

$$= \frac{320}{v} + C.$$

As $v = 20\sqrt{3}$ when $t = 0$,

$$0 = \frac{320}{20\sqrt{3}} + C$$

$$-\frac{16}{\sqrt{3}} = C.$$

$$t = \frac{320}{v} - \frac{16}{\sqrt{3}}.$$

$$t = \frac{320\sqrt{3} - 16v}{v\sqrt{3}}.$$

$\times 3v$.

$$\therefore 3tv = 960 - 16v\sqrt{3}$$

$$3tv + 16v\sqrt{3} = 960$$

$$v(3t + 16\sqrt{3}) = 960$$

$$v = \frac{960}{3t + 16\sqrt{3}}.$$

$$\frac{dx}{dt} = \frac{960}{3t + 16\sqrt{3}}.$$

$$x = \int \frac{960}{3t + 16\sqrt{3}}\,dt$$

$$x = 320\ln(3t + 16\sqrt{3}) + C$$

As $x = 0$ when $t = 0$

$$0 = 320\ln 16\sqrt{3} + C$$

$$-320\ln(16\sqrt{3}) = C.$$

$$x = 320\ln\left(\frac{3t + 16\sqrt{3}}{16\sqrt{3}}\right) = 320\ln\left(\frac{3t}{16\sqrt{3}} + 1\right).$$

Q.(16)(b)(iv) Horizontal distance for max. height:

i.e. Find x when $t = \frac{40}{7} \tan^{-1}\left(\frac{5}{14}\right) \doteqdot 1.96\ldots$

$$x = 320 \ln\left[\frac{3 \times 1.96\ldots + 1}{16\sqrt{3}}\right]$$

$$= 61.577\ldots$$

→ The cannonball has travelled 61.6 m horizontally to achieve its maximum height.

END OF SOLUTIONS!!!