GOSFORD H.S. 2004

Extension 2 Mathematics

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\int xe^{x^2}\,dx$ **1**

(b) Find $\int \frac{1+x}{1+x^2}\,dx$ **2**

(c) By completing the square find $\int \frac{dx}{\sqrt{6x-x^2}}$ **2**

(d) Decompose into partial fractions $\frac{5}{(x+3)(2x+1)}$ and hence **2**

find $\int \frac{5}{(x+3)(2x+1)}\,dx$ **1**

(e) Use the substitution $t=\tan x$, to find **3**

$$\int \frac{dx}{13-5\cos 2x}$$

(f) Given that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ prove that

$\int_0^\pi x\cos 2x\,dx = \int_0^\pi (\pi - x)\cos 2x\,dx$ and hence **2**

evaluate $\int_0^\pi x\cos 2x\,dx$ **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given $z=\frac{1+\sqrt{3}i}{1+i}$, determine

(i) $|z|$ **1**

(ii) $Arg(z)$ **2**

(b) Find the square root of $21-20i$ in the form $a+ib$ **3**

(c) What is the locus in the Argand diagram, of the point Z which represents the number z, where $z\bar{z}-2(z+\bar{z})=5$ **2**

(d) The point A represents the complex number α and the point Z_1 represents the complex number z_1. The point Z_1 is rotated about A through a right angle in the positive direction to take up the position Z_2, representing the complex number z_2. Show that $z_2=(1-i)\alpha+iz_1$. **3**

(e) Find the Cartesian equation of the locus of z if

(i) $\left|\frac{z-1}{z+1}\right|=\frac{1}{2}$ **2**

(ii) $Arg\left(\frac{z-1}{z+3}\right)=\frac{\pi}{2}$ **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1,y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ **2**

(ii) Prove that the tangents to a hyperbola at the end points of the latus rectum through a focus S meet at the foot of the directrix corresponding to S. **3**

(b) The base of a solid is a circle of radius 1 unit. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid. **3**

(c) Given the equation $x^2+xy+y^2=12$

(i) Show that $\frac{dy}{dx}=\frac{-(y+2x)}{x+2y}$ **2**

(ii) Deduce that vertical tangents exist at $(4,-2)$ and $(-4,2)$ and horizontal tangents exist at $(2,-4)$ and $(-2,4)$. **2**

(iii) Show that the curve is symmetrical about $y=x$ **1**

(ii) Sketch the curve showing these tangents and the intercepts on the coordinate axes. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use the substitution $x=5\tan\theta$ to find **3**

$$\int\frac{dx}{\left(25+x^2\right)^{\frac{3}{2}}}$$

(b) Find $\int\sin^{-1}x\,dx$ **2**

(c) If α,β and γ are the roots of $x^3+px+q=0$, find, in terms of the coefficients

(i) $\alpha+\beta+\gamma$ **1**

(ii) $(\alpha+\beta-2\gamma)(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)$ **2**

(d) Find the condition that $x^3-3p^2x+q=0$ has a repeated root. **3**

(e) If α,β and γ are the roots of $x^3+qx+r=0$ form the equation whose roots are α^2,β^2 and γ^2 **2**

(f) Given that $1-2i$ is a zero of the polynomial $P(x)=x^3-5x^2+11x-15$

(i) Explain why $1+2i$ is also a zero. **1**

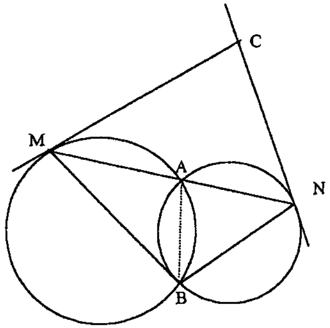
(ii) Find the other zero. **1**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is $bx\cos\theta+ay\sin\theta=ab$. Two points are taken on the minor axis at the same distance from the centre as the foci. Prove that the sum of the squares of the perpendiculars from these points to the tangent at $(a\cos\theta, b\sin\theta)$ is a constant. **4**

(b) A torus (doughnut) is formed by rotating the circle $x^2+y^2=1$ around the line $x=3$. Use the method of cylindrical shells to find the volume of the torus. **4**

(c) Two circles intersect at A and B. A line through A cuts the circles at M and N The tangents at M and N meet each other at C. Prove that M, C, N, B are concyclic. **4**



(d) Prove that for all(real) values of x, **3**

$xe^{-x}<1$

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$ **1**

Evaluate $\int_0^{\frac{\pi}{2}} \sin 5x\cos 3x\,dx$ **2**

(b) A particle is projected vertically upwards in a medium which exerts a resistance to the motion which is proportional to the square of velocity. The velocity of projection is V.

(i) Show that the acceleration ($\ddot{x}$) is given by $\ddot{x}=-(g+kv^2)$ **1**

(ii) Find in terms of V and k the maximum height attained and the time taken to reach this maximum height. **4**

(c) If x, y and z are positive and unequal, prove that

(i) $x+y-2\sqrt{xy}>0$ **1**

(ii) $(x+y)(y+z)(z+x)>8xyz$ **2**

(d)

(i) A body P of mass m travels with constant speed v in a horizontal circular arc with radius of curvature R on a surface inclined at an angle θ to the horizontal. If there is no tendency for the body to slip sideways show that **2**

$\tan\theta=\frac{v^2}{Rg}$

(ii) A railway line is taken around a bend of radius 1 000 metres. The distance between the rails is 1.5 metres. At what height above the inner rail should the outer rail be raised to eliminate lateral thrust for an engine travelling at a speed of 40 km per hour round the bend? (Take $g=9.8ms^{-2}$) **2**

Question 7 (15 marks) Use a SEPARATE writing booklet — **Marks**

(a) Use Mathematical induction to prove that $5^n \geq 1+4n$ — 3

(b) (i) Differentiate $\sec x$ — 1

(ii) Prove that

$$n\int \tan^n x \sec x\,dx = \tan^{n-1} x \sec x - (n-1)\int \tan^{n-2} x \sec x\,dx.$$ — 3

Hence or otherwise, find as a surd, the value of

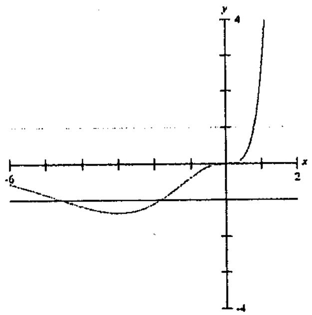
$$\int_0^{\frac{\pi}{4}} \frac{\sin^3 x}{\cos^4 x}dx$$ — 2

(c) If the middle term of the expansion of $(1+x)^{2n}$ is the greatest term, prove by considering expressions for $\frac{T_{n+1}}{T_n}$ and also $\frac{T_{n+2}}{T_{n+1}}$ that x lies between $1-\frac{1}{n+1}$ and $1+\frac{1}{n}$ — 4

(d) The graphs of $y=f(x)$ and $y=\pm 1$ are shown. On the sheets provided draw a neat sketch of

(i) $y^2 = f(x)$ — 1

(ii) $y=\frac{1}{f(x)}$ — 1



Question 8 (15 marks) Use a SEPARATE writing booklet — **Marks**

(a) (i) Express $-4+4i$ in mod arg form. — 1

(ii) By taking $z = r(\cos\theta + i\sin\theta)$ and using de Moivre's theorem show that the roots of $z^5 = -4+4i$ are given by — 3

$$z=\sqrt{2}\left(\cos\frac{k\pi}{20}+i\sin\frac{k\pi}{20}\right) \text{ where } k=3,11,19,27,33$$

(b) Give a sketch of the curve $y=\frac{1}{1+t}$, for $t>-1$. Indicate on your diagram areas which represent $\log(1+x)$ — 1

(i) for $x \geq 0$ — 1

(ii) for $-1 < x \leq 0$ — 1

and hence show that if $x > -1$,

$$\frac{x}{1+x} < \log(1+x) < x.$$ — 2

Deduce that if n is a positive integer,

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}$$ — 2

(c) The roots of $x^5+5x+1=0$ are $\alpha, \beta, \gamma, \delta$ and ε

(i) Write down the values of

$\sum\alpha$, $\sum\alpha\beta$, $\sum\alpha\beta\gamma$, $\sum\alpha\beta\gamma\delta$ and $\alpha\beta\gamma\delta\varepsilon$ — 1

(ii) Prove that the sum of the eleventh powers of roots of $x^5+5x+1=0$ is zero. (that is $\alpha^{11}+\beta^{11}+\gamma^{11}+\delta^{11}+\varepsilon^{11}=0$) — 3

Extension 2 Mathematics TRIAL HSC 2004

GOSFORD HIGH.

Question 1

a) $\int x e^{x^2} dx$

$= \frac{1}{2}\int 2x e^{x^2} dx$

$= \frac{1}{2} e^{x^2} + c$ ✓

b) $\int \frac{1+x}{1+x^2} dx = \int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx$

$= \tan^{-1} x + \frac{1}{2}\log(1+x^2) + C$

c) $\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{-(x^2-6x)}}$

$= \int \frac{dx}{\sqrt{9-(x^2-6x+9)}}$

$= \int \frac{dx}{\sqrt{9-(x-3)^2}}$

$= \sin^{-1}\frac{(x-3)}{3} + C$

d) $\frac{5}{(x+3)(2x+1)} = \frac{a}{x+3} + \frac{b}{2x+1}$

$= \frac{a(2x+1) + b(x+3)}{(x+3)(2x+1)}$

$\therefore 5 = a(2x+1) + b(x+3)$

let $x = -3$

$5 = a(-6+1) + b \times 0$

$5a = -5$

$a = -1$

let $x = -\frac{1}{2}$

$5 = a(2(-\frac{1}{2})+1) + b(-\frac{1}{2}+3)$

$5 = 0 + \frac{5}{2} b$

$b = 2$

$\therefore a = -1, b = 2$

$\therefore \frac{5}{(x+3)(2x+1)} = \frac{-1}{x+3} + \frac{2}{2x+1}$

$\therefore \int \frac{5}{(x+3)(2x+1)} dx = \int \left(\frac{-1}{x+3} + \frac{2}{2x+1}\right) dx$

$= -\ln(x+3) + \ln(2x+1) + c$

$= \ln\left\{\frac{2x+1}{x+3}\right\} + c$

e) $\int \frac{dx}{13 - 5\cos 2x}$

let $t = \tan x$

$\frac{dt}{dx} = \sec^2 x$

$\frac{dx}{dt} = \frac{1}{\sec^2 x}$

$\frac{dx}{dt} = \frac{1}{1+\tan^2 x}$

$\frac{dx}{dt} = \frac{1}{1+t^2}$

$dx = \frac{dt}{1+t^2}$

$= \int \frac{\frac{dt}{1+t^2}}{13 - 5\frac{(1-t^2)}{1+t^2}}$

$= \int \frac{dt}{13+13t^2-5+5t^2}$

$= \int \frac{dt}{8+18t^2}$

$= \frac{1}{18}\int \frac{dt}{\frac{8}{18}+t^2}$

$= \frac{1}{18}\int \frac{dt}{\frac{4}{9}+t^2}$

$= \frac{1}{18} \times \frac{1}{(\frac{2}{3})} \tan^{-1}\frac{t}{(\frac{2}{3})} + c$

$= \frac{1}{18} \times \frac{3}{2} \tan^{-1}\left(\frac{3}{2}\tan x\right) + c$

$= \frac{1}{12} \tan^{-1}\left\{\frac{3\tan x}{2}\right\} + c.$

f) $\int_0^{\pi} x\cos 2x\, dx = \int_0^{\pi} (\pi - x)\cos 2(\pi - x) dx$

$= \int_0^{\pi} (\pi - x)\cos(2\pi - 2x) dx$

$= \int_0^{\pi} (\pi - x)\cos 2x\, dx$

Since $\cos(360° - A) = \cos A$

$$\int_0^{\pi} x\cos 2x\,dx = \int_0^{\pi} \pi\cos 2x\,dx - \int_0^{\pi} x\cos 2x\,dx$$

$$\therefore 2\int_0^{\pi} x\cos 2x\,dx = \pi\int_0^{\pi}\cos 2x\,dx$$

$$= \pi \times \frac{1}{2}\left[\sin 2x\right]_0^{\pi}$$

$$= \frac{\pi}{2}\{\sin 2\pi - \sin 0\}$$

$$= \frac{\pi}{2}(0-0)$$

$$= 0$$

$$\therefore \int_0^{\pi} x\cos 2x\,dx = 0$$

Question 2.

a) $z = \dfrac{1+\sqrt{3}i}{1+i}$

(i) $|z| = \left|\dfrac{1+\sqrt{3}i}{1+i}\right|$

$$= \frac{|1+\sqrt{3}i|}{|1+i|}$$

$$= \frac{\sqrt{1^2+(\sqrt{3})^2}}{\sqrt{1^2+1^2}}$$

$$= \frac{2}{\sqrt{2}}$$

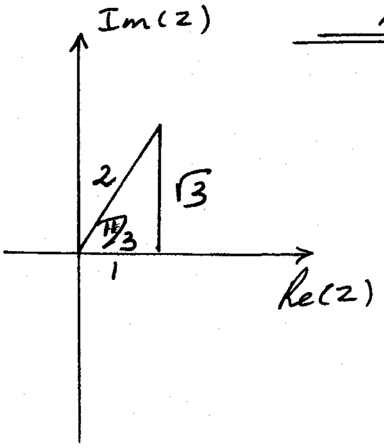
$$= \sqrt{2}.$$

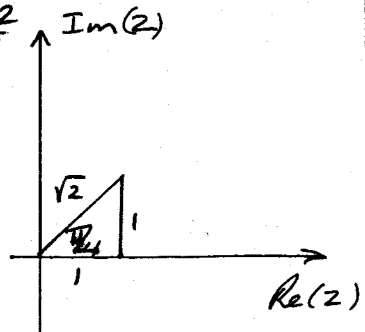
(ii) $\operatorname{Arg} z = \operatorname{Arg}\left(\dfrac{1+\sqrt{3}i}{1+i}\right)$

$$= \underbrace{\operatorname{Arg}(1+\sqrt{3}i)}_{\pi/3} - \underbrace{\operatorname{Arg}(1+i)}_{\pi/4}$$

$$=$$

$$= \frac{\pi}{12}$$





b) ~~Let~~

let $a+ib = \sqrt{21-20i}$

$(a+ib)^2 = 21-20i$

$a^2+2iab+i^2b^2 = 21-20i$

$a^2-b^2+2iab = 21-20i$

$a^2-b^2 = 21 \qquad 2ab = -20$

$\qquad b = -\dfrac{10}{a}$

$\therefore a^2 - \left(-\dfrac{10}{a}\right)^2 = 21$

$a^2 - \dfrac{100}{a^2} = 21$

$a^4 - 21a^2 - 100 = 0$

$(a^2-25)(a^2+4) = 0$

$a^2-25=0 \quad$ OR $\quad a^2+4=0$

$a = \pm 5 \qquad$ No real solutions

If $a=5$, $b=-2$

If $a=-5$, $b=2$

$\therefore \sqrt{21-20i} = \pm(5-2i)$

c) $z\bar{z} - 2(z+\bar{z}) = 5$

$(x+iy)(x-iy) - 2(x+iy+x-iy) = 5$

$x^2 - i^2y^2 - 4x - 5 = 0$

$x^2+y^2-4x-5=0$

$x^2-4x+4+y^2 = 5+4$

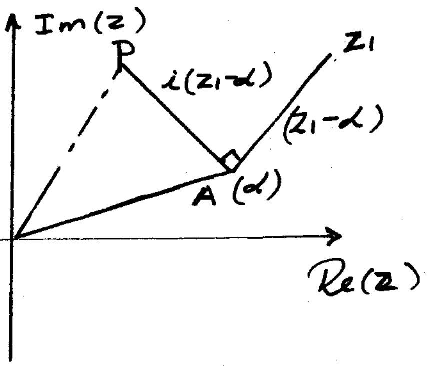
$(x-2)^2+y^2 = 9$

Circle centre $(2,0)$ Radius 3.

d) $\vec{AZ_1} = z_1 - \alpha$

$\vec{AP} = i(z_1-\alpha)$

$\vec{OP} = \vec{OA} + \vec{AP}$



$\therefore OP$

$= \alpha + i(z_1-\alpha)$

$= \alpha + iz_1 - i\alpha$

$= (1-i)\alpha + iz_1$

e) (i)

$$\left|\frac{z-1}{z+1}\right| = \frac{1}{2}$$

$$\frac{|z-1|}{|z+1|} = \frac{1}{2}$$

$$2|z-1| = |z+1|$$

$$2|x+iy-1| = |x+iy+1|$$

$$2|(x-1)+iy| = |(x+1)+iy|$$

$$2\sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}$$

$$4\{(x-1)^2+y^2\} = (x+1)^2+y^2$$

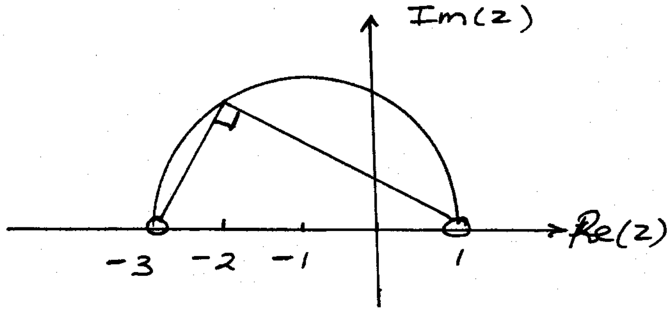
$$4(x^2-2x+1+y^2) = x^2+2x+1+y^2$$

$$4x^2-8x+4+4y^2 = x^2+2x+1+y^2$$

$$3x^2+3y^2-10x+3=0$$

(ii) $\arg\left\{\frac{(z-1)}{(z+3)}\right\} = \frac{\pi}{2}$

$$\arg(z-1) - \arg(z+3) = \frac{\pi}{2}$$



$$(x--1)^2+y^2 = 2^2, \quad y>0$$

$$\underline{\underline{(x+1)^2+y^2=4, \quad y>0.}}$$

<u>Question 3.</u>

a) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{2x}{a^2} = \frac{2y}{b^2}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}.$$

at (x_1,y_1) $\frac{dy}{dx} = \frac{b^2x_1}{a^2y_1}$

$\therefore$ the equation of the tangent is

$$y-y_1 = \frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$$

$$b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2$$

$\div$ by a^2b^2

$$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

But (x_1,y_1) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$\therefore$ Equ. of Tgt at (x_1,y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

(ii) Solve $x = ae$ & $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$(e^2-1) = \frac{y^2}{b^2}$$

$$y^2 = b^2(e^2-1)$$

But $b^2 = a^2(e^2-1) \Rightarrow e^2-1 = \frac{b^2}{a^2}$

$$y^2 = \frac{b^2 \times b^2}{a^2}$$

$$\therefore y = \pm\frac{b^2}{a}$$

$\therefore$ End points of the latus rectum are

$$\left(ae, \frac{b^2}{a^2}\right), \left(ae, -\frac{b^2}{a^2}\right)$$

$\therefore$ tangents at the end points of latus rectum are

P.T.O.

$$\frac{xae}{a^2} - \frac{y}{b^2} \times \frac{b^2}{a^2} = 1$$

i.e. $\frac{xe}{a} - \frac{y}{a^2} = 1 \quad ---1)$

and

$$\frac{xae}{a^2} - \frac{y}{b^2}\left(\frac{-b^2}{a^2}\right) = 1$$

$$\frac{xe}{a} + \frac{y}{a^2} = 1 \quad ---2)$$

Add Equs 1 & 2

$\therefore \frac{2xe}{a} = 2$

$x = \frac{a}{e}$

at $x = \frac{a}{e}$, from equ 1

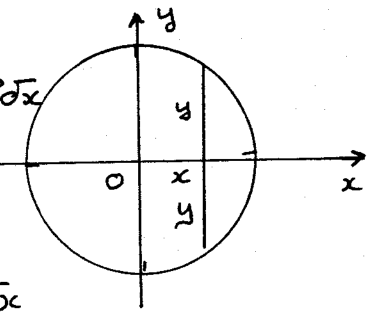
$\frac{a}{e} \times \frac{e}{a} - \frac{y}{a^2} = 1$

$1 - \frac{y}{a^2} = 1$

$0 = \frac{y}{a^2}$

$\therefore y = 0$

$\therefore$ tangents at the end pts of the latus rectum intersect at $(\frac{a}{e}, 0)$ as required.

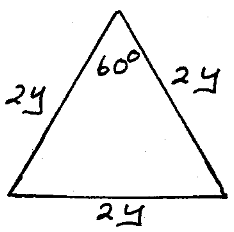


$\delta V = \frac{1}{2} \times 2y \times 2y \sin 60^\circ \delta x$

$\delta V = \frac{1}{2} \times 4y^2 \times \frac{\sqrt{3}}{2} \delta x$

$= \sqrt{3} y^2 \delta x$

$= \sqrt{3}(1-x^2)\delta x$



$V \doteq \sum_{-1}^{1} \sqrt{3}(1-x^2)\delta x$

$V = \lim_{\delta x \to 0} \sum_{-1}^{1} \sqrt{3}(1-x^2)\delta x$

$= \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx$

$= 2\int_0^1 \sqrt{3}(1-x^2)\,dx$

$= 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1$

$= 2\sqrt{3}\left\{\left(1-\frac{1}{3}\right) - (0-0)\right\}$

$= 2\sqrt{3} \times \frac{2}{3}$

$= \frac{4\sqrt{3}}{3}$ cubic units.

c) (i) $x^2 + xy + y^2 = 12$

$2x + x\frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = 0$

$(x+2y)\frac{dy}{dx} = -(2x+y)$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

(ii) Vertical tangents exist when denominator of $\frac{dy}{dx} = 0$ i.e. $x + 2y = 0$

i.e. $y = -\frac{1}{2}x$

Solve $x^2 + xy + y^2 = 12$

and $y = -\frac{1}{2}x$

$x^2 + x\left(-\frac{1}{2}x\right) + \left(-\frac{1}{2}x\right)^2 = 12$

$x^2 - \frac{x^2}{2} + \frac{x^2}{4} = 12$

$4x^2 - 2x^2 + x^2 = 48$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

If $x = 4, y = -2$ & if $x = -4, y = 2$

$\therefore$ Vertical tangents exist at $(4,-2)$ and $(-4,2)$.

Horizontal Tangents exist when the denominator of $\frac{dy}{dx} = 0$ $\therefore 2x + y = 0$

$y = -2x$

Solve

$x^2+xy+y^2=12$

and

$y=-2x$

$\therefore x^2+x(-2x)+(-2x)^2=12$

$x^2-2x^2+4x^2=12$

$3x^2=12$

$x^2=4$

$x=\pm 2$

If $x=2, y=-4$ and if $x=-2$ $y=4$

$\therefore$ horizontal tangents exist at $(2,-4)$ and at $(-2,4)$.

(iii) x^2+xy

$x^2+xy+y^2=12$.

Interchang x and y

$\therefore y^2+yx+x^2=12$

i.e. we obtain the same curve $\therefore$ the curve is symmetrical about $y=x$.

Solve $x^2+xy+y^2=12$ and $y=x$

$x^2+x^2+x^2=12$

$3x^2=12$

$x^2=4$

$x=\pm 2$

$\therefore y=x$ & $x^2+xy+y^2=12$ intersect at $(2,2)$ and $(-2,-2)$.

$x^2+xy+y^2=12$ cuts the x axis when $y=0$

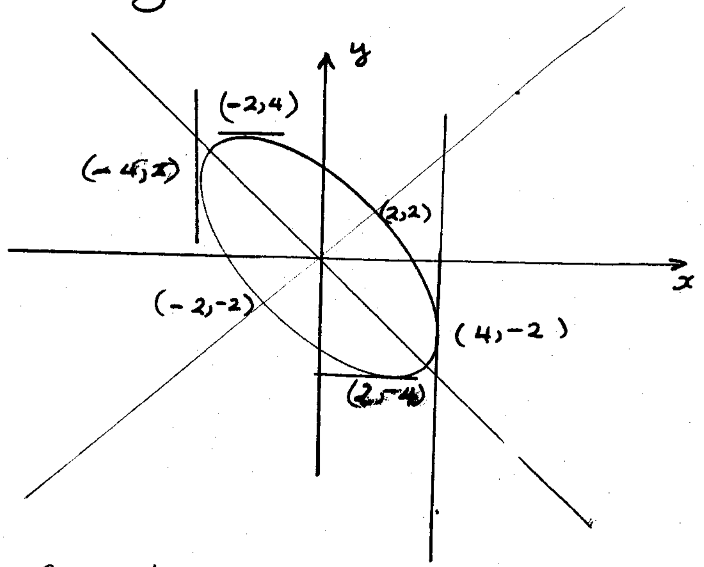
$\therefore x^2=12$

$x=\pm 2\sqrt{3}$

The curve cuts the y axis when $x=0$

$\therefore y^2=12$

$y=\pm 2\sqrt{3}$.



Question 4

a) $\int \frac{dx}{(25+x^2)^{3/2}}$ let $x=5\tan\theta$

$\frac{dx}{d\theta}=5\sec^2\theta$.

$dx=5\sec^2\theta\, d\theta$

$=\int \frac{5\sec^2\theta\, d\theta}{(25+25\tan^2\theta)^{3/2}}$

$=\int \frac{5\sec^2\theta\, d\theta}{125(1+\tan^2\theta)^{3/2}}$

$=\frac{1}{25}\int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}}$

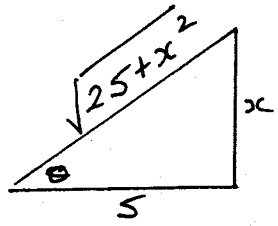
$=\frac{1}{25}\int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$

$=\frac{1}{25}\int \frac{1}{\sec\theta}\, d\theta$

$=\frac{1}{25}\int \cos\theta\, d\theta$

$=\frac{1}{25}\sin\theta+C$.

$=\frac{1}{25}\frac{x}{\sqrt{25+x^2}}+C$.



b) $\int \sin^{-1}x\, dx=\int \frac{d}{dx}(x)\sin^{-1}x\, dx$.

$\therefore \int \sin^{-1}x\, dx = x\sin^{-1}x-\int x\times\frac{1}{\sqrt{1-x^2}}\, dx$

$=x\sin^{-1}x-\int x(1-x^2)^{-\frac{1}{2}}dx$

$=x\sin^{-1}x-\frac{1}{-2}\int -2x(1-x^2)^{-\frac{1}{2}}dx$

$= x\sin^{-1}x + \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \underline{\underline{x\sin^{-1}x + \sqrt{1-x^2} + c}}$.

c)(i) $x^3+px+q=0$

i.e. $x^3+0x^2+px+q=0$

has roots α, β, γ

$\alpha+\beta+\gamma = -\frac{b}{a}$

$= -\frac{0}{1}$

$= 0$

(ii) $\alpha+\beta+\gamma=0$

$\therefore \alpha+\beta = -\gamma$

similarly

$\alpha+\gamma = -\beta$

and $\beta+\gamma = -\alpha$.

$\therefore (\alpha+\beta-2\gamma)(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)$

$= (-\gamma-2\gamma)(-\alpha-2\alpha)(-\beta-2\beta)$

$= (-3\gamma)(-3\alpha)(-3\beta)$

$= -27\alpha\beta\gamma$

$= -27\left(-\frac{d}{a}\right)$

$= -27 \times \frac{-q}{1}$

$= 27q$.

d) $x^3-3p^2x+q=0$

let $P(x) = x^3-3p^2x+q$

$P'(x) = 3x^2-3p^2$.

$P(x)$ has a repeated zero if $P(\alpha)=0$ & $P'(\alpha)=0$

where α is the repeated zero.

If $P'(x)=0$

$3x^2-3p^2=0$

$\therefore x^2=p^2$

$x=\pm p$.

If this is the repeated root then it satisfies the equation

$x^3-3p^2x+q=0$

$\therefore (\pm p)^3-3p^2(\pm p)+q=0$

$\pm p\times p^2-3p^2(\pm p) = -q$

$\pm p(p^2-3p^2) = -q$

$\pm p(-2p^2) = -q$

Square both sides.

$p^2\times 4p^4 = q^2$

$4p^6 = q^2$

e) $x^3+qx+r=0$ has roots α, β, γ

$\therefore \alpha^3+q\alpha+r=0$

We want an equation with roots $\alpha^2, \beta^2, \gamma^2$

Let $X=\alpha^2 \Rightarrow \alpha=\sqrt{X}$

Substitute $\sqrt{x}$ for α

in $\alpha^3+q\alpha+r=0$

$\therefore (\sqrt{X})^3+q(\sqrt{X})+r=0$

$X\sqrt{X}+q\sqrt{X} = -r$

$\sqrt{X}(X+q) = -r$

Square both sides

$X(X^2+2qX+q^2) = r^2$

$\therefore$ Required equation is

$X^3+2qX^2+q^2X-r^2=0$.

f)(i) The complex conjugate $1+2i$ is also a zero because the coefficients are real.

(ii) let the other zero be α

$$\therefore \alpha(1-2i)(1+2i) = \frac{-d}{a}$$

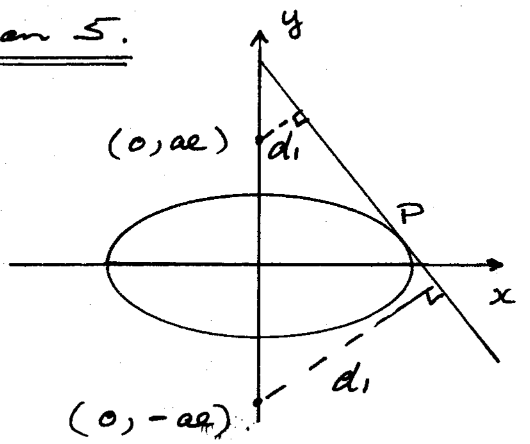
$$\alpha(1-4i^2) = \frac{-{-15}}{1}$$

$$\alpha(1+4) = 15$$

$$5\alpha = 15$$

$$\alpha = 3.$$

Question 5.



tangent is

$$bx\cos\theta + ay\sin\theta - ab = 0$$

$$d_1 = \left| \frac{0 + aae\sin\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right|$$

$$= \left| \frac{a^2e\sin\theta - ab}{\sqrt{b^2(1-\sin^2\theta) + a^2\sin^2\theta}} \right|$$

$$= \left| \frac{a^2e\sin\theta - ab}{\sqrt{b^2 + (a^2-b^2)\sin^2\theta}} \right|$$

But

$$b^2 = a^2(1-e^2)$$

$$\therefore a^2e^2 = a^2 - b^2$$

$$\therefore d_1 = \left| \frac{a^2e\sin\theta - ab}{\sqrt{b^2 + a^2e^2\sin^2\theta}} \right|$$

Similarly

$$d_2 = \left| \frac{-a^2e\sin\theta - ab}{\sqrt{b^2 + a^2e^2\sin^2\theta}} \right|$$

$$d_1^2 + d_2^2$$

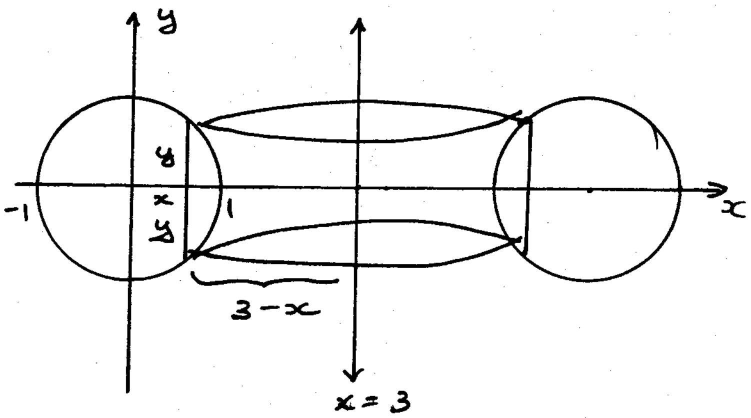
$$= \frac{a^4e^2\sin^2\theta - 2a^3be\sin\theta + a^2b^2}{b^2 + a^2e^2\sin^2\theta} + \frac{a^4e^2\sin^2\theta + 2a^2be\sin\theta + a^2b^2}{b^2 + a^2e^2\sin^2\theta}$$

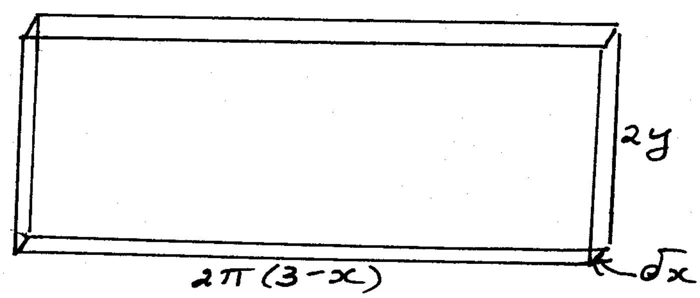
$$= \frac{2a^4e^2\sin^2\theta + 2a^2b^2}{b^2 + a^2e^2\sin^2\theta}$$

$$= \frac{2a^2\{a^2e^2\sin^2\theta + b^2\}}{b^2 + a^2e^2\sin^2\theta}$$

$$= 2a^2$$

which is a constant.





$$\delta V = 2\pi(3-x) \times 2y \times \delta x$$

$$V \doteq \sum_{-1}^{1} 2\pi(3-x)\,2y\,\delta x$$

$$= 4\pi \sum_{-1}^{1} (3-x)\,y\,\delta x$$

$$= 4\pi \sum_{-1}^{1} (3-x)\sqrt{1-x^2}\,\delta x$$

$$V = \lim_{\delta x \to 0} 4\pi \sum_{-1}^{1} (3-x)\sqrt{1-x^2}\,\delta x.$$

$$= 4\pi \int_{-1}^{1} (3-x)\sqrt{1-x^2}\,dx$$

$$= 4\pi \int_{-1}^{1} 3\sqrt{1-x^2}\,dx - 4\pi \int_{-1}^{1} x\sqrt{1-x^2}\,dx$$

$$= 12\pi \int_{-1}^{1} \sqrt{1-x^2}\,dx \quad - 4\pi \times 0$$

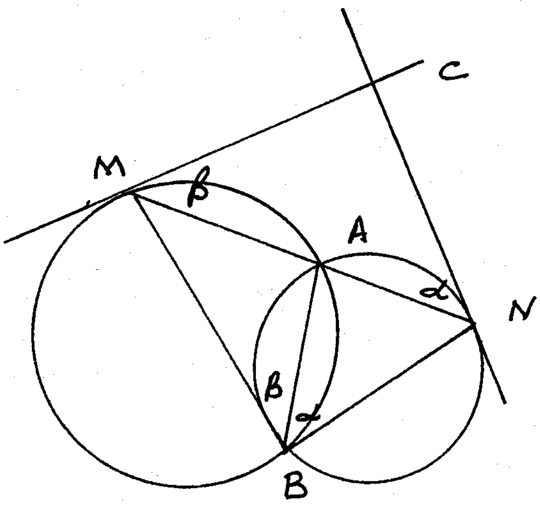
$$= 12\pi \times \tfrac{1}{2}\pi r^2 \qquad 0$$

$$= 6\pi \times \pi \times 1^2$$

$$= 6\pi^2$$

Since $x\sqrt{1-x^2}$ is an odd function

$= 6\pi^2$ Cubic Units.



Let $\angle MNC = \alpha$ and $\angle NMC = \beta$

$\angle ABN = \angle MNC$ (alt. seg. theorem)
$= \alpha$

$\angle MBA = \angle NMC$ (alt. seg. theorem)
$= \beta$

$\angle MCN = 180 - (\alpha + \beta)$ $\angle$ sum of $\triangle$

$\therefore \angle MBN = \alpha + \beta$

$\therefore \angle MCN + \angle MBN = 180°$

Hence MCNB is a cyclic quad. since the opp. $\angle$'s are supplementary

d) If $xe^{-x} < 1$
$0 < 1 - xe^{-x}$

Consider $f(x) = 1 - xe^{-x}$

$$f'(x) = 0 - \{x(-e^{-x}) + e^{-x}.1\}$$
$$= xe^{-x} - e^{-x}$$
$$= e^{-x}(x-1)$$

Stat. pts occur when $\frac{dy}{dx} = 0$ i.e. $e^{-x}(x-1) = 0$

i.e. $x - 1 = 0$
$x = 1$

Note $e^{-x} > 0$ for all x

at $x = 1$, $y = f(x)$
$= 1 - e^{-1}$
$= 1 - \frac{1}{e}$

$\therefore$ Stat. pt at $(1, 1 - \frac{1}{e})$

$$f''(x) = e^{-x}.1 + (x-1)(-e^{-x})$$
$$= e^{-x}(1 - x + 1)$$
$$= e^{-x}(2 - x)$$

at $x = 1$, $f''(x) = e^{-1}(2-1)$
> 0

$\therefore$ Min at $(1, 1 - \frac{1}{e})$

i.e. $f(x) > 1 - \frac{1}{e}$ for all x

i.e. $f(x) > 0$ for all x

i.e. $1 - xe^{-x} > 0$
$1 > xe^{-x}$
$xe^{-x} < 1$
as required

Question 6

a) $\sin(A+B) + \sin(A-B)$

$= \sin A \cos B + \cos A \sin B$
$\quad + \sin A \cos B - \cos A \sin B$

$= 2\sin A \cos B$

$\int_0^{\pi/2} \sin 5x \cos 5x \, dx$

$= \frac{1}{2}\int_0^{\pi/2} (\sin 8x + \sin 2x)\, dx$

$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\pi/2}$

$= -\frac{1}{16}\left[\cos 8x + 4\cos 2x\right]_0^{\pi/2}$

$= -\frac{1}{16}\left\{\cos 4\pi + 4\cos\pi - (\cos 0 + 4\cos 0)\right\}$

$= -\frac{1}{16}\left\{1 - 4 - (1+4)\right\}$

$= -\frac{1}{16} \times -8$

$= \frac{1}{2}$

b) (i) $m\ddot{x} = -mg - mkv^2$

$\ddot{x} = -g - kv^2$

$\ddot{x} = -(g + kv^2)$

(ii) $v\frac{dv}{dx} = -(g+kv^2)$

$\frac{dv}{dx} = -\frac{(g+kv^2)}{v}$

$\frac{dx}{dv} = -\frac{v}{g+kv^2}$

$x = \int -\frac{v}{g+kv^2}\, dv$

$x = -\frac{1}{2k}\int \frac{2kv}{g+kv^2}\, dv$

$x = -\frac{1}{2k}\ln(g+kv^2) + c$

at $x=0$, $v=V$

$0 = -\frac{1}{2k}\ln(g+kV^2) + c$

$c = \frac{1}{2k}\ln(g+kV^2)$

$x = -\frac{1}{2k}\ln(g+kv^2) + \frac{1}{2k}\ln(g+kV^2)$

$x = \frac{1}{2k}\ln\left\{\frac{g+kV^2}{g+kv^2}\right\}$

Maximum height is attained when $v=0$

$\therefore x_{max} = \frac{1}{2k}\ln\left\{\frac{g+kV^2}{g}\right\}$

$\frac{dv}{dt} = -(g+kv^2)$

$\frac{dt}{dv} = \frac{-1}{g+kv^2}$

$t = \int \frac{-1}{g+kv^2}\, dv$

$= -\frac{1}{k}\int \frac{1}{\frac{g}{k}+v^2}\, dv$

$t = -\frac{1}{k} \times \frac{1}{\sqrt{\frac{g}{k}}}\tan^{-1}\left(\frac{v}{\sqrt{\frac{g}{k}}}\right) + c_1$

$t = -\frac{1}{k} \times \frac{\sqrt{k}}{\sqrt{g}}\tan^{-1}\frac{\sqrt{k}\,v}{\sqrt{g}} + c_1$

at $t=0$, $v=V$

$\therefore 0 = \frac{-1}{\sqrt{kg}}\tan^{-1}\frac{\sqrt{k}}{\sqrt{g}}V + c_1$

$c_1 = \frac{1}{\sqrt{kg}}\tan^{-1}\frac{\sqrt{k}}{\sqrt{g}}V$

$t = \frac{-1}{\sqrt{kg}}\tan^{-1}\frac{\sqrt{k}}{\sqrt{g}}v + \frac{1}{\sqrt{kg}}\tan^{-1}\frac{\sqrt{k}}{\sqrt{g}}V$

max. height is reached when $v=0$

$\therefore t = \frac{1}{\sqrt{kg}}\tan^{-1}\frac{\sqrt{k}}{\sqrt{g}}V.$

c) (i)

$(x-y)^2 \geq 0$

$\therefore x^2-2xy+y^2 \geq 0$

$x^2+y^2 \geq 2xy$

Add $2xy$ to both sides

$x^2+2xy+y^2 \geq 4xy$

$(x+y)^2 \geq 4xy$

$\therefore x+y \geq 2\sqrt{xy}$

$\underline{\underline{x+y-2\sqrt{xy} \geq 0}}$

(ii) $x+y \geq 2\sqrt{xy}$

$x+z \geq 2\sqrt{xz}$

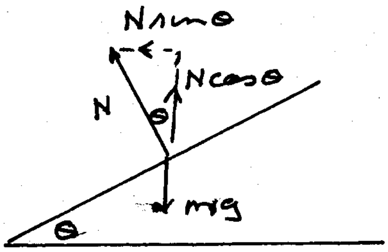
$y+z \geq 2\sqrt{yz}$

$\therefore (x+y)(x+z)(y+z) \geq 8\sqrt{xyxzyz}$

i.e. $(x+y)(x+z)(y+z) \geq 8\sqrt{x^2y^2z^2}$

$\underline{\underline{(x+y)(x+z)(y+z) \geq 8xyz}}$

d)



$N\sin\theta = \frac{mv^2}{R}$

$N\cos\theta = mg$

$\frac{N\sin\theta}{N\cos\theta} = \frac{\frac{mv^2}{R}}{mg}$

$\tan\theta = \frac{v^2}{Rg}$

(ii)

$\tan\theta = \frac{v^2}{Rg}$

$\frac{h}{1{\cdot}5} = \sin\theta$

But θ is small

$\therefore \sin\theta \doteqdot \tan\theta$

$\therefore \frac{h}{1{\cdot}5} = \tan\theta$

$h = 1{\cdot}5 \tan\theta$

$= 1{\cdot}5 \frac{v^2}{Rg}$.

$= \frac{1{\cdot}5 \times \left(\frac{40 \times 1000}{60 \times 60}\right)^2}{1000 \times 9{\cdot}8}$

$= 0{\cdot}018896447$ metres

$\underline{\underline{= 18{\cdot}9 \text{ mm}}}$.

Question 7.

a) Step 1.

If $n = 1$

$5^1 \geq 1+4\times1$ which is true

Step 2.

Prove that the result is true for $n = k$

i.e. $5^k \geq 1+4k$.

Step 3

Prove that the result is true for $n = k+1$

i.e. prove that

$5^{k+1} \geq 1+4(k+1)$

Proof: $5^{k+1} = 5 \cdot 5^k$

$\therefore 5^{k+1} \geq 5(1+4k)$

$5^{k+1} \geq 5+5\times 4k$

$\therefore 5^{k+1} \geq 5+4k$

$5^{k+1} \geq 1+4k+4$

$5^{k+1} \geq 1+4(k+1)$

Hence the result is true for $n=k+1$ if it is true for $n=k$.

Step 4: Since the result is true for $n=1$ then it is true for $n=1+1$ i.e. for $n=2$ and thus for $n=3$ and so on for all positive integral values of n.

b)(i) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(ii) $\int \tan^n x \sec x \, dx$

$= \int \tan^{n-1} x \sec x \tan x \, dx$

$= \int \tan^{n-1} x \frac{d}{dx}(\sec x) \, dx$

$= \sec x \tan^{n-1} x - \int \sec x (n-1) \tan^{n-2} x \sec^2 x \, dx$

$\therefore \int \tan^n x \sec x \, dx = \sec x \tan^{n-1} x - (n-1) \int \sec x \tan^{n-2} x (1+\tan^2 x) dx$

$= \sec x \tan^{n-1} x - (n-1) \int \sec x \tan^{n-2} x \, dx - (n-1) \int \sec x \tan^n x \, dx$

$\therefore \int \tan^n x \sec x \, dx + (n-1) \int \sec x \tan^n x \, dx = \sec x \tan^{n-1} x - (n-1) \int \tan^{n-2} x \sec x \, dx$

$n \int \tan^n x \sec x \, dx = \sec x \tan^{n-1} x - (n-1) \int \tan^{n-2} x \sec x \, dx$

$\int \tan^n x \sec x \, dx = \frac{1}{n} \sec x \tan^{n-1} x - \frac{(n-1)}{n} \int \tan^{n-2} x \sec x \, dx$

$\int_0^{\pi/4} \frac{\sin^3 x}{\cos^4 x} dx$

$= \int_0^{\pi/4} \sec x \tan^3 x \, dx$

$= \frac{1}{3}\left[\sec x \tan^2 x\right]_0^{\pi/4} - \frac{2}{3} \int_0^{\pi/4} \tan x \sec x \, dx$

$= \frac{1}{3}\left\{\sec\frac{\pi}{4} \tan^2 \frac{\pi}{4} - \sec 0 \tan^2 0\right\} - \frac{2}{3}\left[\sec x\right]_0^{\pi/4}$

$= \frac{1}{3}\left\{\sqrt{2}\times 1^2 - 1\times 0\right\} - \frac{2}{3}\left\{\sec\frac{\pi}{4} - \sec 0\right\}$

$= \frac{1}{3}\sqrt{2} - \frac{2}{3}(\sqrt{2}-1)$

$= \frac{\sqrt{2}}{3} - \frac{2\sqrt{2}}{3} + \frac{2}{3}$

$= \frac{1}{3}(2-\sqrt{2})$

c) See next page.

c) T_{n+1} is the middle term in the expansion of $(1+x)^{2n}$.

$$\frac{T_{n+1}}{T_n} = \frac{{}^{2n}C_n\, x^n}{{}^{2n}C_{n-1}\, x^{n-1}}$$

$$= \frac{(2n)!}{n!(2n-n)!} \times \frac{(n-1)!\,(2n-(n-1))!\,.\,x}{(2n)!}$$

$$= \frac{n+1}{n}\, x$$

$$\frac{T_{n+2}}{T_{n+1}} = \frac{{}^{2n}C_{n+1}\, x^{n+1}}{{}^{2n}C_n\, x^n}$$

$$= \frac{(2n)!}{(n+1)!\,(2n-(n+1))!} \times \frac{n!\,(2n-n)!\,.\,x}{(2n)!}$$

$$= \frac{n!\,n!}{(n+1)!\,(n-1)!}\, x$$

$$= \frac{n}{n+1}\, x$$

If $T_{n+1} > T_n \Leftarrow$ i.e. $\frac{T_{n+1}}{T_n} > 1$

$$\frac{n+1}{n}\, x > 1$$

$$(n+1)x > n$$

$$x > \frac{n}{n+1}$$

$$\text{i.e. } x > \frac{1+n-1}{n+1}$$

$$x > 1 - \frac{1}{n+1}$$

If $T_{n+1} > T_{n+2} \Leftarrow$ i.e. $\frac{T_{n+2}}{T_{n+1}} < 1$

$$\frac{n}{n+1}\, x < 1$$

$$nx < (n+1)$$

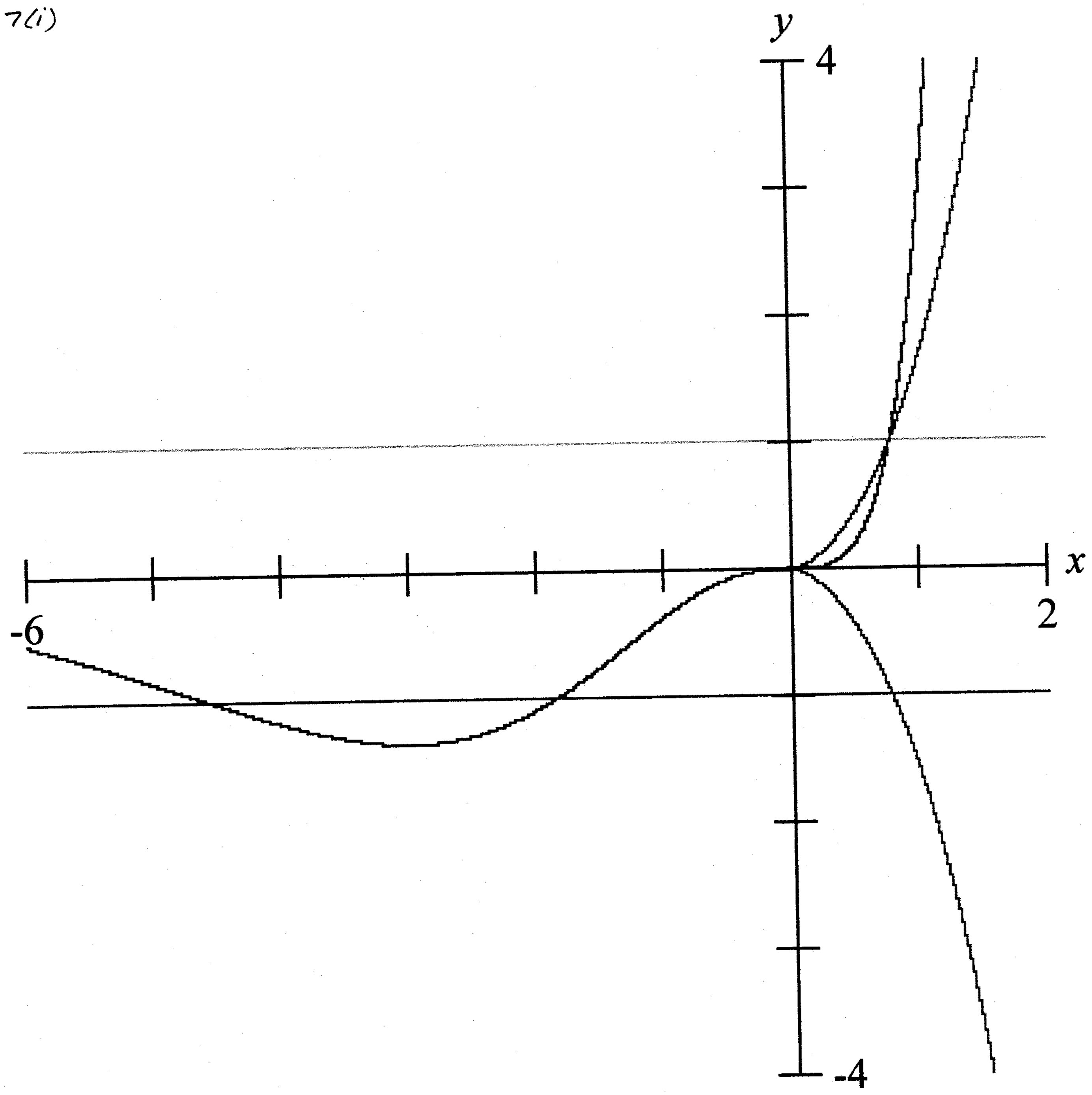
$$x < \frac{n+1}{n}$$

$$x < 1 + \frac{1}{n}$$

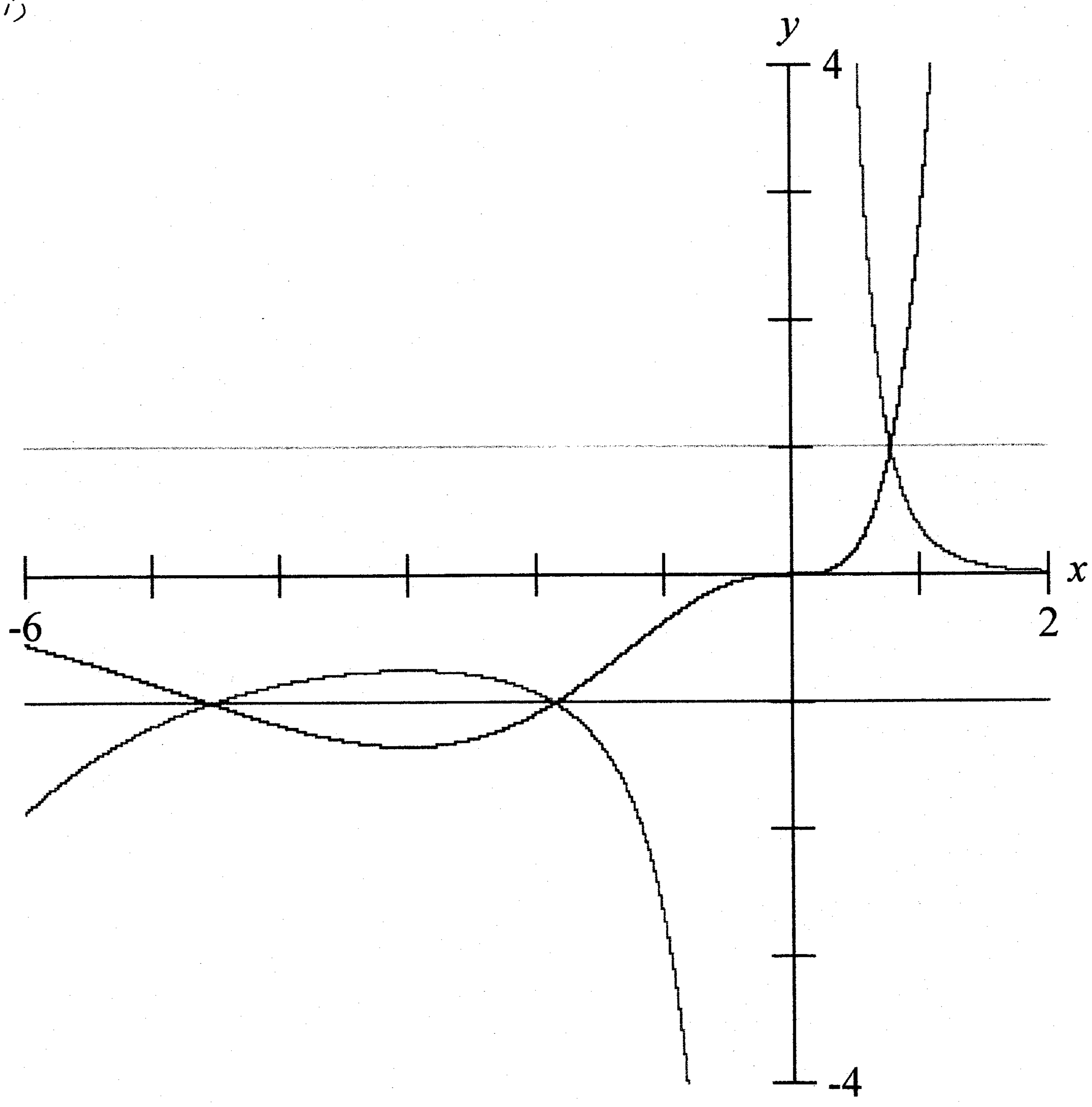
$\therefore$ x lies between $1 - \frac{1}{n+1}$ and $1 + \frac{1}{n}$.

Q 7 d)

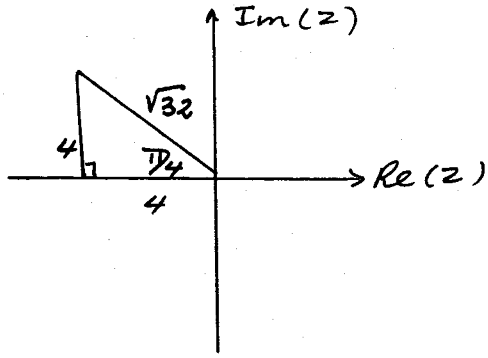
7(i)



Q7d)
7(ii)



Question 8.



a) $z^5 = -4+4i$

$= \sqrt{32}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

$= 2^{\frac{5}{2}}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

Let $z = r(\cos\theta + i \sin\theta)$

$\therefore r^5(\cos\theta + i\sin\theta)^5 = 2^{\frac{5}{2}}\left(\cos\left(\frac{3\pi}{4}+2n\pi\right) + i \sin\left(\frac{3\pi}{4}+2n\pi\right)\right)$

$r^5(\cos 5\theta + i \sin 5\theta) = 2^{5/2}\left\{\cos\left(\frac{3\pi}{4}+2n\pi\right) + i\sin\left(\frac{3\pi}{4}+2n\pi\right)\right\}$

$\therefore r^5 = 2^{5/2}$ and $5\theta = \frac{3\pi}{4} + 2n\pi$

$r = 2^{\frac{1}{2}}$ $\quad = \frac{3\pi + 8n\pi}{4}$

$r = \sqrt{2}$

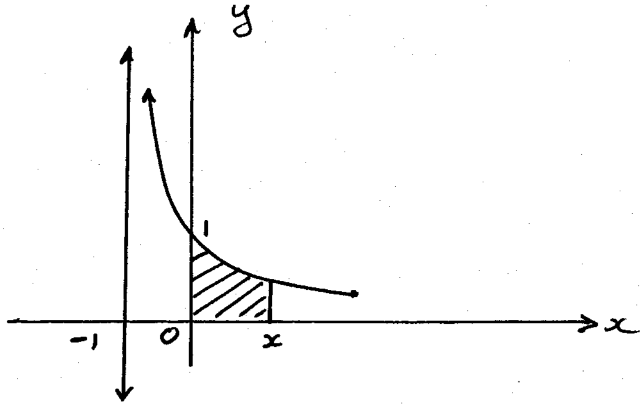
$\theta = \frac{\pi}{20}(8n+3)$ where $k = 0, 1, 2, 3, 4$

$\therefore \theta = \frac{3\pi}{20}, \frac{11\pi}{20}, \frac{19\pi}{20}, \frac{27\pi}{20}, \frac{35\pi}{20}$

$\therefore$ Roots of $z^5 = -4+4i$ are of the form

$z = \sqrt{2}\left\{\cos \frac{k\pi}{20} + i \sin \frac{k\pi}{20}\right\}$ where $k = 3, 11, 19, 27, 35$

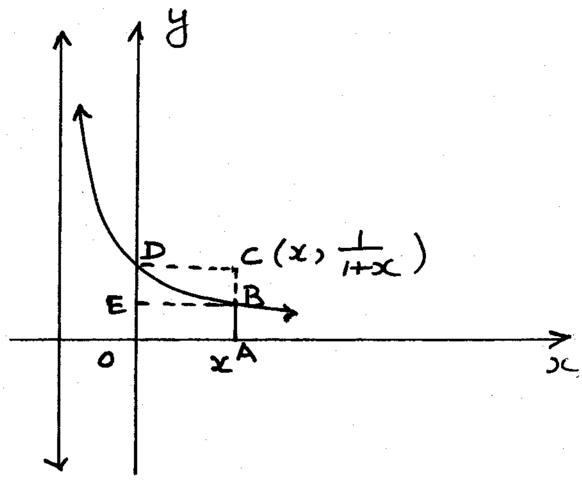
b)



$\int_0^x \frac{1}{1+t}\,dt = \left[\ln(1+t)\right]_0^x$

$= \ln(1+x) - \ln 1$

$= \underline{\underline{\ln(1+x)}}$.



Area OABE $< \int_0^x \frac{1}{1+t}\,dt <$ Area OACD

$x \cdot \frac{1}{1+x} < \int_0^x \frac{1}{1+t}\,dt < x \cdot 1$

$\therefore \frac{x}{1+x} < \int_0^x \frac{1}{1+t}\,dt < x.$

i.e. $\frac{x}{1+x} < \ln(1+x) < x$

Let $x = \frac{1}{n}$ $\quad \therefore \frac{\frac{1}{n}}{1+\frac{1}{n}} < \log\left(1+\frac{1}{n}\right) < \frac{1}{n}$

$\frac{1}{n+1} < \log\left(\frac{n+1}{n}\right) < \frac{1}{n}$

$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}$

8c) let the roots be $\alpha, \beta, \gamma, \delta, \varepsilon$.

Now $\quad \Sigma\alpha =$

$$x^5 + 0x^4 + 0x^3 + 0x^2 + 5x + 1 = 0$$

$$\therefore \Sigma\alpha = 0$$
$$\Sigma\alpha\beta = 0$$
$$\Sigma\alpha\beta\gamma = 0$$
$$\Sigma\alpha\beta\gamma\delta = 5$$
$$\alpha\beta\gamma\delta\varepsilon = -1$$

$$x^5 = -5x - 1$$
$$\therefore x^{10} = 25x^2 + 10x + 1$$
$$\therefore \alpha^{10} = 25\alpha^2 + 10\alpha + 1$$
$$\alpha^{11} = 25\alpha^3 + 10\alpha^2 + \alpha$$

hence $\quad \Sigma\alpha^{11} = 25\Sigma\alpha^3 + 10\Sigma\alpha^2 + \Sigma\alpha$

$$= 25\Sigma\alpha^3 + 10 \times 0 + 0$$
$$= 25\Sigma\alpha^3$$

Now $\quad x^5 = -5x - 1$

$$\alpha^5 = -5\alpha - 1$$

$\div$ by α^2

$$\alpha^3 = -\frac{5}{\alpha} - \frac{1}{\alpha^2}$$

Similar results hold for $\beta, \gamma, \delta, \varepsilon$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 = -5\left\{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \frac{1}{\varepsilon}\right\} - \left\{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} + \frac{1}{\varepsilon^2}\right\}$$

$$= -5\left\{\frac{\Sigma\alpha\beta\gamma\delta}{\alpha\beta\gamma\delta\varepsilon}\right\} - \frac{\Sigma\alpha^2\beta^2\gamma^2\delta^2}{(\alpha\beta\gamma\delta\varepsilon)^2}$$

$$= -\frac{5 \times 5}{-1} - \left\{\frac{(\Sigma\alpha\beta\gamma\delta)^2 - 2\Sigma(\alpha\beta\gamma\delta\,\alpha\beta\gamma\varepsilon)}{(-1)^2}\right\}$$

$$= +25 - \{5^2 - 2\alpha\beta\gamma\delta\varepsilon\,\Sigma\alpha\beta\gamma\}$$

$$= 25 - (25 - 2(-1) \times 0)$$
$$= 25 - 25$$
$$= 0 \quad \text{as required}$$