Question 1

a) Find $\int \sqrt{x^2 - 2x^4}\, dx$. 2.

b) Find $\int \frac{dx}{\sqrt{28 - 12x - x^2}}$ 3.

c) By the use of partial fractions find $\int \frac{x+1}{x^3 + x^2 - 6x} dx$. 3.

d) Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \sin x}$. 3.

e) i) If $I_n = \int_1^e x\ (\ln x)^n\, dx$ for $n = 0,1,2.....$ Use integration by parts to

show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for n = 1,2,3,.... 2.

ii) Hence find the value of I_2 2.

Question 2.

a) If $Z_1 = 2 + i$ and $Z_2 = 4 - 3i$ find in the form $a + ib$:

i) $Z_1 + Z_2$ 1.

ii) $Z_1.Z_2$ 2.

iii) $\frac{Z_1}{Z_2}$ 2.

b) write in modulus argument form $-\sqrt{3} - i$ 3.

c) Let u and v be two complex numbers, where $u = -2 + i$ and v is defined by $|v| = 3$ and $\arg v = \frac{\pi}{3}$

i)On an argand diagram plot the points A and B representing the complex numbers u and v respectively 2.

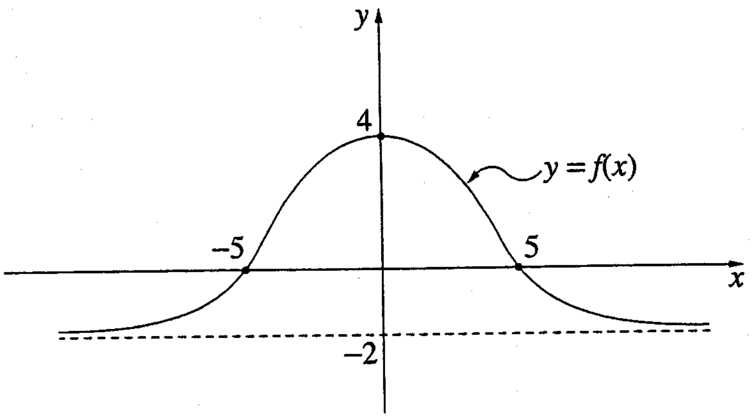
ii) Plot the points C and D representing the complex numbers $u - v$ and iu respectively. Indicate any geometric relationship between the four points A,B,C and D. 3.

d) Describe the set of points in the complex plane that satisfies
$|Z+1| = |Z-i|$ 2.

Question 3.

a) Sketch the graph of $f(x) = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}$ clearly indicating any maximum or minimum points and any points of inflection. 4.

b) The diagram shows the graph of $y = f(x)$



Draw separate sketches, on the answer sheet provided, of the graphs of the following.

i) $y = \dfrac{1}{f(x)}$ 2.

ii) $y = (f(x))^2$ 2.

iii) $y^2 = f(x)$ 2.

c)

i)On the same graph do a neat sketch of the region in the first quadrant bounded by the curves $y^2 = x$ and $y = x^3$. 2.

ii)Use the method of cylindrical shells to find the volume of the solid formed by revolving this region about the line $x = -1$ 3.

Question 4.

a) If α,β,γ are the roots of the cubic equation $x^3 - px + q = 0$ find in terms of p and q the value of:

i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ 3.

ii) $\alpha^3+\beta^3+\gamma^3$ 3.

b) Solve the equation $Z^5 + 16Z = 0$, expressing each solution in the form $a + ib$ where a and b are real numbers. 3.

c) If α is a non-real double root of $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$ factorise $P(x)$ completely as linear factors. 4.

d) In how many ways can 6 boys and 4 girls be arranged in a row so that no two girls are together? 2.

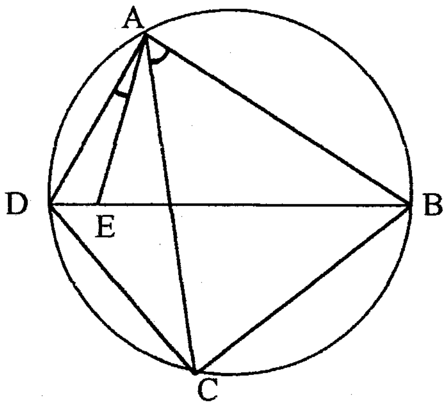
Question 5.

a) Let $P(ct, \frac{c}{t})$ be a point on the rectangular hyperbola $xy = c^2$.

i) Show that the equation of the tangent at P is given by $x + t^2 y = 2ct$. 2.

ii) Show that the area between the asymptotes and the tangent at P is a constant. 2.

b)



ABCD is a cyclic quadrilateral. E is a point on the chord BD, such that angle DAE equals angle BAC. Prove that:

i) AB.CD = AC.BE 2

ii) BC.DA = AC.DE 2

iii) AB.CD + BC.DA = AC.BD. 2

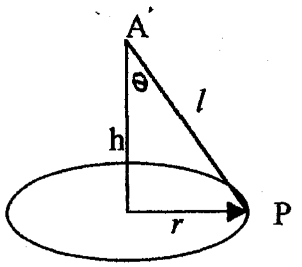
c)

i) Show that $(1-\sqrt{x})^{n-1}.\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$ 2

ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \geq 0$ show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$ 3

Question 6.

a)



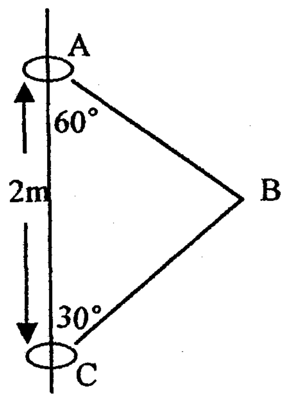
A bob, P, of mass 'm' is suspended from a fixed point A by a light in extensible string of length l. P is observed to perform uniform circular motion with radius r and angular velocity ω in a plane h units below A. If the string makes an angle of θ with the vertical ,T represents the tension in the string and g is acceleration due to gravity show that:

i) $\tan\theta = \dfrac{r\omega^2}{g}$ 2

ii) $h = \dfrac{g}{\omega^2}$ 2

iii) $T = ml\omega^2$ 2

b)



A mass of 10kg, centre B is connected by light rods to sleeves A and C which revolve freely about the vertical axis AC, but do not move vertically.

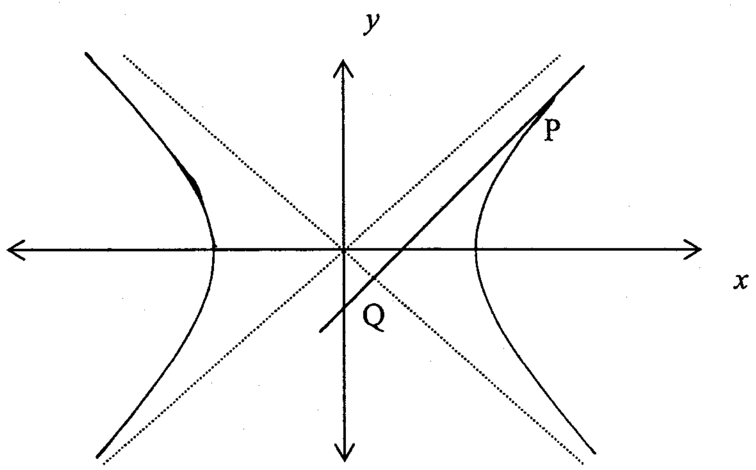
i) Given AC = 2 metres, show that the radius of the circular path of revolution of B is $\dfrac{\sqrt{3}}{2}$ metres. 2.

ii) Find the tensions in the rods AB, BC when the mass makes 90 revolutions per minute about the vertical axis. 4.

c) A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Find the volume of the solid formed if every section perpendicular to the major axis is an isosceles triangle with altitude 6. 3.

Question 7.

a)



Let P be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Let Q be the point of intersection of the tangent at P with an asymptote of the hyperbola. From Q perpendiculars QM and QN are drawn to the co-ordinate axes. Prove that MN passes through P. 4

b) If Z represents the complex number $x+iy$ Sketch on the complex plane $\text{Re}(Z^2)>0$. 3

c) If $0<x<y<\frac{1}{2}$ prove that: $\sqrt{xy}<x+y<\sqrt{x+y}$ 4

d) Prove that if α,β are the roots of the equation $t^2-2t+2=0$ then:

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{\sin n\theta}{\sin^n\theta} \qquad \text{where } \cot\theta=x+1$$ 4

Question 8.

a)A certain projectile moving through air experiences air resistance proportional to the square of its speed.

i) Explain why the equations of motion, with upwards taken as positive, are:

$$\ddot{x} = -g - kv^2 \text{, when moving upwards,}$$

1

$$\ddot{x} = -g + kv^2 \text{, when moving downwards,}$$

where g is the acceleration due to gravity and k is a positive constant.

ii)Suppose that the projectile is fired vertically upwards from the ground with an initial speed of V metres per second.

α) By replacing $\ddot{x}$ by $v\frac{dv}{dx}$ and integrating, show that the maximum height H reached by the projectile is: 3

$$H = \frac{1}{2k}\log(1 + \frac{kV^2}{g})$$

β) By replacing $\ddot{x}$ by $\frac{dv}{dt}$ and integrating, show that the time T taken to reach this maximum height is:

$$T = \frac{1}{\sqrt{gk}}\tan^{-1}(\frac{V\sqrt{k}}{\sqrt{g}}).$$ 3

i) If the projectile is dropped from a height L, then the height x above the ground and the time t elapsed when the velocity is v are given by:

$$x = L + \frac{1}{2k}\ln(1 - \frac{kv^2}{g})$$

$$t = \frac{1}{2\sqrt{gk}}\ln(\frac{\sqrt{g} - v\sqrt{k}}{\sqrt{g} + v\sqrt{k}}).$$

(you do not need to prove these equations.)

Suppose that the projectile is fired vertically upwards from the ground with an initial speed $\frac{\sqrt{g}}{\sqrt{k}}$ and eventually falls back to the ground. Show that the total flight time of the projectile is: 3

$$\frac{1}{4\sqrt{gk}}(\pi + 2\ln(3 + 2\sqrt{2})) \text{ seconds.}$$

b) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$. Show that:

$$p = \frac{1}{2a}\{P(a) - P(-a)\} \qquad \text{and} \qquad q = \frac{1}{2}\{P(a) + P(-a)\}$$ 3

Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for the cases:

i) n even 1

ii) n odd 1

END OF PAPER

2005 EXTENSION 2 TRIAL.

Question 1

a) $\int \sqrt{x^2 - 2x^4}\, dx$

$= \int \sqrt{x^2(1-2x^2)}\, dx$

$= \int x(1-2x^2)^{1/2}\, dx$

$= -\frac{1}{4}\int -4x(1-2x^2)^{\frac{1}{2}}\, dx = -\frac{1}{4}\times\frac{2}{3}(1-2x^2)^{\frac{3}{2}} + c$

$= -\frac{1}{6}(1-2x^2)^{3/2} + c$

b) $\int \frac{dx}{\sqrt{28-12x-x^2}}$

$= \int \frac{dx}{\sqrt{28-(x^2+12x)}}$

$= \int \frac{dx}{\sqrt{28-(x^2+12x+36-36)}}$

$= \int \frac{dx}{\sqrt{28-(x+6)^2+36}}$

$= \int \frac{dx}{\sqrt{64-(x+6)^2}}$

$= \sin^{-1}\left(\frac{x+6}{8}\right) + C.$

c) $\frac{x+1}{x^3+x^2-6x}$

$= \frac{x+1}{x(x^2+x-6)}$

$= \frac{x+1}{x(x+3)(x-2)}$

Let $\frac{x+1}{x(x+3)(x-2)} = \frac{a}{x} + \frac{b}{x+3} + \frac{c}{x-2}$

$\therefore\ x+1 = a(x+3)(x-2) + bx(x-2) + cx(x+3)$

let $x=0$. $1 = -6a$

$\therefore a = -\frac{1}{6}$

let $x=2$. $3 = 10c$

$\therefore c = \frac{3}{10}$

let $x=-3$. $-2 = 15b$

$\therefore b = -2/15$

$\therefore \int \frac{x+1}{x(x+3)(x-2)}\, dx = \int \frac{-1}{6x} - \frac{2}{15(x+3)} + \frac{3}{10(x-2)}\, dx$

$-\frac{1}{6}\ln x - \frac{2}{15}\ln(x+3) + \frac{3}{10}\ln(x-2) + C$

d) $\int_0^{\pi/2} \frac{dx}{2+\sin x}$

$t = \tan\frac{x}{2}$

$dx = \frac{2}{1+t^2}\,dt$

$x = \frac{\pi}{2},\ t = 1$

$x = 0,\ t = 0$

$= \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\, dt.$

$= \int_0^1 \frac{2}{2(1+t^2)+2t}\, dt$

$= \int_0^1 \frac{1}{1+t^2+t}\, dt$

$= \int_0^1 \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}$

$= \int_0^1 \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}$

$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(t+\frac{1}{2})}{\sqrt{3}}\right)\right]_0^1$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$

e) i) $I_n = \int_1^e x(\ln x)^n\,dx$

$= \int_1^e (\ln x)^n \cdot \frac{d}{dx}\left(\tfrac{1}{2}x^2\right) dx$

$= \left[\tfrac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{n}{2}\int_1^e x^2 \cdot (\ln x)^{n-1} \cdot \frac{1}{x}\,dx$

$= \tfrac{1}{2}e^2 - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\,dx$

$= \tfrac{1}{2}e^2 - \frac{n}{2} I_{n-1}$

ii) $I_2 = \int_1^e x(\ln x)^2\,dx$

$= \frac{e^2}{2} - I_1$

Now $I_1 = \int_1^e x \ln x\,dx$

$= \int_1^e \ln x \cdot \frac{d}{dx}\left(\tfrac{1}{2}x^2\right) dx$

$= \left[\tfrac{1}{2}x^2 \ln x\right]_1^e - \int_1^e \tfrac{1}{2}x^2 \cdot \frac{1}{x}\,dx$

$= \frac{e^2}{2} - \int_1^e \tfrac{1}{2}x\,dx$

$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e$

$= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)$

$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$

$= \frac{e^2}{4} + \frac{1}{4}$

$\therefore I_2 = \frac{e^2}{2} - \left(\frac{e^2}{4} + \frac{1}{4}\right)$

$= \frac{e^2}{2} - \frac{e^2}{4} - \frac{1}{4}$

$= \frac{e^2}{4} - \frac{1}{4}$

Question 2.

a) $Z_1 = 2+i \quad Z_2 = 4-3i$

i) $Z_1 + Z_2$

$= 6 - 2i$

ii) $Z_1 . Z_2$

$= (2+i)(4-3i)$

$= 11 - 2i$

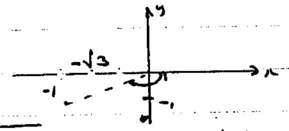
iii) $\frac{Z_1}{Z_2} = \frac{2+i}{4-3i}$

$= \frac{2+i}{4-3i} \times \frac{4+3i}{4+3i}$

$= \frac{5+9i}{16+3}$

$= \frac{5}{19} + \frac{9i}{19}$.

b) $-\sqrt{3} - i$

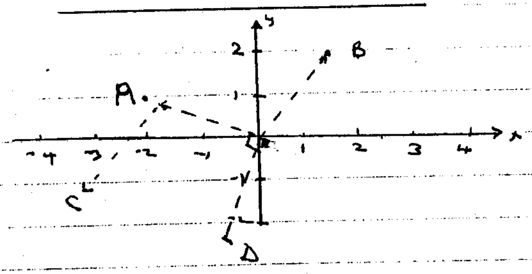


$|-\sqrt{3} - i| = \sqrt{3+1}$ $\qquad$ $\arg(-\sqrt{3} - i) = -\frac{5\pi}{6}$

$= 2$.

$\therefore -\sqrt{3} - i = 2\left(\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right)$

c) i)



d) $|z+1| = |z-i|$

$|x+iy+1| = |x+iy-i|$

$\sqrt{(x+1)^2+y^2} = \sqrt{x^2+(y-1)^2}$

$x^2+2x+1+y^2 = x^2+y^2-2y+1$

$2x = -2y$

$x = -y$.

Q3.)

a) $f(x) = 4x^{\frac{1}{3}} + x^{\frac{4}{3}}$

$f'(x) = \frac{4}{3}x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}}$

$f''(x) = -\frac{8}{9}x^{-\frac{5}{3}} + \frac{4}{9}x^{-\frac{2}{3}}$

turning points when $f'(x) = 0$

$\frac{4}{3}x^{-\frac{2}{3}} + \frac{4}{3}x^{\frac{1}{3}} = 0$

$\frac{4}{3}x^{-\frac{2}{3}}(1+x) = 0$

$\therefore\ x = 0$ or $x = -1$

$y = 0$ $\quad$ $y = -3$

$\cancel{f''(0) = 0}$ $\quad$ $0 < f''(-1)$

$\therefore$ ~~(0,0) pt. of inflection~~ $\quad$ $\therefore$ $(-1,-3)$ is a min

point of inflection $f''(x) = 0$

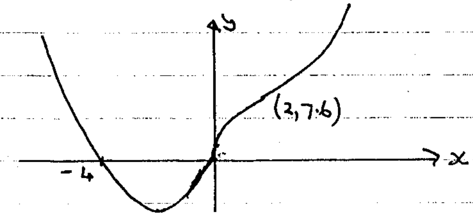
$-\frac{8}{9}x^{-\frac{5}{3}} + \frac{4}{9}x^{-\frac{2}{3}} = 0$

$-\frac{4}{9}x^{-\frac{5}{3}}(2-x) = 0$

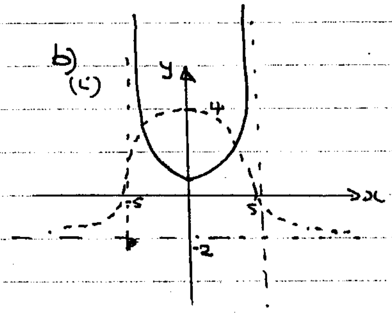
$\therefore\ x = 0$ or $x = 2$

$y = 0$ $\quad$ $y = 7.6$

$f''(-1) > 0$ $\quad f''(1) < 0$ $\quad\quad f''(1) < 0,\ 0 < f''(3)$

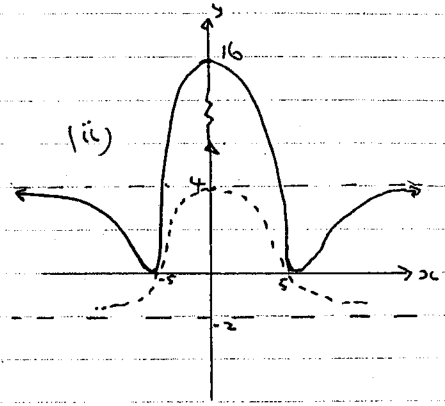


b) (i)



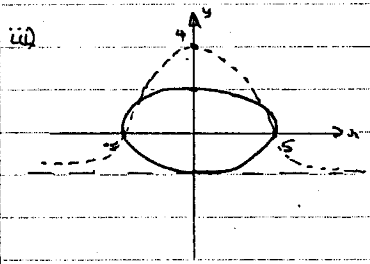
$y = \frac{1}{f(x)}$

(ii)

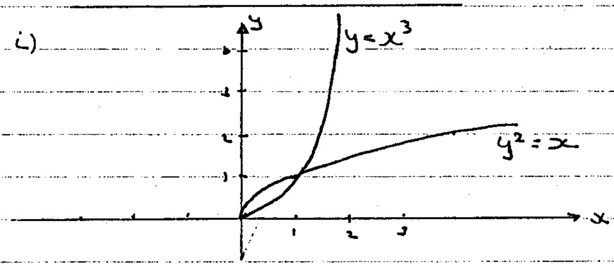


$y = (f(x))^2$.

iii)



c) i)



ii) Volume of a shell $= 2\pi r h \,\Delta x$

$$= 2\pi(1+x)(\sqrt{x}-x^3)\,\Delta x$$

Volume $\doteqdot \sum_{x=0}^{1} 2\pi(1+x)(\sqrt{x}+x^3)\,\Delta x$

$$V = 2\pi\int_0^1 (1+x)(\sqrt{x}-x^3)\,dx$$

$$= 2\pi\int_0^1 x^{1/2}-x^3+x^{3/2}-x^4\,dx$$

$$= 2\pi\left[\frac{2x^{3/2}}{3}-\frac{x^4}{4}+\frac{2x^{5/2}}{5}-\frac{x^5}{5}\right]_0^1$$

$$= 2\pi\left\{\left(\tfrac{2}{3}-\tfrac{1}{4}+\tfrac{2}{5}-\tfrac{1}{5}\right)-0\right\}$$

$$= 2\pi\left(\frac{37}{60}\right)$$

$$= \frac{37\pi}{30} \text{ cubic units.}$$

Question 4

a) $x^3 - px + q = 0$

i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$

$$= \frac{\gamma^2\beta^2+\alpha^2\gamma^2+\alpha^2\beta^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{(\alpha\beta+\alpha\gamma+\beta\gamma)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{(\alpha\beta\gamma)^2}$$

$$= \frac{p^2}{q^2}$$

ii) If α, β, γ are roots then:

$\alpha^3 - p\alpha + q = 0$ ---(1)

$\beta^3 - p\beta + q = 0$ ---(2)

$\gamma^3 - p\gamma + q = 0$ ---(3)

(1)+(2)+(3) $\alpha^3+\beta^3+\gamma^3 - p(\alpha+\beta+\gamma) + 3q = 0$

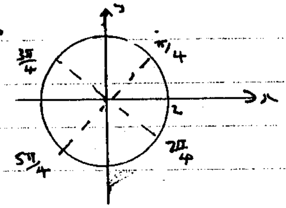
$\alpha^3+\beta^3+\gamma^3 = p(\alpha+\beta+\gamma) - 3q$

$= -3q.$

b) $z^5 + 16z = 0$

$z(z^4+16) = 0$

$z = 0$ or $z^4 = -16$



$z_1 = 0$

$z_2 = 2(\cos \pi/4 + i\sin \pi/4) = \sqrt{2}+i\sqrt{2}$

$z_3 = 2(\cos 3\pi/4 + i\sin 3\pi/4) = -\sqrt{2}+i\sqrt{2}$

$z_4 = 2(\cos 5\pi/4 + i\sin 5\pi/4) = -\sqrt{2}-i\sqrt{2}$

$z_5 = 2(\cos 7\pi/4 + i\sin 7\pi/4) = \sqrt{2}-i\sqrt{2}.$

c) $P(x) = x^4 - 4x^3 + 14x^2 - 20x + 25$

If α is a double root then $\bar{\alpha}$ is also a double root.

$\therefore \alpha+\alpha+\bar{\alpha}+\bar{\alpha} = -b/a$ and $\alpha.\alpha.\bar{\alpha}\bar{\alpha} = e/a$

$= 4$ $\quad (\alpha\bar{\alpha})^2 = 25$

$\alpha\bar{\alpha} = 5$

Now if $\alpha = a+ib$ and $a^2+b^2 = 5$

then $4a = 4$ ie $1+b^2 = 5$

$a = 1$ $\quad b = \pm 2.$

$\therefore P(x) = (x+1+2i)^2(x+1-2i)^2$

d) At any one time the girls cannot occupy 3 of the 10 positions. Therefore their arrangement is ${}^7P_4$. But for every girl arrangement the boys can be arranged in $6!$ ways.

Therefore the total arrangements $= {}^7P_4 . 6!$

$= 604800.$

Ques 5.

a) i) $xy = c^2$

$y = c^2/x$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

at $x = ct$ $\frac{dy}{dx} = -\frac{1}{t^2}$

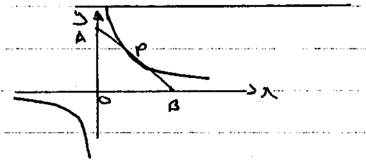
$\therefore$ eqn. of tangent

$y - c/t = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$.

ii)



asymptotes are the 'x' and 'y' axes.

Let the tangent at P intersect the axes at A and B.

for A $(x=0)$ $\therefore t^2y = 2ct$

$y = \frac{2c}{t}$

$\therefore A\left(0, \frac{2c}{t}\right)$

for B $(y=0)$ $x = 2ct$

$\therefore B(2ct, 0)$

Area between the tangent and the asymptotes

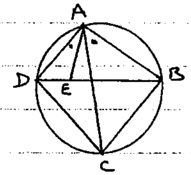
$= $ Area of ΔAOB

$= \frac{1}{2} \times 2ct \times \frac{2c}{t}$

$= 2c^2$ units

which is a constant.

b)



(i) taking triangles ABE and ACD

$\angle DAC = \angle DAE + \angle EAC$

and $\angle BAE = \angle BAC + \angle EAC$

but $\angle DAE = \angle BAC$ given

$\therefore \angle DAC = \angle BAE$

also $\angle ACD = \angle DBA$ {angles at the circumference standing on the same arc are equal

$\therefore \Delta ABE \ ||| \ \Delta ACD$ (A.A.A)

$\therefore \frac{AB}{AC} = \frac{BE}{CD}$

$\therefore AB.CD = BE.AC$.

ii) taking triangles ABC and AED

$\angle DAE = \angle BAC$ given

$\angle EDA = \angle BCA$ angles at the circumference standing on the same arc are equal.

$\therefore \Delta ABC \ ||| \ \Delta AED$ (A.A.A.)

$\therefore \frac{BC}{DE} = \frac{AC}{DA}$

$\therefore BC.DA = AC.DE$.

c) $AB.CD + BC.DA = AC.BE + AC.DE$

$= AC(BE + DE)$

$= AC.BD$.

c) i) $(1-\sqrt{x})^{n-1}.\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^{n}$

R.H.S $= (1-\sqrt{x})^{n-1}\left(1-(1-\sqrt{x})\right)$

$= (1-\sqrt{x})^{n-1}.\sqrt{x}$

$=$ L.H.S.

ii) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

$= \int_0^1 (1-\sqrt{x})^n . \frac{d}{dx}(x) . \, dx$

$= \left[x(1-\sqrt{x})^n\right]_0^1 - \int_0^1 x . n(1-\sqrt{x})^{n-1} \times \frac{-1}{2\sqrt{x}} \, dx$

$= 0 - \frac{n}{2}\int -\sqrt{x}(1-\sqrt{x})^{n-1} \, dx$

$= \frac{n}{2}\left[\int_0^1 (1-\sqrt{x})^{n-1} dx - \int_0^1 (1-\sqrt{x})^n \, dx\right]$

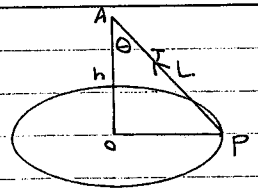
$I_n = \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$

$(1+\frac{n}{2}) I_n = \frac{n}{2} I_{n-1}$

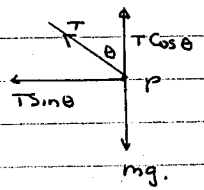
$(\frac{2+n}{2}) I_n = \frac{n}{2} I_{n-1}$

$I_n = \frac{n}{2+n} I_{n-1}$.

Q6) a)



resolving forces at P.



Vertically:

$T\cos\theta - mg = 0$

$T\cos\theta = mg$ ...(1)

horizontally

$T\sin\theta = mr\omega^2$ ...(2)

(i) (2) ÷ (1) $\tan\theta = \frac{mr\omega^2}{mg}$

$\tan\theta = \frac{r\omega^2}{g}$

ii) from ΔAOP

$\tan\theta = \frac{r}{h}$

$\therefore \frac{r}{h} = \frac{r\omega^2}{g}$

$\therefore h\omega^2 = g$

$h = \frac{g}{\omega^2}$

iii) from (2) $T\sin\theta = mr\omega^2$

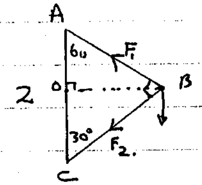
$T = \frac{mr\omega^2}{\sin\theta}$

$= \frac{mr\omega^2}{r/L}$

$= \frac{mr\omega^2 L}{r}$

$= mL\omega^2$

b) i)



In ΔABC:

$$\frac{AB}{AC} = \sin 30^\circ$$
$$AB = AC\sin 30$$
$$= 2\times\tfrac{1}{2}$$
$$= 1$$

also In ΔAOB

$$\frac{OB}{AB} = \sin 60$$
$$OB = AB\sin 60$$
$$= 1\times\tfrac{\sqrt3}{2}$$
$$= \sqrt3/2$$

$\therefore$ the radius of the circular path of B is $\sqrt3/2$ metres

ii) The mass performs circular motion of radius $(r) = \sqrt3/2$ m.

With angular velocity $= 90\times2\pi\times\frac{1}{60}$

$= 3\pi$ rad/sec.

resolving forces horizontally:

$$F_1\cos30^\circ + F_2\cos60^\circ = \frac{\sqrt3}{2}F_1 + \frac12 F_2$$

using $F = mr\omega^2$

$$\frac{\sqrt3}{2}F_1 + \frac12 F_2 = 10.\frac{\sqrt3}{2}(3\pi)^2$$
$$\therefore \quad \sqrt3F_1 + F_2 = 90\sqrt3\pi^2 \quad \cdots (1)$$

Vertically (resultant force $= 0$)

$$F_1\sin30 - F_2\sin60 - 10g = 0$$
$$\tfrac12F_1 - \tfrac{\sqrt3}{2}F_2 - 10g = 0$$
$$F_1 - \sqrt3F_2 = 20g \quad \cdots (2)$$

(1) $\times\sqrt3$ $\quad 3F_1 + \sqrt3F_2 = 270\pi^2 \cdots (3)$

(2) + (3) $\quad 4F_1 = 270\pi^2 + 20g$

$$F_1 = 5/2(27\pi^2 + 2g)N \quad (\doteqdot 715N)$$

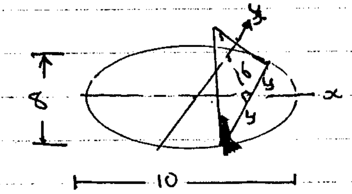
Now.

(2) $\times\sqrt3$ $\quad \sqrt3F_1 - 3F_2 = 20\sqrt3g \cdots (4)$

(1) − (4) $\quad 4F_2 = 90\sqrt3\pi^2 - 20\sqrt3g$

$$= 10\sqrt3(9\pi^2 - 2g)$$
$$F_2 = \frac{5\sqrt3}{2}(9\pi^2 - 2g) \quad (\doteqdot 300N)$$

(c)



$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Volume of a Slice $= (\tfrac12\times2y\times6)\Delta x$

$= 6y\,\Delta x.$

$$\text{Volume} \doteqdot \sum_{x=-5}^{5} 6y\,\Delta x$$

$$V = \int_{-5}^{5} 6y\,dx$$
$$= \int_{-5}^{5} 6\left(4\sqrt{1-\frac{x^2}{25}}\right)dx$$
$$= 24\int_{-5}^{5}\sqrt{1-\frac{x^2}{25}}\,dx$$
$$= 24\int_{-5}^{5}\sqrt{\frac{25-x^2}{25}}\,dx$$
$$= 24\int_{-5}^{5}\frac{\sqrt{25-x^2}}{5}\,dx$$
$$= \frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx$$
$$= \frac{24}{5}\times\frac12\pi r^2 \quad (\sqrt{25-x^2} \rightarrow \text{semi-circle})$$
$$= \frac{24}{10}\times\pi\times5^2$$
$$= 60\pi \text{ cubic units}$$

Ques 7.

a) Let P be the point $(a\sec\theta, b\tan\theta)$

$\therefore$ the equation at P is: $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ ....(1)

the equation of the asymptote: $y = \dfrac{-bx}{a}$ ...(2)

Subst. (2) into (1):

$$\frac{x\sec\theta}{a} + \frac{\frac{bx}{a}\tan\theta}{b} = 1$$

$$\frac{x\sec\theta}{a} + \frac{x\tan\theta}{a} = 1$$

$$x(\sec\theta + \tan\theta) = a$$

$$x = \frac{a}{\sec\theta + \tan\theta}$$

Subst into (2): $y = \dfrac{-b}{a}\left(\dfrac{a}{\sec\theta + \tan\theta}\right)$

$$= \frac{-b}{\sec\theta + \tan\theta}$$

$\therefore$ Q is the point $\left(\dfrac{a}{\sec\theta+\tan\theta}, \dfrac{-b}{\sec\theta+\tan\theta}\right)$

$\therefore$ M is the point $\left(\dfrac{a}{\sec\theta+\tan\theta}, 0\right)$

and N is the point $\left(0, \dfrac{-b}{\sec\theta+\tan\theta}\right)$

Now gradient of MN $= \dfrac{\frac{b}{\sec\theta+\tan\theta}}{\frac{a}{\sec\theta+\tan\theta}}$

$$= \frac{b}{a}.$$

$\therefore$ the equation of MN: $y - 0 = \dfrac{b}{a}\left(x - \dfrac{a}{\sec\theta+\tan\theta}\right)$

$$y = \frac{bx}{a} - \frac{b}{\sec\theta+\tan\theta}$$

If P lies on the line MN it must satisfy the equation.

ie $b\tan\theta = \dfrac{ba\sec\theta}{a} - \dfrac{b}{\sec\theta+\tan\theta}$

$$= b\sec\theta - \frac{b}{\sec\theta+\tan\theta}$$

$$= b\left(\sec\theta - \frac{1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\sec\theta(\sec\theta+\tan\theta) - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\sec^2\theta + \sec\theta\tan\theta - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{1+\tan^2\theta + \sec\theta\tan\theta - 1}{\sec\theta+\tan\theta}\right)$$

$$= b\left(\frac{\tan^2\theta + \sec\theta\tan\theta}{\sec\theta+\tan\theta}\right)$$

$$= b\tan\theta\left(\frac{\tan\theta + \sec\theta}{\sec\theta+\tan\theta}\right)$$

$$= b\tan\theta.$$

$\therefore$ P lies on the line.

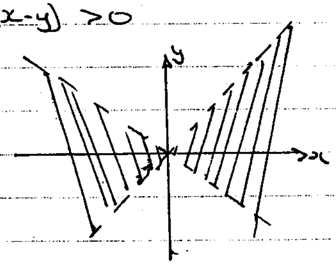
b) $\mathrm{Re}(z^2) > 0$

$\mathrm{Re}((x+iy)^2) > 0$

$\mathrm{Re}(x^2 - y^2 + 2ixy) > 0$

$x^2 - y^2 > 0$

$(x+y)(x-y) > 0$



c) $(\sqrt{x}-\sqrt{y})^2 \geq 0$

$x+y-2\sqrt{xy} \geq 0$

$x+y \geq 2\sqrt{xy}$

$\therefore \quad x+y \geq \sqrt{xy}$ ----- (1)

If $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ then

$x+y \leq 1$

$\therefore \sqrt{x+y} \geq x+y$ ..(2) as the square root of a number between 0 and 1 is greater than the number.

$\therefore$ from (1) and (2)

$\sqrt{xy} \leq x+y \leq \sqrt{x+y}.$

d) $t^2-2t+2=0$

$t = \dfrac{2 \pm \sqrt{4-8}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

$= \dfrac{2 \pm 2i}{2}$

$= 1 \pm i.$

let $\alpha = 1+i, \quad \beta = 1-i$

$\therefore \quad x+\alpha = (\cot\theta - 1) + 1 + i$ also $x+\beta = (\cot\theta-1)+(1-i)$

$= \cot\theta + i \qquad\qquad = \cot\theta - i$

$= \dfrac{\cos\theta + i\sin\theta}{\sin\theta} \qquad\qquad = \dfrac{\cos\theta - i\sin\theta}{\sin\theta}$

$\therefore \quad \dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta}$

$= \dfrac{\dfrac{(\cos\theta+i\sin\theta)^n}{\sin^n\theta} - \dfrac{(\cos\theta-i\sin\theta)^n}{\sin^n\theta}}{2i}$

$= \dfrac{\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta}{2i\sin^n\theta}$

$= \dfrac{2i\sin n\theta}{2i\sin^n\theta} = \dfrac{\sin n\theta}{\sin^n\theta}.$

Ques 8.

a) (i) The acceleration due to gravity is always downwards and so is always $-g$. The acceleration due to air resistance is proportional to v^2 and so has magnitude kv^2 (for positive constant of proportionality k)

But air resistance acts opposite to the direction of motion. So when the projectile is rising, the air resist acts downwards, and $\ddot{x} = -g-kv^2$ and when the projectile is falling, the air resistance acts upwards and $\ddot{x} = -g+kv^2$

ii) α) $\ddot{x} = -g-kv^2$

$v\dfrac{dv}{dx} = -g-kv^2$

$\dfrac{dv}{dx} = \dfrac{-g-kv^2}{v}$

$\dfrac{dx}{dv} = -\dfrac{v}{g+kv^2}$

$\dfrac{dx}{dv} = -\dfrac{1}{2k} \times \dfrac{2kv}{g+kv^2}$

$x = -\frac{1}{2k}\log(g+kv^2) + c.$

when $x=0$, $v=V$ so: $0 = -\frac{1}{2k}\log(g+kV^2)+c$

$\therefore \quad c = \frac{1}{2k}\log(g+kV^2)$

$\therefore \quad x = -\frac{1}{2k}\log(g+kv^2) + \frac{1}{2k}\log(g+kV^2)$

$= \frac{1}{2k}\log\left(\dfrac{g+kV^2}{g+kv^2}\right)$

at max height H, $v=0$

$H = \frac{1}{2k}\log\left(\dfrac{g+kV^2}{g}\right)$

$= \frac{1}{2k}\log\left(1+\dfrac{kV^2}{g}\right)$

(β) $\ddot{x} = -g - kv^2$

$$\frac{dv}{dt} = -g - kv^2$$

$$\frac{dt}{dv} = -\frac{1}{g+kv^2}$$

$$t = -\frac{1}{\sqrt{gk}} \tan^{-1} v\sqrt{k/g} + C_1$$

When $t=0$, $v=V$ $\quad 0 = -\frac{1}{\sqrt{gk}} \tan^{-1} V\sqrt{\frac{k}{g}} + C_1$

$$\therefore \quad C_1 = \frac{1}{\sqrt{gk}} \tan^{-1} V\sqrt{\frac{k}{g}}$$

$$\therefore \quad t = \frac{1}{\sqrt{gk}}\left(\tan^{-1} V\sqrt{k/g} - \tan^{-1} v\sqrt{\frac{k}{g}}\right)$$

at max height $v=0$

$$T = \frac{1}{\sqrt{gk}} \tan^{-1} V\sqrt{k/g}$$

ii) for the rising time T and max height H, subst

$$V = \sqrt{g/k}$$

$$T = \frac{1}{\sqrt{gk}} \tan^{-1} 1$$

$$= \frac{\pi}{4\sqrt{gk}}$$

$$H = \frac{1}{2k} \log 2$$

Let U be the velocity when the projectile strikes the ground again. then substitute $x=0$ and $L = \frac{1}{2k}\log$ into the given equation for x:

$$0 = \log 2 + \log\left(1 - \frac{kU^2}{g}\right)$$

$$0 = \log\left(2 - \frac{2kU^2}{g}\right)$$

$$1 = 2 - \frac{2kU^2}{g}$$

$$\frac{2kU^2}{g} = 1$$

$$U = -\sqrt{\frac{g}{2k}}$$

Substitute this value of U for v in the given equation for t:

Falling time $= \frac{1}{2\sqrt{gk}} \log\left(\frac{\sqrt{2g} - U\sqrt{2k}}{\sqrt{2g} + U\sqrt{2k}}\right)$

$$= \frac{1}{2\sqrt{gk}} \log\left(\frac{\sqrt{2g} + \sqrt{g}}{\sqrt{2g} - \sqrt{g}}\right)$$

$$= \frac{1}{2\sqrt{gk}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{1}{2\sqrt{gk}} \log(3+2\sqrt{2})$$

$\therefore$ total time in flight $= \frac{\pi}{4\sqrt{gk}} + \frac{1}{2\sqrt{gk}}\log(3+2\sqrt{2})$

$$= \frac{1}{4\sqrt{gk}}\left(\pi + 2\log(3+2\sqrt{2})\right) \text{ Seconds}$$

b) $\quad P(x) = (x^2-a^2)\,Q(x) + px+q$

$$= (x-a)(x+a)\,Q(x) + px+q$$

$$\therefore P(a) = pa+q \quad \cdots (1)$$

$$P(-a) = -pa+q \quad \cdots (2)$$

(1)+(2) $\quad 2q = P(a)+P(-a)$ $\qquad$ (1)−(2): $\quad 2ap = P(a)-P(-a)$

$\quad q = \frac{1}{2}(P(a)+P(-a))$ $\qquad$ $p = \frac{1}{2a}(P(a)-P(-a))$

When $P(x) = x^n - a^n$ then

i) when n is even, $P(a)=0$ and $P(-a)=0$

$\therefore$ the remainder $=0$.

ii) When n is odd, $P(a)=0$ and $P(-a) = -2a^n$

$\therefore \quad pa+q=0 \quad \cdots (1)$ $\qquad$ (1)−(2) $\quad 2ap = 2a^n$

$\quad -pa+q = -2a^n \quad \cdots (2)$ $\qquad$ $ap = a^n$

(1)+(2) $\quad 2q = -2a^n$ $\qquad$ $p = a^{n-1}$

$\quad q = -a^n$

$\therefore$ the remainder is $a^{n-1}x - a^n$.