Candidate Number ____________________

GOSFORD HIGH SCHOOL

2007
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Each question should be started on a new page.
- All necessary working should be shown in every question

Total marks: - 120

- Attempt Questions 1 -8
- All questions are of equal value

Question 1 (15 Marks) **Marks**

a) Find $\int \frac{x}{\sqrt{9-4x^2}}dx$ **2**

b) Find $\int_1^e x^5 \log_e x \, dx$ **3**

c) (i) Find real numbers a, b and c such that **2**

$$\frac{8}{(x+2)(x^2+4)} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}$$

(ii) Hence show that $\int_0^2 \frac{8dx}{(x+2)(x^2+4)} = \frac{1}{2}\log 2 + \frac{\pi}{4}$ **4**

d) Use the substitution $x = 3\sin\theta$ to evaluate $\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{(9-x^2)^{\frac{3}{2}}}$ **4**

Question 2 (15 Marks) Begin a New Booklet

a) The zeros of $(x-1)(x+i)$ are obviously 1 and –i. These are not complex conjugates. How do you explain this? **2**

b) Let $\alpha = -1 + i\sqrt{3}$

(i) Find the exact value of $|\alpha|$ and $\arg \alpha$

(ii) Find the exact value of α^7 in the form $a+ib$ where a and b are real. **4**

c) Find the square roots of $-5-12i$ in the form $a + ib$. **4**

d) The equation $|z-1-3i|+|z-9-3i|=10$ corresponds to an ellipse in the Argand diagram.

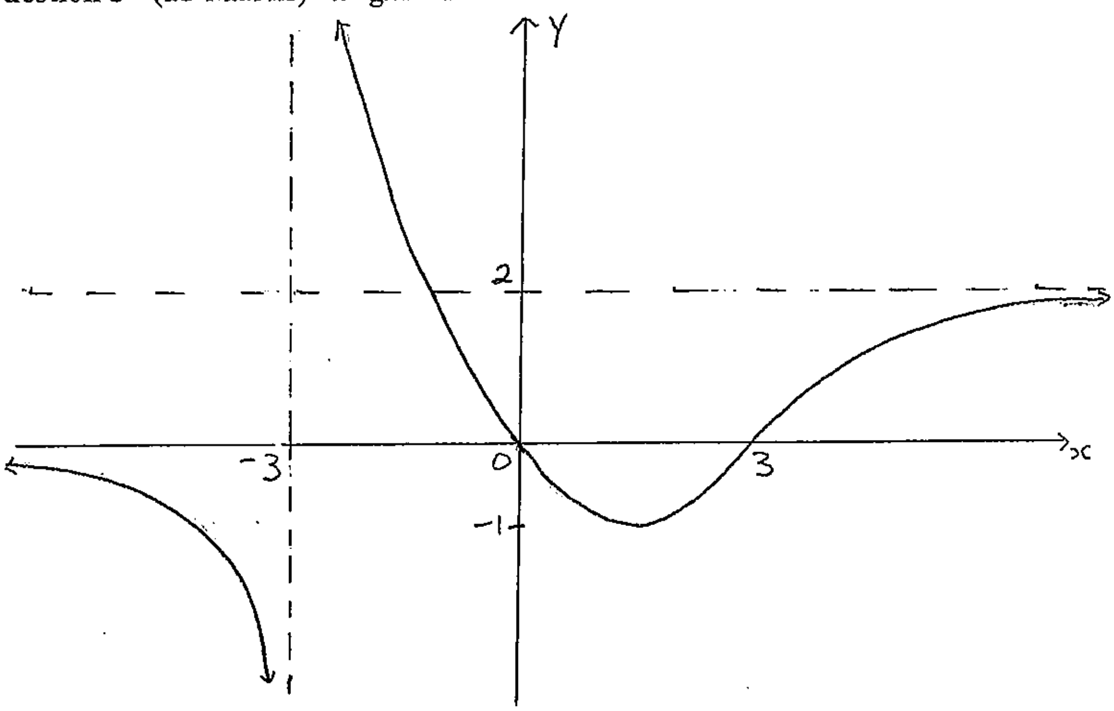
(i) Write down the complex number corresponding to the centre of the ellipse. **1**

(ii) Sketch the ellipse, and state the lengths of the major and minor axes. **3**

(iii) Write down the range of values of arg(z) for complex numbers z corresponding to points on the ellipse. **1**

Question 3 (15 Marks) Begin a New Booklet Marks

a)



The diagram shows the graph of $y = f(x)$.

Draw separate half page sketches of:

(i) $y = (f(x))^2$ 2

(ii) $y = \sqrt{f(x)}$ 2

(iii) $y^2 = f(x)$ 1

(iv) $y = \dfrac{1}{f(x)}$ 2

(v) $y = xf(x)$ 2

b) (i) Prove that the tangent at a point (x_1, y_1) to $xy = c^2$ is
$$xy_1 + x_1 y = 2c^2$$ 2

(ii) P is a point of intersection of the rectangular hyperbolas
$$x^2 - y^2 = a^2 \quad \text{and} \quad xy = c^2.$$

The tangent at P to the first hyperbola meets its asymptotes in A and C, and
The tangent at P to the second hyperbola meets its asymptotes in B and D. 4
Prove that ABCD is a square.

Question 4 (15 Marks) Begin a New Booklet

a) A particle P of mass m moves with constant angular velocity ω on a circle of radius r. Its position at time t is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta \qquad \text{where} \quad \theta = \omega t.$$

(i) Show that there is an inward radial force of magnitude $mr\omega^2$ acting on P. **3**

(ii) A telecommunications satellite, of mass m, orbits Earth with constant angular velocity ω at a distance r from the centre of Earth. The gravitational force exerted by Earth on the satellite is $\dfrac{Am}{r^2}$ where A is a constant.

By considering all other forces on the satellite to be negligible, show that

$$r = \sqrt[3]{\frac{A}{\omega^2}}$$ **1**

b) It is given that x, y, z are positive numbers. Prove that

(i) $x^2 + y^2 \geq 2xy$ **1**

(ii) $x^2 + y^2 + z^2 - xy - yz - zx \geq 0$ **2**

Multiply both sides of the inequality in (ii) by $(x+y+z)$ to show

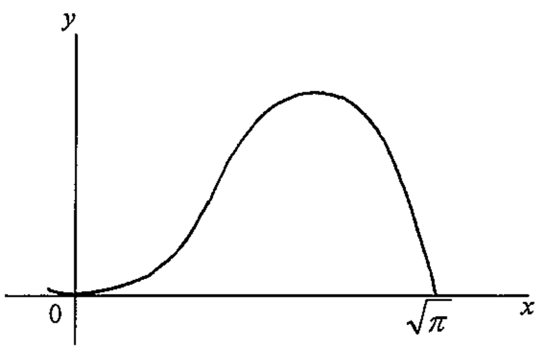
(iii) $x^3 + y^3 + z^3 \geq 3xyz$ **2**

Deduce from (iii) or prove otherwise, that

(iv) $(x+y+z)(x^{-1}+y^{-1}+z^{-1}) \geq 9$ **2**

c) The curve $y = \sin(x^2)$ from $x = 0$ to $x = \sqrt{\pi}$ is rotated about the y-axis. **4**

Sketch a typical cylindrical shell and use this method to find the volume formed.



Question 5 (15 Marks) Begin a New Booklet

a) Prove that if two polynomials P(x) and Q(x) have a common factor of (x–a), then (x–a) is also a factor of P(x) – Q(x). **2**

Hence find the value of k if x^3+x^2-5x+k and $x^3-8x^2+13x-2k$ have a common double root. What is this double root? **3**

b) (i) Find the expansion of $\cos 5\theta$ and $\sin 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. **4**

(ii) Use the results of (i) to obtain an expression for $\tan 5\theta$ in terms of of powers of $\cos\theta$ and $\sin\theta$. Hence prove

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1-10\tan^2\theta + 5\tan^4\theta}$$ **3**

(iii) Hence solve the equation $t^5-5t^4-10t^3+10t^2+5t-1=0$ **3**

Question 6 (15 Marks) Begin a New Booklet

a) A particle of mass m falls vertically from rest, from a point O, in a medium whose resistance is mkv, where k is a positive constant and v its velocity.

(i) Obtain an expression for its velocity after t seconds. **3**

An equal particle is projected vertically upwards from O with initial velocity u, in the same medium. This particle is released simultaneously with the first particle.

(ii) Show that the velocity of the first particle when the second particle is momentarily at rest, is given by $\dfrac{Vu}{V+u}$ where V is the terminal velocity of the first particle. **5**

b) A solid has as its base the circle $x^2+y^2=a^2$ in the xy plane. Find the volume of the solid such that every cross-section by a plane parallel to the x-axis is a square with one side in the xy-plane. **4**

c) If m and n are positive integers and $m \neq n$, show that

$$\int_0^{\pi} \cos mx \cos nx dx = 0$$ **3**

Question 7 (15 Marks) Begin a New Booklet

a) The curves $y = \cos x$ and $y = \tan x$ intersect at a point P whose x coordinate is α.

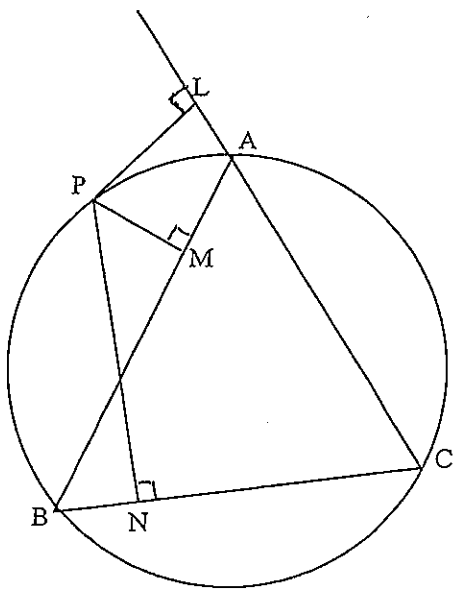
(i) Show that the curves intersect at right angles at P **3**

(ii) Show that $\sec^2 \alpha = \dfrac{1+\sqrt{5}}{2}$ **2**

b) If $U_n = \displaystyle\int_0^1 (1-x^2)^n\,dx$ show that $U_n = \dfrac{2n}{2n+1}U_{n-1}$ **4**

Hence evaluate U_4 **1**

c) In the diagram, ABC is a triangle inscribed in a circle. P is a point on the minor arc AB. L, M, and N are the feet of the perpendiculars from P to CA (produced), AB and BC respectively.



(i) State a reason why P, M, A and L are concyclic points. **1**

(ii) State a reason why P, B, N and M are concyclic points. **1**

(iii) Join BP, PA, LM and MN.

Use the three cyclic quadrilaterals to prove that L, M and N are collinear.

Hint: Let $\angle$ PBN $= \alpha$ **3**

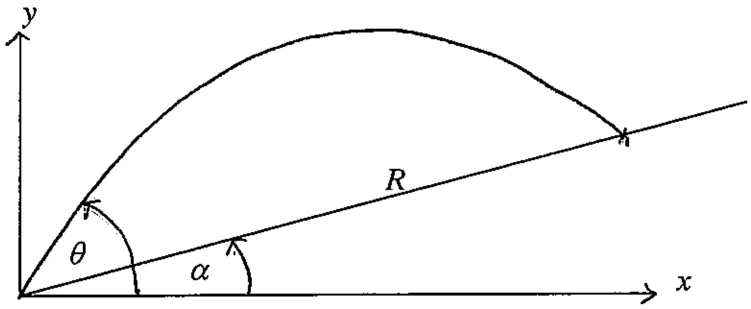
Question 8 (15 Marks) Begin a New Booklet

a) From a diagram, show that $\sin x < x < \tan x$ if $0 < x < \frac{\pi}{2}$.

Hence prove that $\int_0^{\frac{\pi}{6}} x^2 \sin x \, dx < \frac{\pi^4}{2^6.3^4} < \int_0^{\frac{\pi}{6}} x^2 \tan x \, dx.$ **3**

b) Solve for x $\tan^{-1} 3x - \tan^{-1} 2x = \tan^{-1} \frac{1}{5}$ **3**

c) A projectile is fired with velocity V , at an angle θ to the horizontal, up a plane inclined at an angle α to the horizontal, (where α and V are constants).



Neglecting air resistance, and using g for acceleration due to gravity,

(i) Write down the equations of motion for the projectile. **1**

(ii) Show that the equation of the trajectory is

$$y = -\frac{g}{2V^2} x^2 \sec^2\theta + x \tan\theta$$ **2**

(iii) Write down the equation of the inclined plane.

By solving simultaneously for x , and noting that $R = x\sec\alpha$, **2**

find an expression for the range R on the inclined plane.

(v) Hence show that $\frac{dR}{d\theta} = \frac{2V^2}{g}(\cos 2\theta + \sin 2\theta \tan\alpha) \sec\alpha$ **2**

(vi) Find the value of θ (in terms of α) which gives the maximum range. **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \ x > 0$

Solutions to 2007 Trial HSC Extension 2

Question 1

a) $$\int \frac{x}{\sqrt{9-4x^2}}\,dx = -\frac{1}{8}\int -8x\,(9-4x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{8}\cdot 2\,(9-4x^2)^{\frac{1}{2}} = -\frac{1}{4}\sqrt{9-4x^2} + C$$

b) $$\int_1^e x^5 \ln x\,dx \qquad u = \ln x \qquad dv = x^5\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = \frac{x^6}{6}$$

$$\int_1^e x^5 \ln x\,dx = \left[\frac{x^6}{6}\ln x\right]_1^e - \int_1^e \frac{x^6}{6}\cdot\frac{1}{x}\cdot dx$$

$$= \left[\frac{x^6}{6}\ln e\right]_1^e - \frac{1}{6}\int_1^e x^5\,dx$$

$$= \left[\frac{x^6}{6}\ln x - \frac{x^6}{36}\right]_1^e$$

$$= \left(\frac{e^6}{6} - \frac{e^6}{36}\right) - \left(0 - \frac{1}{36}\right) = \frac{5e^6 - 1}{36}$$

c) i) $$\frac{8}{(x+2)(x^2+4)} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}$$

$$= \frac{a(x^2+4) + (x+2)(bx+c)}{(x+2)(x^2+4)}$$

$$\therefore\ 8 \equiv a(x^2+4) + (x+2)(bx+c)$$

Sub $x=-2$: $\quad 8 = 8a \implies a = 1$

Equate coeffs of x^2

$$0 = a + b \implies b = -1$$

Sub $x = 0$

$$8 = 4a + 2c$$

$$4 = 2a + c \implies c = 2$$

ii) $$\int_0^2 \frac{1}{x+2} + \frac{-x+2}{x^2+4}\,dx$$

$$= \int_0^2 \frac{1}{x+2} - \frac{1}{2}\cdot\frac{2x}{x^2+4} + \frac{2}{x^2+4}\,dx$$

ii) cont $= \left[\ln(x+2) - \frac{1}{2}\ln(x^2+4) + 2\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \left(\ln 4 - \frac{1}{2}\ln 8 + \tan^{-1}1\right) - \left(\ln 2 - \frac{1}{2}\ln 4 + \tan^{-1}0\right)$

$= 2\ln 2 - \frac{3}{2}\ln 2 + \frac{\pi}{4} - \ln 2 + \ln 2 - 0$

$= \frac{1}{2}\ln 2 + \frac{\pi}{4}$ as required.

d) $\displaystyle\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{(9-x^2)^{3/2}}$ Put $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$. When $x=0$, $\theta=0$; when $x=\frac{3}{\sqrt{2}}$, $\theta=\frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{3\cos\theta\, d\theta}{(9-9\sin^2\theta)^{3/2}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{3\cos\theta\, d\theta}{(9\cos^2\theta)^{3/2}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{3\cos\theta\, d\theta}{27\cos^3\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{9\cos^2\theta}$$

$$= \frac{1}{9}\int_0^{\frac{\pi}{4}} \sec^2\theta\, d\theta$$

$$= \frac{1}{9}\left[\tan\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{9}\left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= \frac{1}{9}$$

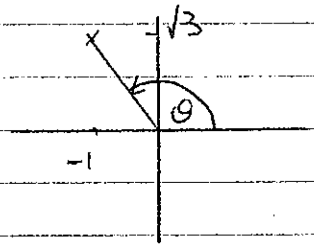
Question 2

a) $(x-1)(x+i) = x^2 + (i-1)x - i$

$P(x) = x^2 + (i-1)x - i$

The coefficients of $P(x)$ are not all real. Therefore the zeros are not in conjugate pairs.

b) i) $\alpha = -1 + i\sqrt{3}$



$|\alpha|^2 = 1^2 + 3$

$|\alpha| = 2$

$\theta = \arg\alpha = \pi + \tan^{-1} - \sqrt{3}$

$= \dfrac{2\pi}{3}$

ii) $-1 + i\sqrt{3} = 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$\alpha^7 = 2^7\left(\text{cis}\,\dfrac{2\pi}{3}\right)^7$

$= 128\ \text{cis}\,\dfrac{14\pi}{3}$ But $\dfrac{14\pi}{3} = \dfrac{2\pi}{3}$

$= 128\ \text{cis}\,\dfrac{2\pi}{3}$

$= 128\,(-1 + i\sqrt{3})$

$= -128 + 128i\sqrt{3}$

$\therefore a = -128,\quad b = 128\sqrt{3}$

c) $z^2 = -5 - 12i = (a + ib)^2$

$a^2 - b^2 = -5$

$2ab = -12 \implies ab = -6$

$a^2 - \dfrac{36}{a^2} = -5$

$a^4 + 5a^2 - 36 = 0$

$(a^2+9)(a^2-4) = 0$

$a^2 = -9$ or 4

$a = \pm 2$ (real vals)

$$b = -\frac{6}{a} = \mp\frac{6}{2} = \mp 3$$

$$\therefore \sqrt{-5-12i} = 2-3i \quad \text{or} \quad -2+3i$$

d)

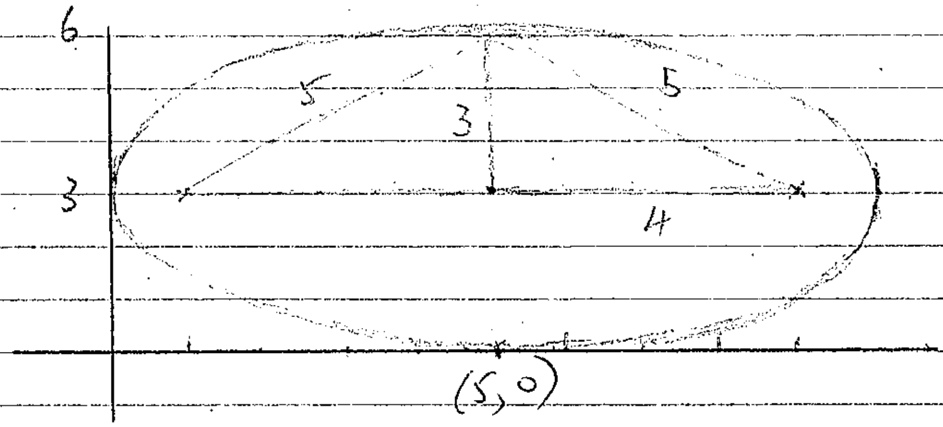
$$|z-1-3i| + |z-9-3i| = 10$$

$$|z-(1+3i)| + |z-(9+3i)| = 10$$

Distance of z from $(1+3i)$ + distance from $(9+3i)$ $= 10$

i) Centre of ellipse is mid point ie $(5+3i)$

ii)



Major axis 10 units
Minor axis 6 units

iii) At $(5,0)$ $\arg z = 0$

At $(0,3)$ $\arg z = \frac{\pi}{2}$

$$0 \le \arg z \le \frac{\pi}{2}$$

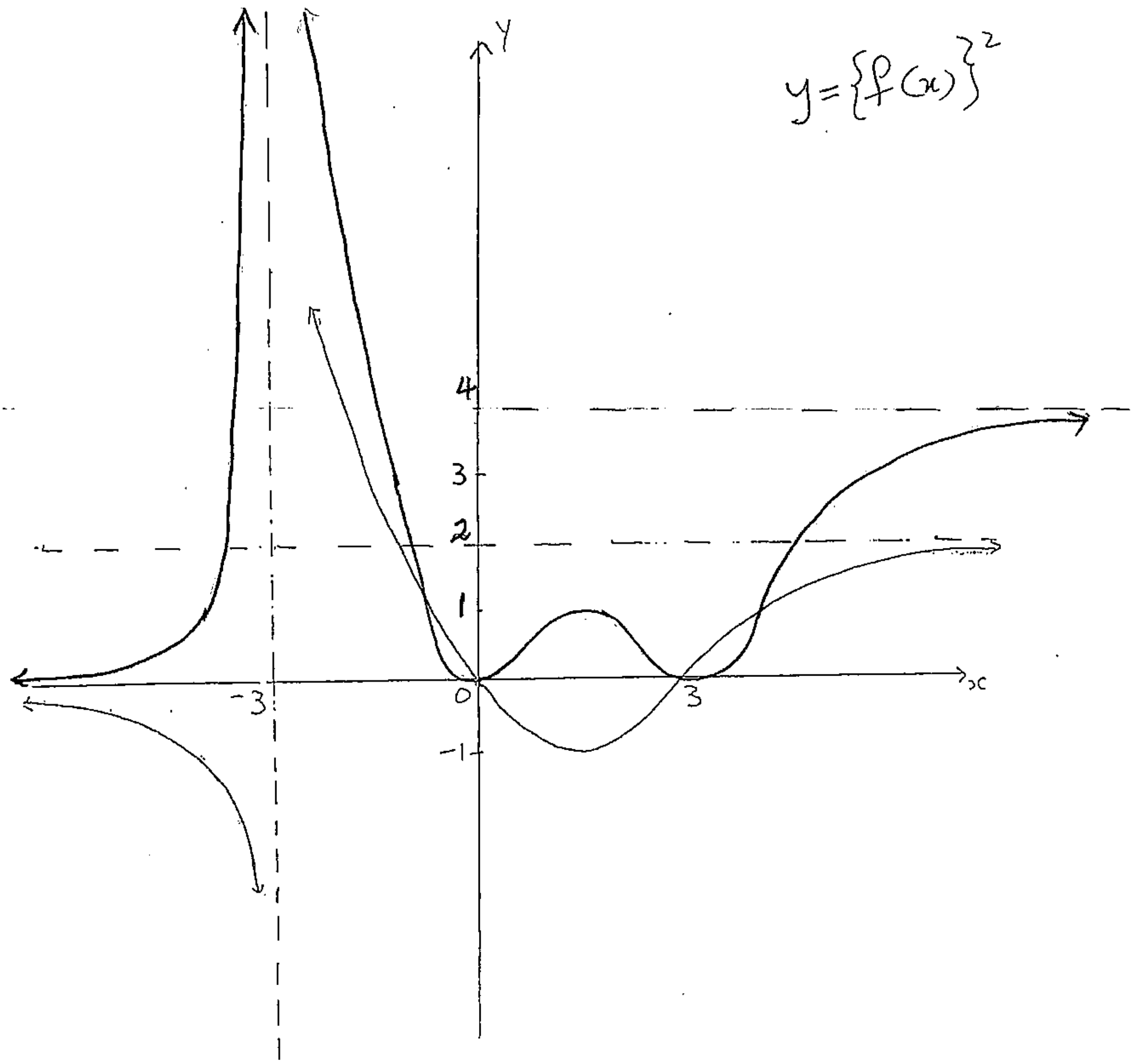
(Note equation of ellipse is

$$\frac{(x-5)^2}{25} + \frac{(y-3)^2}{9} = 1$$
)

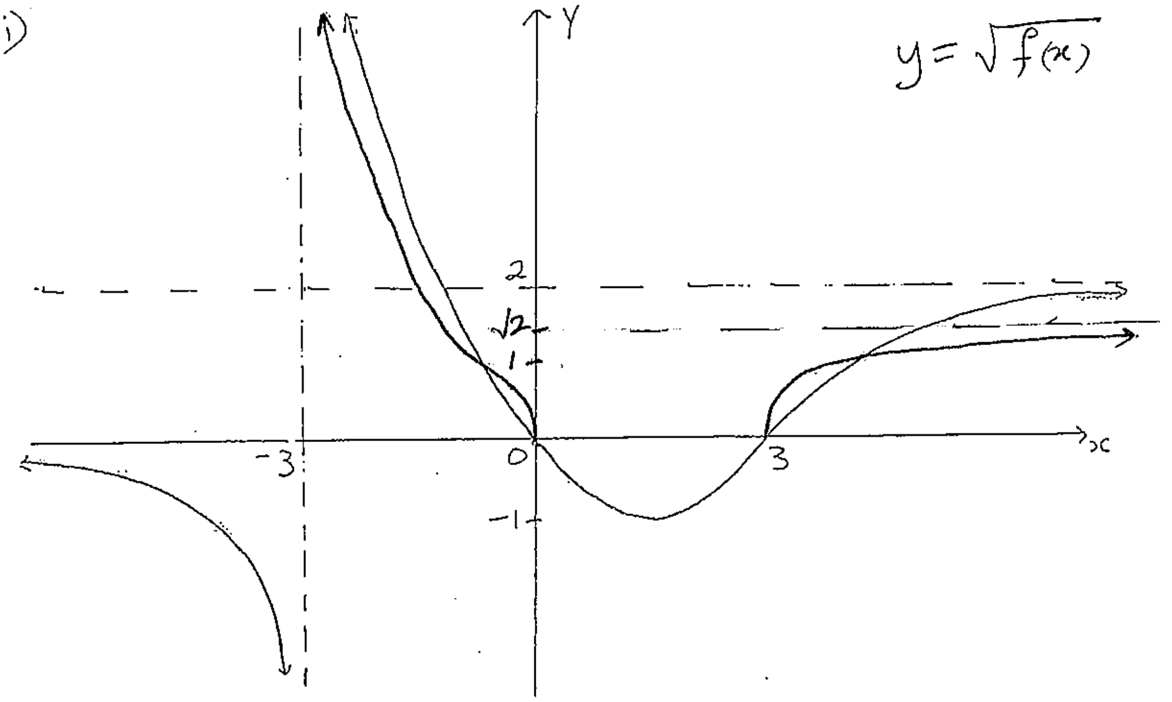
Question 3

a)

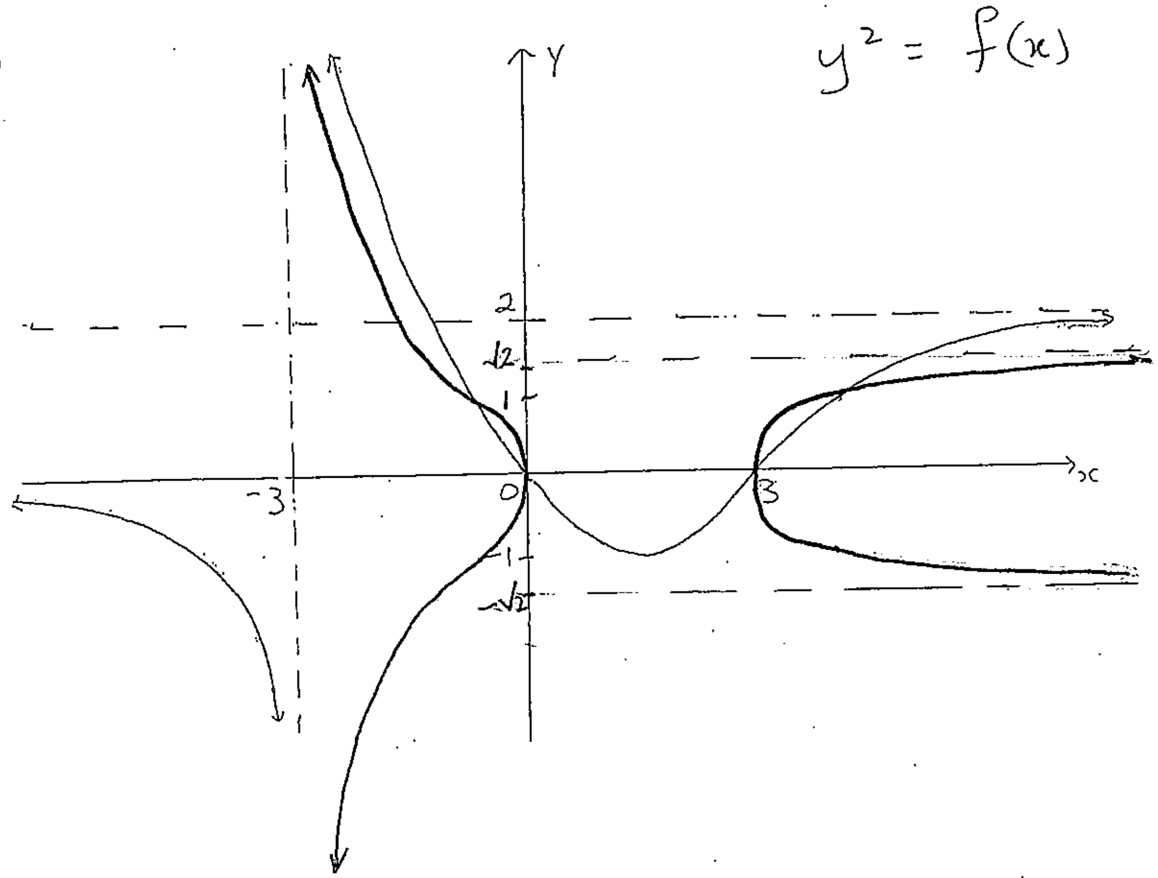
i)



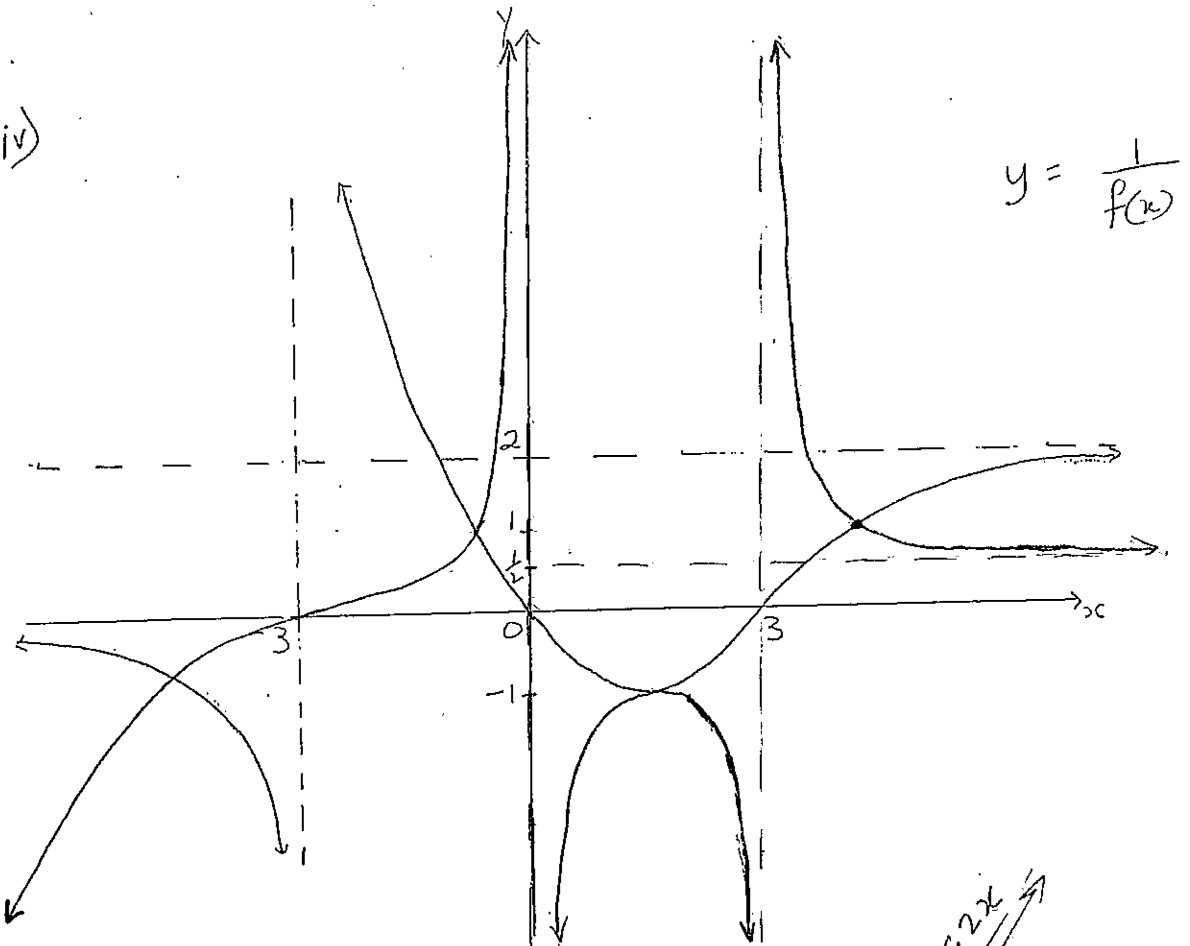
ii)



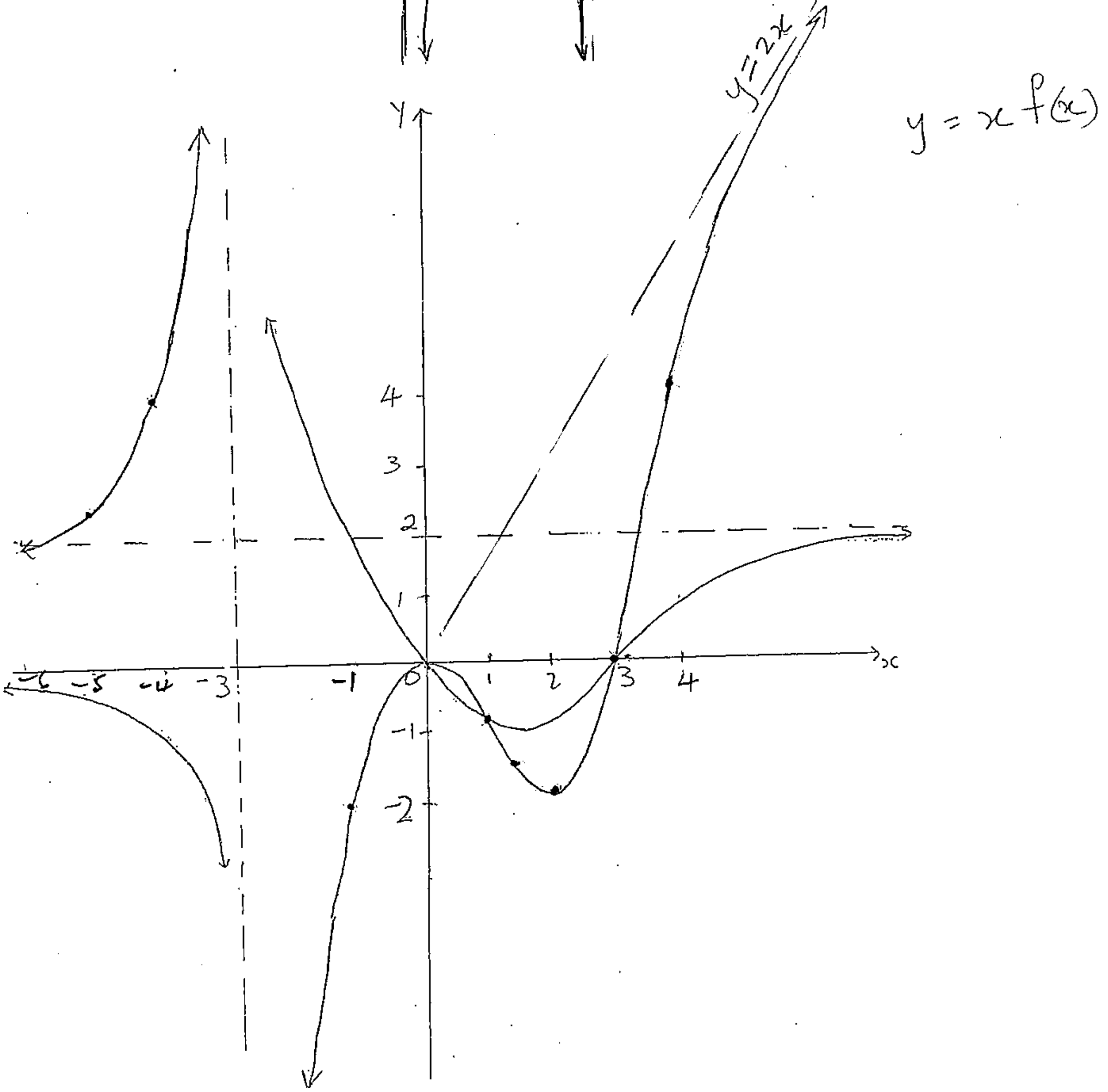
iii)



iv)



v)



Questron 3b

i) $xy = c^2$

$x\dfrac{dy}{dx} + y \cdot 1 = 0$ (Differentiating implicitly)

$\therefore \dfrac{dy}{dx} = -\dfrac{y}{x} = \boxed{-\dfrac{y_1}{x_1}}$ for $P(x_1, y_1)$

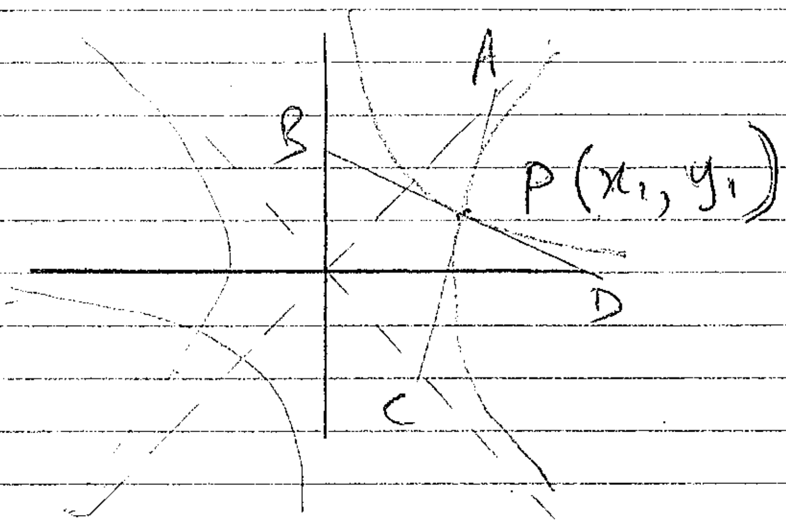
$\therefore$ Tangent is

$y - y_1 = -\dfrac{y_1}{x_1}(x - x_1)$

$x_1 y - x_1 y_1 = -y_1 x + x_1 y_1$

$x y_1 + x_1 y = 2x_1 y_1 = 2c^2$ as required.

ii)



For $x^2 - y^2 = a^2$

$2x - 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{2x}{2y} = \dfrac{x}{y}$

$\therefore$ at P, $\dfrac{dy}{dx} = \boxed{\dfrac{x_1}{y_1}}$

Tangent is

$y - y_1 = \dfrac{x_1}{y_1}(x - x_1)$

$y_1 y - y_1^2 = x_1 x - x_1^2$

$x_1^2 - y_1^2 = x_1 x - y_1 y$

$a^2 = x_1 x - y_1 y$.

This meets $y = x$ at A: $a^2 = x(x_1 - y_1)$

A is $\left(\dfrac{a^2}{x_1 - y_1}, \dfrac{a^2}{x_1 - y_1}\right)$

this meets $y = -x$ at C: $a^2 = x(x_1 + y_1)$

C is $\left(\dfrac{a^2}{x_1 + y_1}, \dfrac{-a^2}{x_1 + y_1}\right)$

Tangent $xy_1 + x_1 y = 2c^2$ meets y axis at B

$$B: \left(0, \frac{2c^2}{x_1}\right) = (0, 2y_1) \quad \text{as } y_1 = \frac{c^2}{x_1}$$

and x axis at D

$$D: \left(\frac{2c^2}{y_1}, 0\right) = (2x_1, 0) \quad \text{as } \frac{c^2}{y_1} = x_1$$

Now Mid Point of BD $= \left(\frac{0+2x_1}{2}, \frac{2y_1+0}{2}\right)$

$$= (x_1, y_1) \quad \text{ie } P.$$

Mid Point of AC

$$x = \frac{\frac{a^2}{x_1 - y_1} + \frac{a^2}{x_1 + y_1}}{2} \qquad y = \frac{\frac{a^2}{x_1 - y_1} + \frac{-a^2}{x_1 + y_1}}{2}$$

$$= \frac{a^2(x_1 + y_1 + x_1 - y_1)}{2(x_1 - y_1)(x_1 + y_1)} \qquad y = \frac{a^2(x_1 + y_1 - x_1 + y_1)}{2(x_1 - y_1)(x_1 + y_1)}$$

$$= \frac{2a^2 x_1}{2(x_1^2 - y_1^2)} \qquad = \frac{2a^2 y_1}{2(x_1^2 - y_1^2)}$$

But $x_1^2 - y_1^2 = a^2$

$$\therefore x = \frac{2a^2 x_1}{2a^2} \qquad y = \frac{2a^2 y_1}{2a^2}$$

$$x = x_1 \qquad y = y_1$$

$\therefore$ Mid Point of AC is (x_1, y_1) ie P.

$\therefore$ Diagonals bisect each other $\therefore$ ABCD is a parallelogram.

But Gradients of tangents $m_1 \times m_2 = -\frac{y_1}{x_1} \times \frac{x_1}{y_1} = -1$

$\therefore$ Diagonals are perpendicular.

$\therefore$ ABCD is a rhombus.

Now length of BD :

$$d^2 = 4x_1^2 + 4y_1^2$$

$$d = 2\sqrt{x_1^2 + y_1^2}$$

Length of AC :

$$d^2 = \left(\frac{a^2}{x_1+y_1} - \frac{a^2}{x_1-y_1}\right)^2 + \left(\frac{-a^2}{x_1+y_1} - \frac{a^2}{x_1-y_1}\right)^2$$

$$= \left\{\frac{a^2(x_1-y_1-x_1-y_1)}{(x_1+y_1)(x_1-y_1)}\right\}^2 + \left\{\frac{-a^2(x_1-y_1+x_1+y_1)}{(x_1+y_1)(x_1-y_1)}\right\}^2$$

$$= \left\{\frac{a^2(-2y_1)}{x_1^2-y_1^2}\right\}^2 + \left\{\frac{-a^2(2x_1)}{x_1^2-y_1^2}\right\}^2$$

$$= \left\{\frac{a^2(-2y_1)}{a^2}\right\}^2 + \left\{\frac{-a^2(2x_1)}{a^2}\right\}^2$$

$$= 4y_1^2 + 4x_1^2$$

$$d = 2\sqrt{x_1^2+y_1^2}$$

$\therefore$ BD = AC $\therefore$ equal diagonals.

ABCD is now a square

Question 4

a) i) $x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\dot{x} = \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt}$ $\qquad$ $\dot{y} = \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt}$

$\dot{x} = -r\omega\sin\theta$ $\qquad$ $\dot{y} = r\omega\cos\theta$

as $\theta = \omega t$ & $\frac{d\theta}{dt} = \omega$

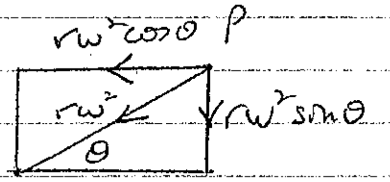
$\ddot{x} = -r\omega\cos\theta\frac{d\theta}{dt}$ $\qquad$ $\ddot{y} = -r\omega\sin\theta\frac{d\theta}{dt}$ as ω is constant

$\ddot{x} = -r\omega^2\cos\theta$ $\qquad$ $\ddot{y} = -r\omega^2\sin\theta$

$$Acc = \sqrt{\ddot{x}^2 + \ddot{y}^2} = \sqrt{r^2\omega^4\cos^2\theta + r^2\omega^4\sin^2\theta}$$

$$a = r\omega^2$$

Now $F = ma$ $\quad\therefore\quad$ $F = mr\omega^2$



Direction of this force

OP is resultant of $\ddot{x}$ & $\ddot{y}$

Force is directed to centre of the circle

ii) Gravitational force $F = \frac{Am}{r^2}$

As ω is constant, this is equal to radial force $mr\omega^2$. (All other forces negligible)

$$mr\omega^2 = \frac{Am}{r^2}$$

$$r^3 = \frac{A}{\omega^2}$$

$$r = \sqrt[3]{\frac{A}{\omega^2}}$$

4b) i) $(x-y)^2 \geq 0$ equality holds when $x=y$

ie $x^2 - 2xy + y^2 \geq 0$

$$x^2 + y^2 \geq 2xy \quad \text{—— } ①$$

ii) $\therefore$ $y^2 + z^2 \geq 2yz$ —— ②

& $z^2 + x^2 \geq 2zx$ —— ③

①+②+③ $x^2+y^2+y^2+z^2+z^2+x^2 \geq 2xy+2yz+2zx$

$$x^2+y^2+z^2 \geq xy+yz+zx$$

or

$$x^2+y^2+z^2-xy-yz-zx \geq 0 \quad \text{—— } ④$$

iii) ④ $\times$ $(x+y+z)$

$$x^3+xy^2+xz^2-x^2y-xyz-zx^2$$
$$+x^2y+y^3+yz^2-xy^2-y^2z-xyz$$
$$+zx^2+zy^2+z^3-xyz-yz^2-z^2x \geq 0$$

$$x^3+y^3+z^3-xyz-xyz-xyz \geq 0$$

$$x^3+y^3+z^3 \geq 3xyz \quad \text{—— } ⑤$$

iv) Put $x = x^{1/3}$, $y = y^{1/3}$, $z = z^{\frac{1}{3}}$

⑤ becomes $x+y+z \geq 3\,x^{1/3}y^{1/3}z^{1/3}$ —— ⑥

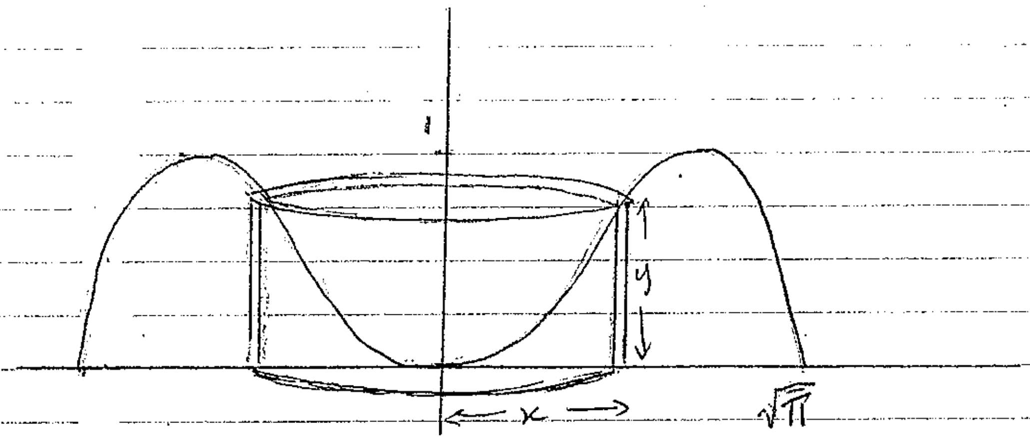
Put $x = x^{-\frac{1}{3}}$, $y = y^{-\frac{1}{3}}$, $z = z^{-\frac{1}{3}}$ in ⑤

⑤ becomes $x^{-1}+y^{-1}+z^{-1} \geq 3\,x^{-\frac{1}{3}}y^{-\frac{1}{3}}z^{-\frac{1}{3}}$ —— ⑦

⑥ $\times$ ⑦ $(x+y+z)(x^{-1}+y^{-1}+z^{-1}) \geq 9\,x^{\frac{1}{3}}x^{-\frac{1}{3}}y^{\frac{1}{3}}y^{-\frac{1}{3}}z^{\frac{1}{3}}z^{-\frac{1}{3}}$

$$(x+y+z)(x^{-1}+y^{-1}+z^{-1}) \geq 9$$

4 c)



$$\Delta V = 2\pi r h \,\Delta x$$

$$\Delta V = 2\pi x y \,\Delta x$$

$$\Delta V = 2\pi x \sin(x^2)\,\Delta x$$

$$V = \pi \int_0^{\sqrt{\pi}} 2x \sin(x^2)\,dx$$

$$= \pi \left[-\cos(x^2) \right]_0^{\sqrt{\pi}}$$

$$= \pi \left(-\cos\pi + \cos 0 \right)$$

$$= \pi \left(1 + 1 \right)$$

$$V = 2\pi \quad u^3$$

Question 5

a) Let $P(x) = (x-a)R(x)$
 + $Q(x) = (x-a)T(x)$

$$P(x) - Q(x) = (x-a)R(x) - (x-a)T(x)$$
$$= (x-a)(Rx - T(x))$$

$\therefore$ $(x-a)$ is a factor of $P(x) - Q(x)$

$$P(x) = x^3 + x^2 - 5x + k$$
$$Q(x) = x^3 - 8x^2 + 13x - 2k$$
have a common double factor $(x-\alpha)^2$

$\therefore$ $P(x) - Q(x)$ has a factor $(x-\alpha)^2$

$$P(x) - Q(x) = x^3 + x^2 - 5x + k - (x^3 - 8x^2 + 13x - 2k)$$
$$= x^3 + x^2 - 5x + k - x^3 + 8x^2 - 13x + 2k$$
$$= 9x^2 - 18x + 3k$$

For a double root of a quadratic $\Delta = 0$

$$18^2 - 4 \times 9 \times 3k = 0$$
$$2^2 9^2 - 2^2 9 \; 3k = 0$$
$$3k = 9$$
$$k = 3$$

$$\therefore P(x) - Q(x) = 9x^2 - 18x + 9$$
$$= 9(x-1)^2$$

Double root is $x = 1$.

b) i) Consider $z = (\cos\theta + i\sin\theta)^5 = 1^5 \operatorname{cis} 5\theta$ By De Moivres theorem

$\therefore$ $\operatorname{cis}5\theta = \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$ (Binomial theorem)

Equating real + imaginary parts:

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

+

$$\sin 5\theta = \sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta$$

$$\therefore \tan 5\theta = \frac{\sin^5\theta - 10\cos^2\theta\sin^3\theta + 5\cos^4\theta\sin\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing top + bottom by $\cos^5\theta$ (provided

$\cos\theta \neq 0$ ie $\theta \neq (2n-1)\frac{\pi}{2}$)

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$ as required.

iii) Put $\tan 5\theta = 1$ and $t = \tan\theta$

$$1 = \frac{t^5 - 10t^3 + 5}{1 - 10t^2 + 5t^4}$$

$$1 - 10t^2 + 5t^4 = t^5 - 10t^3 + 5$$

$t^5 - 5t^4 - 10t^3 + 10t^2 + 5t - 1 = 0$ has solutions equal to solutions of $\tan 5\theta = 1$

$$\tan 5\theta = 1$$

$$5\theta = \frac{\pi}{4} + n\pi = (4n+1)\frac{\pi}{4}$$

$$\theta = (4n+1)\frac{\pi}{20}$$

Principal values: $n=0 \Rightarrow \theta = \frac{\pi}{20}$; $n=1 \Rightarrow \theta = \frac{5\pi}{20} = \frac{\pi}{4}$

$n=2 \Rightarrow \theta = \frac{9\pi}{20}$; $n=-1 \Rightarrow \theta = \frac{-3\pi}{20}$; $n=-2 \Rightarrow \theta = \frac{-7\pi}{20}$

$\therefore$ Solutions are

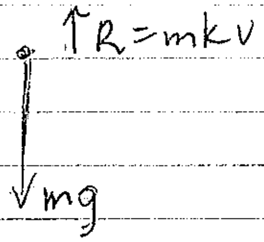
$t = \tan\frac{\pi}{20}$, $\tan\frac{\pi}{4}$, $\tan\frac{9\pi}{20}$, $\tan\frac{-3\pi}{20}$, $\tan\frac{-7\pi}{20}$.

or $t = \tan\frac{\pi}{20}$, $\tan\frac{\pi}{4}$, $\tan\frac{9\pi}{20}$, $\tan\frac{13\pi}{20}$, $\tan\frac{17\pi}{20}$

would be the same values for t.

Question 6

a) Taking Downwards as positive



$$m\ddot{x} = mg - mkv$$

$$\ddot{x} = g - kv$$

$$\frac{dv}{dt} = g - kv$$

$$\frac{dt}{dv} = \frac{1}{g - kv}$$

$$t = -\frac{1}{k}\ln(g - kv) + c_1$$

When $=0$, $v = 0$ $\therefore$ $c_1 = \frac{1}{k}\ln g$

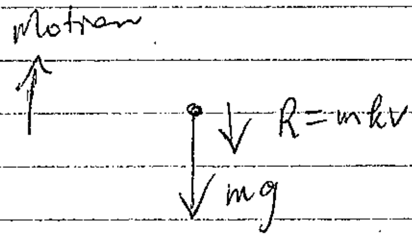
$$\therefore t = -\frac{1}{k}\ln\frac{g - kv}{g}$$

or $$-kt = \ln\frac{g - kv}{g}$$

$$e^{-kt} = \frac{g - kv}{g}$$

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

ii) Taking upwards as positive



$$m\ddot{x} = -mg - mkv$$

$$\ddot{x} = -g - kv$$

$$\frac{dt}{dv} = \frac{-1}{g + kv}$$

$$t = -\frac{1}{k}\ln(g + kv) + c_2$$

When $t = 0$, $v = u$ $\Rightarrow$ $c_2 = \frac{1}{k}\ln(g + ku)$

$$\text{d}\quad t = -\frac{1}{k}\ln\left(\frac{g + kv}{g + ku}\right)$$

When 2nd particle is at rest $v=0$

$$t = -\frac{1}{k} \ln\left(\frac{g}{g+ku}\right)$$

Velocity of 1st particle at this time

$$V = \frac{g}{k}\left(1 - e^{-kt}\right)$$

$$v = \frac{g}{k}\left(1 - e^{\ln\left(\frac{g}{g+ku}\right)}\right)$$

$$v = \frac{g}{k}\left(1 - \frac{g}{g+ku}\right)$$

$$= \frac{g}{k}\left(\frac{g+ku-g}{g+ku}\right)$$

$$v = \frac{g}{k} \cdot \frac{ku}{g+ku} = \frac{gu}{g+ku} \text{ ——— } *$$

Now terminal velocity of first particle is

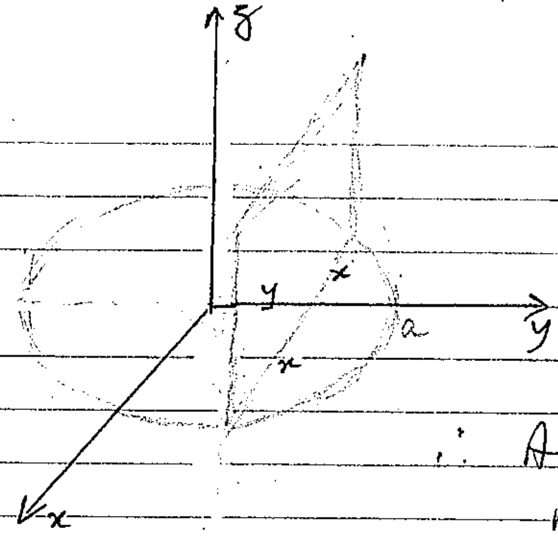
$$\lim_{t \to \infty} \frac{g}{k}\left(1 - e^{-kt}\right)$$

$$V = \frac{g}{k}$$

$\therefore$ * becomes
$$v = \frac{\frac{g}{k} u}{\frac{g}{k} + u}$$

$$v = \frac{Vu}{V+u}$$ as required.

6 b)



At distance y from O

$$x = \sqrt{a^2 - y^2}$$

Side of square is

$$2x = 2\sqrt{a^2 - y^2}$$

$\therefore$ Area of Cross-section

$$A = \left(2\sqrt{a^2-y^2}\right)^2$$

$$= 4\,(a^2 - y^2)$$

$$\Delta V = 4\,(a^2 - y^2)\,\Delta y$$

$$V = \int_{-a}^{a} 4\,(a^2 - y^2)\,dy$$

$$= 2\int_{0}^{a} 4\,(a^2 - y^2)\,dy$$

$$= 8\left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$V = 8\left(a^3 - \frac{a^3}{3}\right)$$

$$V = \frac{16a^3}{3}\ u^3$$

c)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

or $\cos\alpha\cos\beta = \frac{1}{2}\left\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\right\}$

$$\cos mx \cos nx = \frac{1}{2}\left\{\cos(m+n)x + \cos(m-n)x\right\}$$

$$\int_0^{\pi} \cos mx \cos nx\,dx = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi}$$

$$= \frac{1}{2}\,\frac{\sin(m+n)\pi}{m+n} + \frac{1}{2}\,\frac{\sin(m-n)\pi}{m-n} - 0$$

But $\sin k\pi = 0$

$\therefore\ I = 0$

Question 7

a) i) For P, $\cos x = \tan x \implies \cos^2 x = \sin x$

also $y = \cos x$ $\quad$ $y = \tan x$ $\quad$ $\cos^2 \alpha = \sin \alpha$

$y' = -\sin x$ $\quad$ $y' = \sec^2 x$

$m_1 = -\sin \alpha$ $\quad$ $m_2 = \sec^2 \alpha$

$$m_1 . m_2 = -\sin \alpha . \sec^2 \alpha$$
$$= -\sin \alpha . \frac{1}{\cos^2 \alpha}$$

But at P, $\cos^2 \alpha = \sin \alpha$

$\therefore m_1 m_2 = -1$ as required.

curves intersect at right angles at P.

ii) Now solving $\cos^2 \alpha = \sin \alpha$
$$1 - \sin^2 \alpha = \sin \alpha$$
$$\sin^2 \alpha + \sin \alpha - 1 = 0$$
$$\sin \alpha = \frac{-1 \pm \sqrt{1+4}}{2}$$
$$\sin \alpha = \frac{-1 \pm \sqrt{5}}{2}$$

But $\cos^2 \alpha = \sin \alpha$ $\therefore$ positive solution required
$$\sin \alpha = \frac{-1+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$$

$$\cos^2 \alpha = \frac{\sqrt{5}-1}{2}$$

$$\sec^2 \alpha = \frac{2}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1}$$

$$\sec^2 \alpha = \frac{2(\sqrt{5}+1)}{5-1}$$

$$\sec^2 \alpha = \frac{\sqrt{5}+1}{2}$$ as required

7 b) $U_n = \int_0^1 (1-x^2)^n \, dx$

Put $u = (1-x^2)^n$ $\qquad dv = dx$

$du = n(1-x^2)^{n-1} \cdot -2x \, dx \qquad v = x$

$$\int u \, dv = uv - \int v \, du$$

$$\int_0^1 (1-x^2)^n \, dx = \left[x(1-x^2)^n\right]_0^1 - \int_0^1 nx \cdot -2x(1-x^2)^{n-1} \, dx$$

$$= 0 - 2n \int_0^1 -x^2 (1-x^2)^{n-1} \, dx.$$

$$= -2n \int_0^1 (1-x^2-1)(1-x^2)^{n-1} \, dx$$

$$= -2n \left\{ \int_0^1 (1-x^2)^n \, dx - \int_0^1 (1-x^2)^{n-1} \, dx \right\}$$

$$U_n = -2n \left\{ U_n - U_{n-1} \right\}$$

$$= -2n U_n + 2n U_{n-1}$$

$$(2n+1) U_n = 2n U_{n-1}$$

$$U_n = \frac{2n}{2n+1} U_{n-1} \quad \text{as required}$$

$\therefore$ $U_4 = \frac{8}{9} U_3$; $U_3 = \frac{6}{7} U_2$

$U_2 = \frac{4}{5} U_1$; $U_1 = \frac{2}{3} U_0$

$$U_0 = \int_0^1 (1-x^2)^0 \, dx = \int_0^1 1 \, dx = \left[x\right]_0^1 = 1$$

$$\therefore U_4 = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$U_4 = \frac{128}{315}$$

7c) i) $\angle ALP + \angle AMP = 90° + 90° = 180°$

Opposite angles are supplementary.

ii) $\angle PMB = \angle PNB$ (given $90°$ each)

Equal angles subtended by interval PB at points M & N (ie angles in same segment standing on arc PB)

iii) Let $\angle PBN = \alpha$

$\therefore \angle PMN = 180° - \alpha$

(Opposite angle of cyclic quad PMNB)

Also $\angle PAC = 180° - \alpha$

(Opposite angles of cyclic quad PACB)

$\therefore \angle LAP = \alpha$ (LAC is a straight angle)

But $\angle LAP = \angle LMP$ (Angles in same segment standing on arc PL of cyclic quad PLAM)

ie $\angle LMP = \alpha$

$\therefore \angle LMP + \angle PMN = \alpha + 180 - \alpha$

$= 180°$

ie

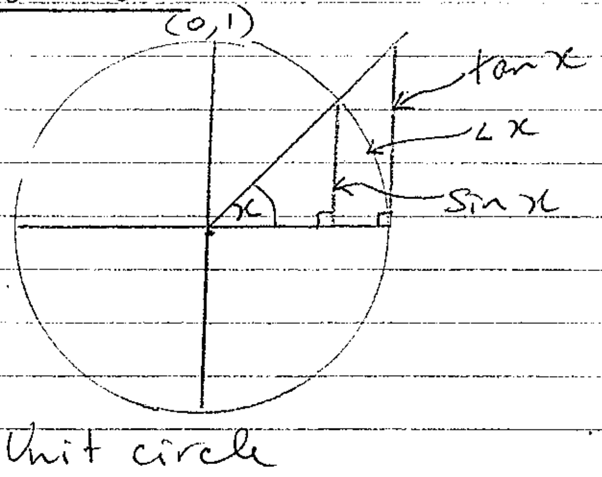
$\angle LMN$ is a straight angle

or

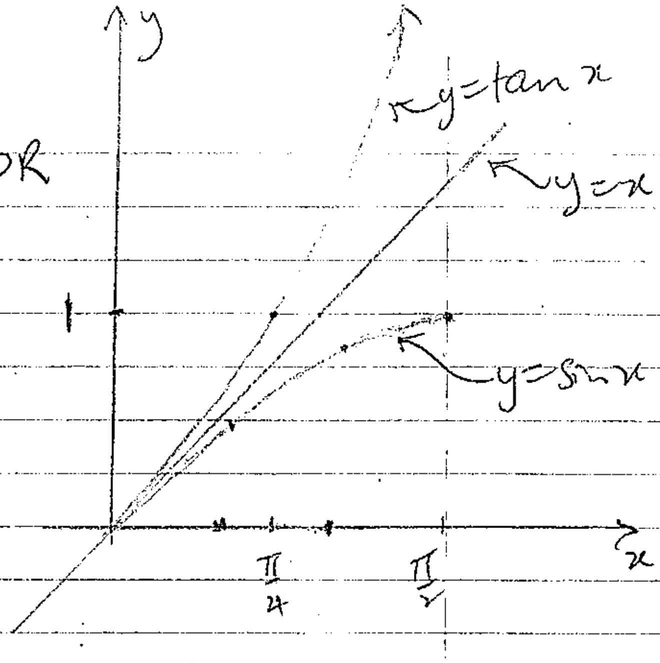
L, M & N are collinear.

Question 8

a)



OR



As all are positive for $0 < x < \frac{\pi}{2}$

area under $\sin x$ < area under x < area under $\tan x$

x^2 is also positive

$\therefore \quad x^2 \sin x < x^3 < x^2 \tan x$

& Areas under these are all positive

$$\therefore \int_0^{\pi/6} x^2 \sin x \, dx < \int_0^{\pi/6} x^3 \, dx < \int_0^{\pi/6} x^2 \tan x \, dx$$

But $$\int_0^{\pi/6} x^3 \, dx = \left[\frac{x^4}{4}\right]_0^{\frac{\pi}{6}} = \frac{\pi^4}{4 \times 6^4} = \frac{\pi^4}{2^6 \cdot 3^4}$$

ie $$\int_0^{\pi/6} x^2 \sin x \, dx < \frac{\pi^4}{2^6 \cdot 3^4} < \int_0^{\pi/6} x^2 \tan x \, dx.$$

b) Put $\alpha = \tan^{-1} 3x \implies \tan\alpha = 3x$

$\quad \beta = \tan^{-1} 2x \implies \tan\beta = 2x$

$\therefore \quad (\alpha - \beta) = \tan^{-1} \frac{1}{5}$

or $\tan(\alpha - \beta) = \frac{1}{5}$

$$\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{1}{5}$$

$$\frac{3x - 2x}{1+6x^2} = \frac{1}{5}$$

$$5x = 1 + 6x^2$$

$$6x^2 - 5x + 1 = 0$$

$$(3x-1)(2x-1) = 0$$

$$x = \frac{1}{3} \text{ or } \frac{1}{2}$$

c) i) $\ddot{x} = 0$ $\quad\quad$ $\ddot{y} = -g$

$\dot{x} = c_1$ $\quad\quad$ $\dot{y} = -gt + c_2$

$t=0$, $\dot{x} = V\cos\theta$, $\quad$ $\dot{y} = V\sin\theta$

$\therefore \dot{x} = V\cos\theta$ $\quad\quad$ $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta + c_3$ $\quad\quad$ $y = -\frac{gt^2}{2} + Vt\sin\theta + c_4$

$t=0$, $x=0$, $y=0$

$\therefore x = Vt\cos\theta$ $\quad\quad$ $y = -\frac{gt^2}{2} + Vt\sin\theta$

ii) $t = \frac{x}{V\cos\theta}$

$$\therefore y = \frac{-gx^2}{2V^2\cos^2\theta} + V.\frac{x}{V\cos\theta}\sin\theta$$

$$y = \frac{-gx^2\sec^2\theta}{2V^2} + x\tan\theta$$

iii) Inclined plane : $y = x\tan\alpha$

$$\therefore x\tan\alpha = \frac{-gx^2\sec^2\theta}{2V^2} + x\tan\theta$$

$$g\sec^2\theta\, x^2 + 2V^2x\tan\alpha - 2V^2x\tan\theta = 0$$

$$x\left(g\sec^2\theta\, x + 2V^2(\tan\alpha - \tan\theta)\right) = 0$$

$$x = 0 \text{ or } x = \frac{2V^2(\tan\theta - \tan\alpha)}{g\sec^2\theta}$$

$$\therefore R = \frac{2V^2}{g}(\tan\theta - \tan\alpha)\cos^2\theta\sec\alpha$$

or $R = \frac{2V^2}{g}\left(\sin\theta\cos\theta - \tan\alpha\cos^2\theta\right)\sec\alpha$

$$R = \frac{2V^2}{g}\left(\tfrac{1}{2}\sin 2\theta - \tan\alpha\cos^2\theta\right)\sec\alpha$$

v) Now $\frac{dR}{d\theta} = \frac{2V^2}{g}\left(\tfrac{1}{2}.2\cos 2\theta - 2\cos\theta(-\sin\theta)\tan\alpha\right)\sec\alpha$

$$\frac{dR}{d\theta} = \frac{2V^2}{g}\left(\cos 2\theta + \sin 2\theta\tan\alpha\right)\sec\alpha$$

vi) For maximum range $\frac{dR}{d\theta} = 0$.

$$\cos 2\theta + \sin 2\theta\tan\alpha = 0$$

$$\sin 2\theta\tan\alpha = -\cos 2\theta$$

$$\tan 2\theta = -\frac{1}{\tan\alpha}$$

$$\tan 2\theta = -\cot\alpha$$

$$\tan 2\theta = -\tan\left(\frac{\pi}{2} - \alpha\right)$$

$\therefore 2\theta = \pi - \left(\frac{\pi}{2} - \alpha\right)$ or $2\pi - \left(\frac{\pi}{2} - \alpha\right)$

$2\theta = \frac{\pi}{2} + \alpha$ or $\frac{3\pi}{2} + \alpha$

$\theta = \frac{\pi}{4} + \frac{\alpha}{2}$ or $\frac{3\pi}{4} + \frac{\alpha}{2}$

But $0 < \theta < \frac{\pi}{2}$ & $\theta > \alpha$

$$\therefore \theta = \frac{\pi}{4} + \frac{\alpha}{2}$$

(Minimum range is obviously when $\theta = \frac{\pi}{2}$, $R = 0$