GOSFORD HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

EXTENSION 2 MATHEMATICS

General Instructions:

- Reading time: 5minutes.
- Working time: 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question should be started on a separate writing booklet.
- All necessary working should be shown in every question.

Total marks: - 120

Attempt all Questions 1- 8.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{dx}{\sqrt{9x^2-1}}$. (2)

(b) Find $\int \frac{dx}{\sqrt{4x-x^2}}$. (2)

(c) Evaluate $\int_0^{\pi} x\ sinx\ dx$. (3)

(d) Find $\int cos^5 x\ sin^2 x\ dx$. (4)

(e) Use the substitution $t = tan\ \frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2 + cos\, x}$. (4)

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) If $z = 2 + i$ and $\omega = 1 - 3i$ find in the form $x + iy$

(i) z^2. (1)

(ii) $z\overline{\omega}$. (1)

(iii) $\frac{z}{\omega}$. (1)

(b)

(i) Express $z = 1 + \sqrt{3}i$ in modulus-argument form. (2)

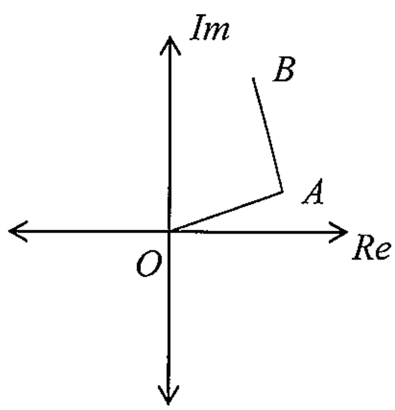
(ii) Show that $(1 + \sqrt{3}i)^6$ is a real number. (2)

(c) For the complex number $z = x + iy$, where x and y are real numbers, find and clearly sketch the curve on an Argand diagram for which

(i) $|z + \bar{z}| \leq 2$. (2)

(ii) $Re(z^2 - 4) = 0$. (3)

(d) The point A in the Argand diagram below represents the complex number $z = a + ib$. The point B represents the complex number $2 + 5i$.

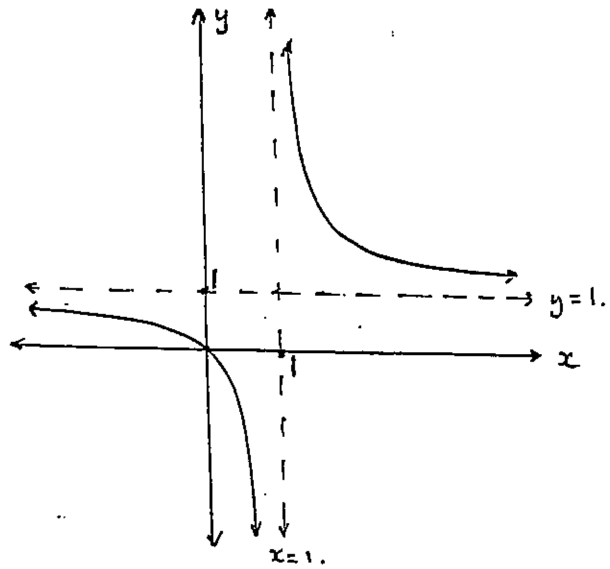


NOT TO SCALE

If the complex number represented by the point C is such that OABC is a square, find C in terms of a and b and hence evaluate a and b. (3)

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The function defined by $y = f(x)$ is drawn below.



Draw separate one-third page sketches of

(i) $y = f(x)$ and $y = f(-x)$. (2)

(ii) $y = f(x)$ and $= \dfrac{1}{f(x)}$. (2)

(iii) $y = f(x)$ and $|y| = f(x)$. (2)

(iv) $y = f(x)$ and $y^2 = f(x)$. (3)

(b) The equation of a curve is $4x^2 + xy + y^2 = 10$. Find the equation of the tangent to the curve at the point (1,2) on it. (3)

(c) Find the number of different ways of arranging any 4 of the letters from the word EXERCISES. (3)

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) When a polynomial $P(x)$ is divided by $(x-3)$ the remainder is 10 and when $P(x)$ is divided by $(x-4)$ the remainder is 13. Determine the remainder when $P(x)$ is divided by $(x-3)(x-4)$. (2)

(b) If α,β and γ are the roots of the equation $x^3 - 7x^2 - 7 = 0$ find the equations whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. (2)

(ii) $\alpha^2, \beta^2, \gamma^2$. (2)

(c)

(i) Express $\frac{2}{x^3+2x}$ in the form $\frac{A}{x} + \frac{Bx+C}{x^2+2}$. (2)

(ii) Show that $\int_1^2 \frac{2}{x^3+2x}\,dx = \frac{1}{2}\,ln2.$ (2)

(d) Consider the equation $z^4 + pz^3 + qz + r = 0$, where p, q & r are real numbers. The sum of the roots of this equation is 6 more than the product of the roots.
If $1 + i$ is a root of the equation, find

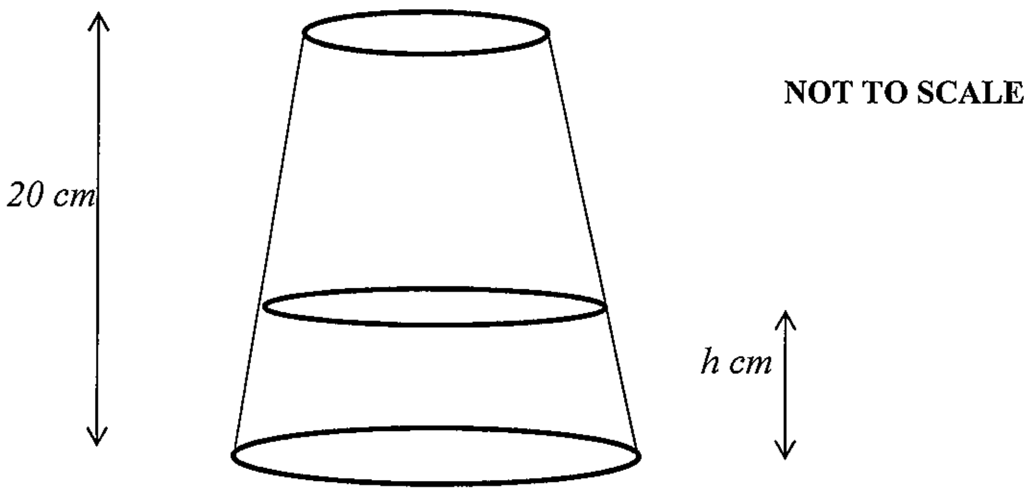
(i) p, q & r. (3)

(ii) all the roots of the equation. (2)

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The region bounded by the x-axis and the curve $y = -2 + 3x - x^2$ is rotated about the line $x = 3$ to form a solid. Use the method of cylindrical shells to find the volume of the solid formed. (5)

(b) The diagram below shows the frustrum of a right cone. (A frustrum of a cone is a cone with its top cut off.) The height of the frustrum is 20 cm and the radii of the base and the top are 15 cm and 10 cm respectively.

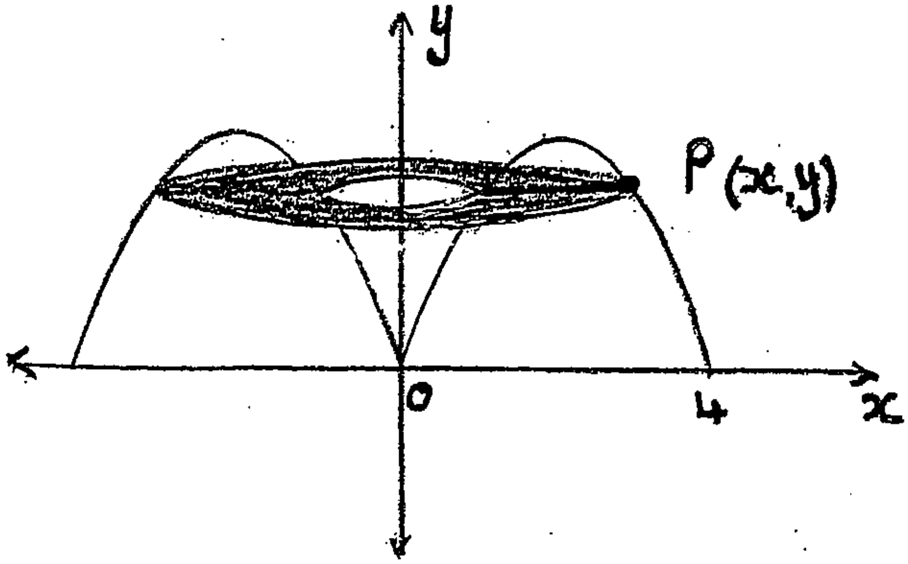


A horizontal cross-section taken at height h cm is a circle of radius r units.

(i) Show that $r = 15 - \frac{h}{4}$. (2)

(ii) Find the volume of the frustrum. (3)

(c) The region bounded by $y = 4x - x^2$ and the x-axis is rotated about the y-axis to form a solid of revolution. If a horizontal line is drawn from the point $P(x, y)$ on the curve, where $2 < x < 4$, to the y-axis it sweeps out an annulus.



(i) Show that the area of the annulus is given by

$A = \pi\left[4\sqrt{16 - 4y}\right].$ (3)

(ii) Hence find the volume of the solid. (2)

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the ellipse $\mathcal{E}$, with equation $\dfrac{x^2}{100} + \dfrac{y^2}{64} = 1$.

(i) Calculate the eccentricity of $\mathcal{E}$. (1)

(ii) Find the coordinates of the foci and the equations of the directrices of $\mathcal{E}$. (2)

(iii) Show that the equation of the tangent at the point $P(x_0, y_0)$ on $\mathcal{E}$ is

$$\frac{x_0 x}{100} + \frac{y_0 y}{64} = 1.$$ (3)

(b) A conic is a rectangular hyperbola with eccentricity $\sqrt{2}$, focus $(2,0)$ and directrix $x = 1$.

(i) Find the equation of this hyperbola. (1)

(ii) Sketch this hyperbola clearly showing the asymptotes and vertices. (1)

(iii) Show that the equation of the normal at the point $P(asec\theta, atan\theta)$ is $xtan\theta + ysec\theta = 2\sqrt{2}sec\theta tan\theta$. (3)

(iv) This normal meets the x-axis at $Q(X, 0)$ and the y-axis at $R(0, Y)$.

If T is the point (X, Y), find the locus of T and describe this locus geometrically. (4)

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of unit mass is projected vertically upwards from ground level with initial speed U. Assume that air resistance is kv, where v is the particle's speed and k is a positive constant. We wish to consider the particle's motion as it falls back to ground level. Let y be the displacement of the particle measured vertically downwards from the point of maximum height, t be the time elapsed after the particle has reached maximum height, and g be the acceleration due to gravity.

(i) Explain why $v(0) = 0$ and $\frac{dv}{dt} = g - kv$ while the particle is in motion. (1)

(ii) Deduce that $v = \frac{g}{k}(1 - e^{-kt})$ for $t \geq 0$. (3)

(iii) By writing $\frac{dv}{dt} = v\frac{dv}{dy}$, deduce from part (i) that
$\frac{g}{k} \log_e \left(\frac{g-kv}{g}\right) + v = -ky.$ (3)

(iv) Using parts (ii) and (iii) deduce that $t = \frac{v+ky}{g}$. (2)

(v) Given that the particle reaches a maximum height
$h = \frac{1}{k}\left[U - \frac{g}{k} \log_e \left(\frac{g+kU}{g}\right)\right]$ in time $t_h = \frac{1}{k} \log_e \left(\frac{g+kU}{g}\right)$, deduce that the total time T that the particle is in the air is $T = \frac{U+V}{g}$, where V is the final speed of the particle when it returns to ground level. (1)

(b) If $I_n = \int_0^1 (x^2 - 1)^n \, dx$, $n = 0,1,2, \ldots$

(i) Show that $I_0 = 1$. (1)

(ii) Prove that $I_n = \frac{-2n}{2n+1} I_{n-1}$. (3)

(iii) Hence evaluate $\int_0^1 (x^2 - 1)^4 \, dx$. (1)

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a)

(i) Use DeMoivre's Theorem to express $cos\ 4\theta$ and $sin\ 4\theta$ in powers of $cos\ \theta$ and $sin\ \theta$. Hence express $tan\ 4\theta$ as a rational function of t, where $t = tan\ \theta$. (4)

(ii) Hence solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$. (3)

(b) A particle is projected from the origin with an initial velocity of $V\ ms^{-1}$ at an angle of α to the horizontal.

(i) Show that the maximum range on the horizontal plane is $\frac{V^2}{g}$ when $\alpha = \frac{\pi}{4}$. (4)

(ii) The particle is now to hit a target which is h metres above its horizontal position when the maximum range in part (i) is reached. If the angle of projection α remains the same, show that the initial velocity must be increased from $V\ ms^{-1}$ to $\frac{V^2}{\sqrt{V^2-gh}}\ ms^{-1}$. (Air resistance is neglected). (4)

END OF PAPER

GLENN,
PLEASE MARK Q1 & 2

QUESTION 1.

a) $\int \frac{dx}{\sqrt{9x^2-1}} = \int \frac{dx}{3\sqrt{x^2-\frac{1}{9}}}$

$= \frac{1}{3}\int \frac{dx}{\sqrt{x^2-\frac{1}{9}}}$ (2)

$= \frac{1}{3} \ln \left| x + \sqrt{x^2-\frac{1}{9}} \right| + c, \quad x > \frac{1}{3} > 0.$

b) $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x^2-4x+4)}}$

$= \int \frac{dx}{\sqrt{4-(x-2)^2}}$ (2)

$= \sin^{-1}\left[\frac{x-2}{2}\right] + c$

c) $\int_0^{\pi} x \sin x \, dx$

Let $u = x \qquad v' = \sin x$

$u' = 1 \qquad v = -\cos x$

$I = \left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x \, dx$

$= (-\pi\cos\pi - 0) + \left[\sin x\right]_0^{\pi}$ (3)

$= \pi - 0 + \sin\pi - \sin 0$

$= \pi - 0 + 0 - 0$

$= \pi$

d) $\int \cos^5 x \sin^2 x \, dx$

$= \int \cos^4 x \sin^2 x \cos x \, dx$

$$= \int (1-\sin^2 x)^2 \sin^2 x \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$I = \int (1-u^2)^2 u^2 \, du$$

$$= \int u^2 (1 - 2u^2 + u^4) \, du$$

4

$$= \int u^2 - 2u^4 + u^6 \, du$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + c$$

$$= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + c$$

e) $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}$ Let $t = \tan\frac{x}{2}$

$dx = \dfrac{2\,dt}{1+t^2}$

When $x=0$, $t=0$

$x = \frac{\pi}{2}$, $t=1$. & $\cos x = \dfrac{1-t^2}{1+t^2}$

$$I = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt$$

$$= \int_0^1 \frac{1}{\frac{2+2t^2+1-t^2}{\cancel{1+t^2}}} \frac{2}{\cancel{1+t^2}} \, dt$$

$$= \int_0^1 \frac{2}{3+t^2} \, dt$$

4

$$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{6} - 0 \right) = \frac{\pi}{3\sqrt{3}}$$

QUESTION 2

a) $z = 2+i, \quad w = 1-3i$

(i) $z^2 = (2+i)^2$

$= 4 + 4i + i^2$ 1

$= 3 + 4i$

(ii) $z\bar{w} = (2+i)(1+3i)$

$= 2 + 6i + i + 3i^2$

$= -1 + 7i$ 1

(iii) $\dfrac{z}{w} = \dfrac{(2+i)}{(1-3i)} \times \dfrac{(1+3i)}{(1+3i)}$

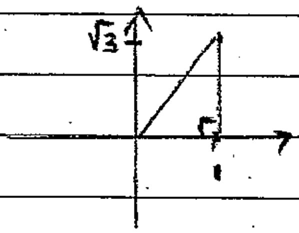
$= \dfrac{-1+7i}{1-9i^2}$

$= \dfrac{-1+7i}{10}$ 1

$= \dfrac{-1}{10} + \dfrac{7}{10}i$

b) $z = 1 + \sqrt{3}i$

(i)



$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$

$\arg z = \tan^{-1}\sqrt{3}$

$= \dfrac{\pi}{3}$

$\therefore z = 2 \operatorname{cis} \dfrac{\pi}{3}$ 2

(ii) $z^6 = 2^6 \operatorname{cis} 6\dfrac{\pi}{3}$

$= 64 \operatorname{cis} 2\pi$

$= 64(\cos 2\pi + i \sin 2\pi)$

$= 64(1 + 0i)$ 2

$= 64$ which is real.

c)

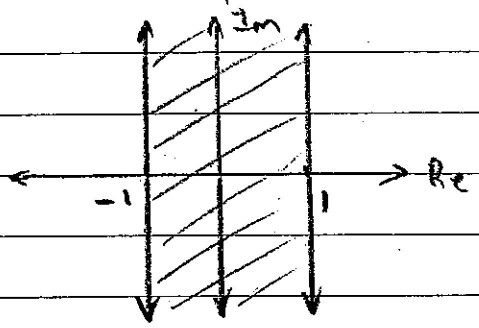
(i) If $z = x + iy$, $\bar{z} = x - iy$

$\therefore z + \bar{z} = 2x$

If $|z + \bar{z}| \leq 2$

$|2x| \leq 2$

$|x| \leq 1$



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(ii) If $z = x + iy$

$z^2 = (x + iy)^2$

$= x^2 + 2xyi + i^2y^2$

$= (x^2 - y^2) + 2xyi$

$\therefore Re(z^2 - 4) = x^2 - y^2 - 4$

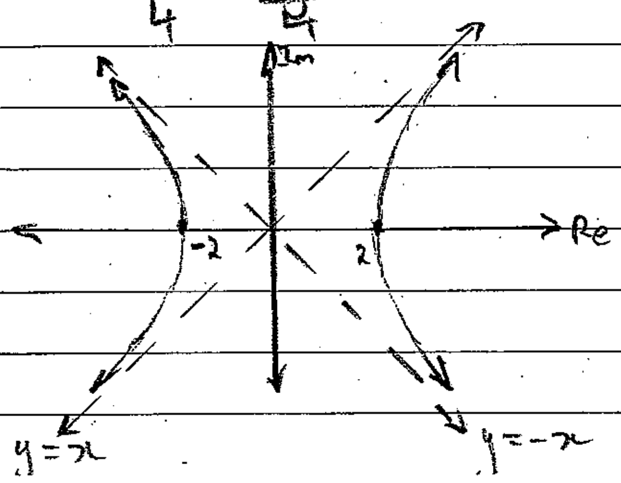
If $Re(z^2 - 4) = 0$

$x^2 - y^2 - 4 = 0$

$x^2 - y^2 = 4$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$

This is a rect hyperbola with asymptotes $y = \pm x$, x intercepts ± 2.



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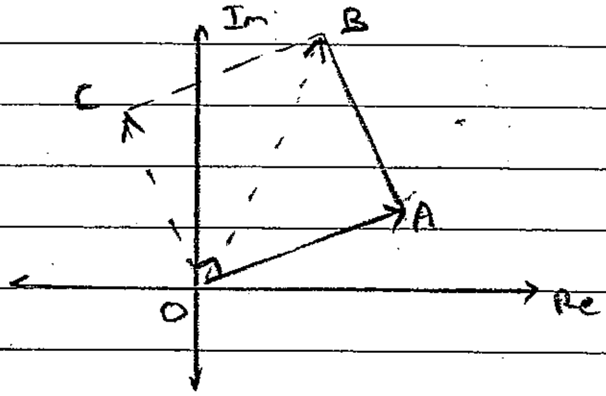
d)



If A is $z = a+ib$

C is $iz = i(a+ib)$

$= ia + i^2 b$

$\therefore C$ is $-b+ia$ (1)

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$

$\therefore 2+5i = a+ib + -b+ia$

$2+5i = a-b + i(a+b)$

$a-b = 2 \quad -(1)$

$a+b = 5 \quad -(2)$

(1) + (2)

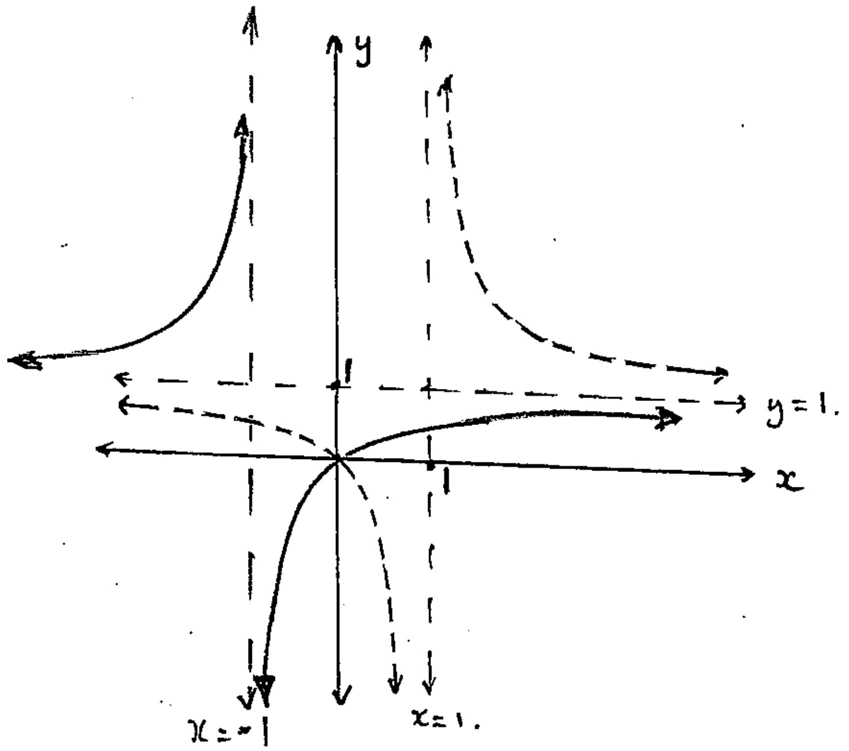
$2a = 7$

$a = \frac{7}{2}$ (2)

$b = \frac{3}{2}$.

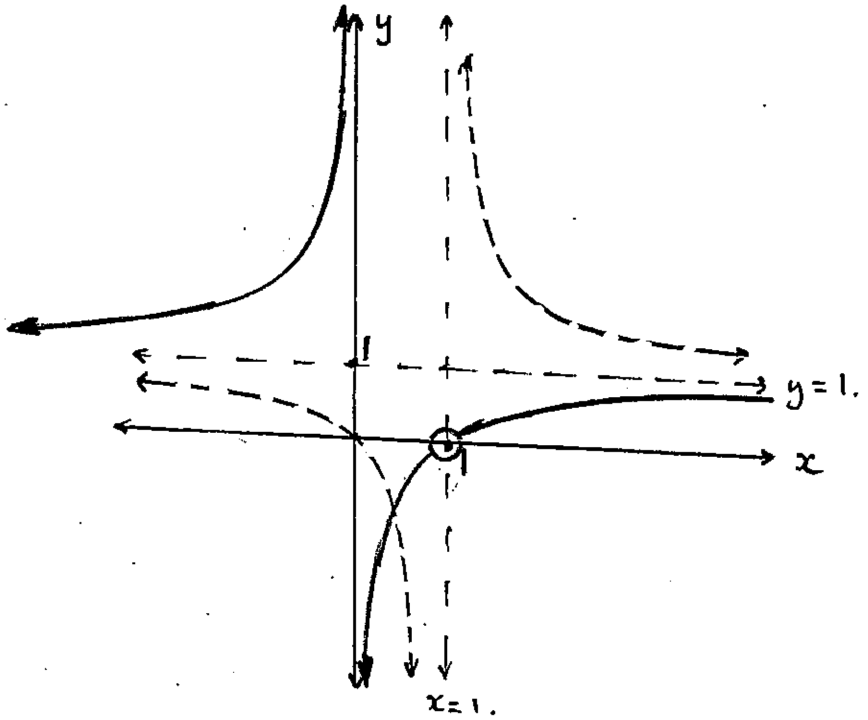
QUESTION 3.

a) (i)



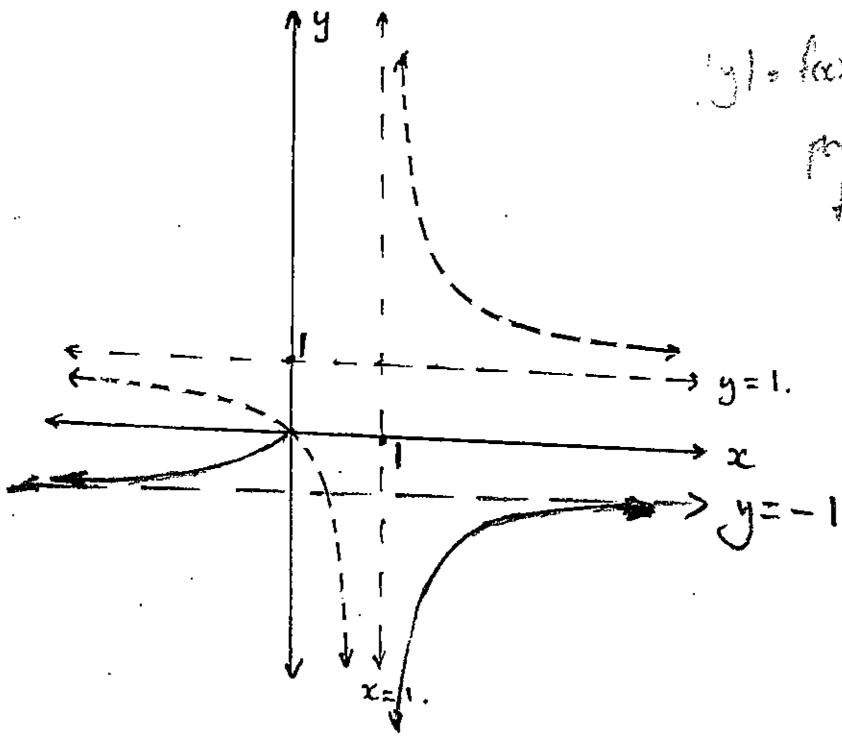
(2)

(ii)



(2)

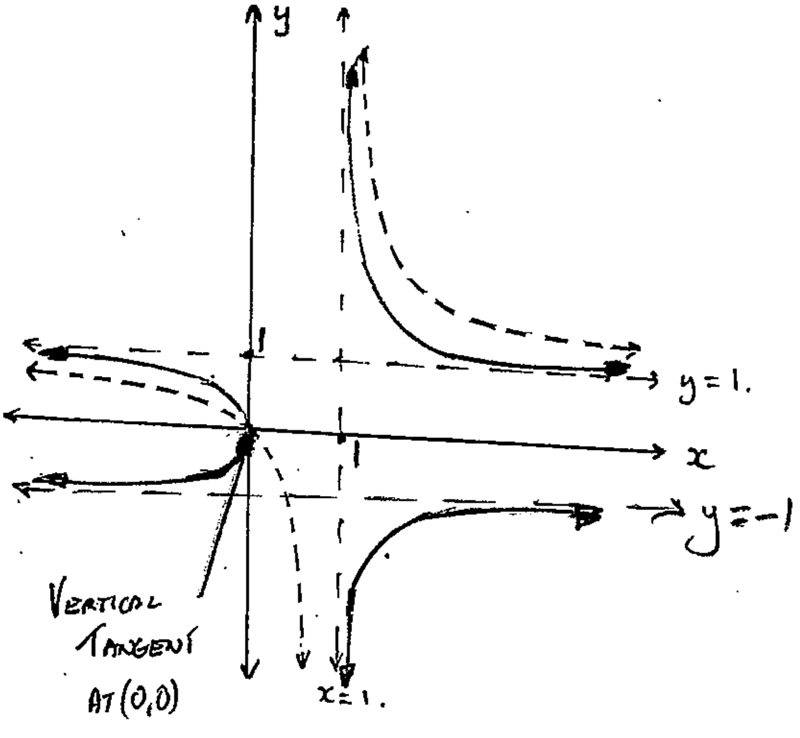
(III)



|y| = f(x) consists of the part of f(x) above the x-axis & its reflection in the x-axis

2

(IV)



$y^2 = f(x)$

$y = \pm\sqrt{f(x)}$

3

b) If $4x^2 + xy + y^2 = 10$

By implicit differentiation

$$8x + y.1 + x.y' + 2y.y' = 0$$

$$y'(x+2y) = -8x-y$$

$$y' = \frac{-8x-y}{x+2y}$$

When $x=1$, $y=2$

$$y' = \frac{-8-2}{1+4}$$

$$= -2$$

2

$\therefore$ Eqn of tangent is

$$y-2 = -2(x-1)$$

$$y-2 = -2x+2$$

$$2x+y-4 = 0$$

c) Groups of 4 letters can be classified as

(i) 3 alike & 1 different

(ii) 2 " & 2 "

(iii) 2 " & 2 others alike

(iv) all different.

(i) 3 E's & one of X, R, C, I, S can be done in $^5C_1$ ways

(ii) 2 E's & two of X, R, C, I, S can be done in $^5C_2$ ways

2 S's & two of E, X, R, C, I can be done in $^5C_2$ ways

(iii) 2 E's & 2 S's can be done in only 1 way.

(iv) 4 letters from E, X, R, C, I, S can be done in $^6C_4$ ways

No of different arrangements of each classification is

(i) $^5C_1 \times \frac{4!}{3!} = 20$

(ii) $(^5C_2 + ^5C_2) \times \frac{4!}{2!} = 240$

3

(iii) $1 \times \frac{4!}{2!.2!} = 6$

(iv) $^6C_4 \times 4! = 360$

$\therefore$ Total no. of different arrangements is 626.

QUESTION 4.

a) If $P(x)$ is divided by $(x-3)(x-4)$ then

$$P(x) = (x-3)(x-4) \cdot Q(x) + R(x)$$

where $R(x) = ax+b$ (deg of $R(x)$ < deg of divisor

Now $P(3) = 10$

$\therefore 10 = 3a+b$ — ①

& $P(4) = 13$

$\therefore 13 = 4a+b$ — ②

2

② − ①

$3 = a$

$b = 1$

$\therefore$ the remainder is $3x+1$

b) (i) Let $y = \frac{1}{x}$, since $x = \alpha, \beta, \gamma$

$y = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\therefore x = \frac{1}{y}$

$\therefore \left(\frac{1}{y}\right)^3 - 7\left(\frac{1}{y}\right)^2 - 7 = 0$

$\frac{1}{y^3} - \frac{7}{y^2} - 7 = 0$

2

$1 - 7y - 7y^3 = 0$

i.e. $0 = 7y^3 + 7y - 1$

$\therefore$ the required eqn is $7x^3 + 7x - 1 = 0$

(ii) Let $y = x^2$, since $x = \alpha, \beta, \gamma$

$y = \alpha^2, \beta^2, \gamma^2$

$x = \sqrt{y}$

$\therefore (\sqrt{y})^3 - 7(\sqrt{y})^2 - 7 = 0$

$$y^{3/2} - 7y - 7 = 0$$
$$y^{3/2} = 7y + 7$$

Square both sides

$$y^3 = (7y+7)^2$$
$$y^3 = 49y^2 + 98y + 49$$

2

$$y^3 - 49y^2 - 98y - 49 = 0$$

$\therefore$ The required eqⁿ is $x^3 - 49x^2 - 98x - 49 = 0$.

c) (i) $$\frac{2}{x^3+2x} = \frac{2}{x(x^2+2)}$$

Let $$\frac{2}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$\therefore \quad 2 = A(x^2+2) + x(Bx+C)$$

When $x=0$, $2 = 2A$
$$A = 1$$

When $x=1$, $2 = 3A + B + C$
$$2 = 3 + B + C$$
$$B + C = -1 \quad — (1)$$

When $x=-1$, $2 = 3A + B - C$
$$2 = 3 + B - C$$
$$B - C = -1 \quad — (2)$$
$$(1) + (2)$$
$$2B = -2$$
$$B = -1$$

2

$$\therefore \quad C = 0$$

$$\therefore \quad \frac{2}{x^3+2x} = \frac{1}{x} - \frac{x}{x^2+2}$$

(ii) $\int_1^2 \frac{2}{x^3+2x}\,dx = \int_1^2 \frac{1}{x} - \frac{x}{x^2+2}\,dx$

$= \left[\ln x - \frac{1}{2}\ln(x^2+2)\right]_1^2$

$= (\ln 2 - \frac{1}{2}\ln 6) - (\ln 1 - \frac{1}{2}\ln 3)$

$= \ln 2 - \ln\sqrt{6} + \ln\sqrt{3}$

$= \ln\left(\frac{2\sqrt{3}}{\sqrt{6}}\right)$

$= \ln\left(\frac{2}{\sqrt{2}}\right)$

$= \ln(\sqrt{2})$ 2

$= \frac{1}{2}\ln 2$

d)(i) If $1+i$ is a root of the eq^n we may sub $1+i$ into the eq^n.

Now $(1+i)^2 = 1+2i+i^2$
$= 2i$
$(1+i)^3 = 2i(1+i)$
$= -2+2i$
$(1+i)^4 = (2i)^2$
$= -4$

$\therefore$ If $(1+i)^4 + p(1+i)^3 + q(1+i) + r = 0$

$-4 + p(-2+2i) + q(1+i) + r = 0$

But $\sum\alpha = -p$ & $\sum\alpha\beta\gamma\delta = r$

Hence $-p = 6+r$
$r = -p-6$

$\therefore\ -4 + p(-2+2i) + q(1+i) - p - 6 = 0$

$-4 - 2p + 2pi + q + qi - p - 6 = 0$

$-10 - 3p + q + i(2p+q) = 0 + 0i$

$\therefore\ 2p + q = 0 \quad — ①$

$-3p + q = 10 \quad — ②$

$① - ②$

$5p = -10$

$p = -2$

3

$\therefore\ q = 4$

$\&\ r = -4.$

(iii) Since $(1+i)$ is a root & p, q, r are real $(1-i)$ is also a root

Let the roots be $\alpha, \beta, 1+i, 1-i$

Now $\sum\alpha = \alpha + \beta + 2$

$\therefore\ \alpha + \beta + 2 = -p$

$\alpha + \beta + 2 = 2$

$\alpha = -\beta \quad — ①$

Also $\sum\alpha\beta\gamma\delta = \alpha\beta(1-i^2)$

$= \alpha\beta \times 2$

$= 2\alpha\beta$

2

$\therefore\ 2\alpha\beta = r$

$2\alpha\beta = -4$

$\alpha\beta = -2 \quad — ②$

Sub ① into ②

$-\beta^2 = -2$

$\beta^2 = 2$

$\beta = \pm\sqrt{2}$

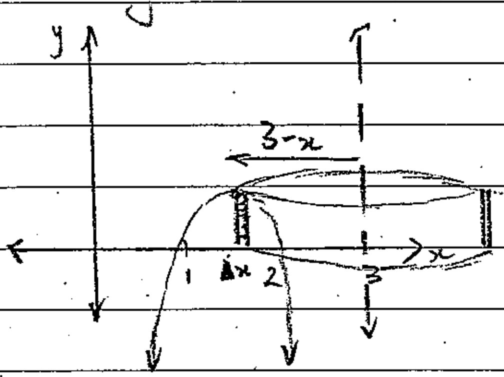
$\alpha = \pm\sqrt{2}$

Roots are $\sqrt{2}$, $\sqrt{2}$, $(1+i)$, $(1-i)$

Question 5.

a) If $y = -2 + 3x - x^2$

$y = (2-x)(x-1)$



$\Delta V = 2\pi xy \cdot \Delta x$

$= 2\pi(3-x)(-2+3x-x^2) \cdot \Delta x$

$V = \lim_{\Delta x \to 0} \sum_1^2 2\pi(3-x)(-2+3x-x^2) \cdot \Delta x$

$= 2\pi \int_1^2 (3-x)(-2+3x-x^2)\, dx$ 5

$= 2\pi \int_1^2 -6 + 9x - 3x^2 + 2x - 3x^2 + x^3\, dx$

$= 2\pi \int_1^2 x^3 - 6x^2 + 11x - 6 \quad dx$

$= 2\pi \left[\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x \right]_1^2$

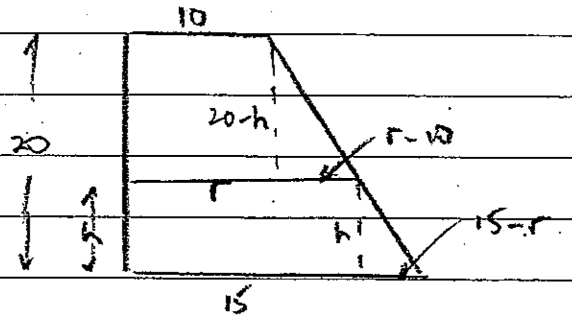
$= 2\pi \left[\left(\frac{2^4}{4} - 2(2)^3 + \frac{11}{2}(2)^2 - 6(2) \right) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6 \right) \right]$

$= 2\pi \left[(4 - 16 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6 \right) \right]$

$= 2\pi \times \frac{1}{4}$

$= \frac{\pi}{2}$ units^3.

b) (i)



Since the Δs are similar

$$\frac{r-10}{15-r} = \frac{20-h}{h}$$

$$hr - 10h = (15-r)(20-h)$$

$$\cancel{hr} - 10h = 300 - 15h - 20r + \cancel{hr}$$

$$20r = 300 - 5h$$

$$r = \frac{300-5h}{20}$$

(2)

$$r = 15 - \frac{h}{4}$$

(ii) Area of cross-section is πr^2

$$\therefore A = \pi\left(15 - \frac{h}{4}\right)^2$$

$$\Delta V = A \cdot \Delta h$$

$$= \pi\left(15 - \frac{h}{4}\right)^2 \cdot \Delta h$$

$$V = \lim_{\Delta h \to 0} \sum_{0}^{20} \pi\left(15 - \frac{h}{4}\right)^2 \cdot \Delta h$$

$$= \int_0^{20} \pi\left(15 - \frac{h}{4}\right)^2 dh$$

$$= \pi\left[\frac{\left(15-\frac{h}{4}\right)^3}{3 \times -\frac{1}{4}}\right]_0^{20}$$

$$= \pi\left[\frac{(15-5)^3}{-3/4} - \frac{(15-0)^3}{-3/4}\right]$$

$$= \pi \times 3166\tfrac{2}{3}$$

(3)

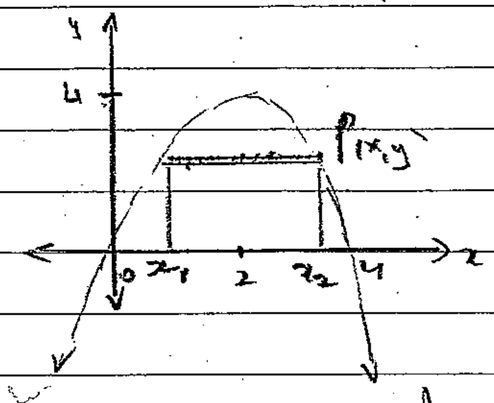
$$= \frac{9500\pi}{3} \text{ units}^3$$

c)(i) If $4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or 4

When $x = 2$, $y = 8 - 4$
$= 4$



The annulus has inner radius x_1, outer radius x_2

Area formula $= \pi(x_2^2 - x_1^2)$

$A = \pi(x_2 + x_1)(x_2 - x_1)$

where x_1 & x_2 are the roots of $y = 4x - x^2$ for various values of y

Since $y = 4x - x^2$

$x^2 - 4x + y = 0$

If x_1, x_2 are the roots

$x_1 + x_2 = \frac{-b}{a}$ $\quad x_1 x_2 = \frac{c}{a}$

$= 4$ $\quad = y$

Now $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2$

$= 4^2 - 4y$

$= 16 - 4y$

$x_2 - x_1 = \sqrt{16 - 4y}$

3

$\therefore A = \pi(4)(\sqrt{16-4y})$

$= \pi\left[4\sqrt{16-4y}\right]$

(ii) $\Delta V = \pi\left[4\sqrt{16-4y}\right] \cdot \Delta y$

$V = \lim_{\Delta y \to 0} \sum_{0}^{4} \pi\left[4\sqrt{16-4y}\right] \Delta y$

$= 4\pi \int_0^4 \sqrt{16-4y}\, dy$

$= 4\pi \int_0^4 2\sqrt{4-y}\, dy$

$= 8\pi \int_0^4 (4-y)^{\frac{1}{2}}\, dy$

$$= 8\pi \left[\frac{(4-y)^{3/2}}{\frac{3}{2} \times -1} \right]_0^4$$

$$= 8\pi \left[-\frac{2}{3} (4-y)^{3/2} \right]_0^4$$

$$= 8\pi \left[-\frac{2}{3} \times 0 + \frac{2}{3} \times 4^{3/2} \right]$$

2

$$= 8\pi \times \frac{2}{3} \times 8$$

$$= \frac{128\pi}{3} \text{ units}^3.$$

QUESTION 6.

a) (i) If $\frac{x^2}{100} + \frac{y^2}{64} = 1$

$a = 10, \quad b = 8.$

$b^2 = a^2(1 - e^2)$

$64 = 100(1 - e^2)$

$\frac{64}{100} = 1 - e^2$

$e^2 = 1 - \frac{64}{100}$ (1)

$e^2 = \frac{36}{100}$

$e = \frac{3}{5}, \quad e > 0.$

(ii) Foci are $(\pm ae, 0)$

$ae = 10 \times \frac{3}{5}$

$= 6$

Foci are $(\pm 6, 0)$

Directrices are $x = \pm \frac{a}{e}$ (2)

$\frac{a}{e} = 10/\frac{3}{5}$

$= 50/3$

Directrices are $x = \pm \frac{50}{3}$.

(iii) If $\frac{x^2}{100} + \frac{y^2}{64} = 0$

$\frac{2x}{100} + \frac{2y}{64} \cdot y' = 0$

$\frac{2y}{64} \cdot y' = \frac{-2x}{100}$

$$y' = \frac{\frac{-\cancel{2}x}{100}}{\frac{\cancel{2}y}{64}}$$

$$= \frac{-64x}{100y}$$

At $P(x_0, y_0)$, $y' = \frac{-64}{100}\frac{x_0}{y_0}$

$\therefore$ the eqn of the tangent at P is

$$y - y_0 = \frac{-64}{100}\frac{x_0}{y_0}(x - x_0)$$

$$\frac{y_0(y-y_0)}{-64} = \frac{x_0}{100}(x - x_0)$$

3

$$-\frac{y_0 y}{64} + \frac{y_0^2}{64} = \frac{x_0 x}{100} - \frac{x_0^2}{100}$$

$$\frac{x_0^2}{100} + \frac{y_0^2}{64} = \frac{x_0 x}{100} + \frac{y_0 y}{64}$$

$$\therefore \quad 1 = \frac{x_0 x}{100} + \frac{y_0 y}{64}$$ since (x_0, y_0) lies on E

ie. $$\frac{x_0 x}{100} + \frac{y_0 y}{64} = 1.$$

b) (i) Since the focus is $(ae, 0)$

$$ae = 2$$

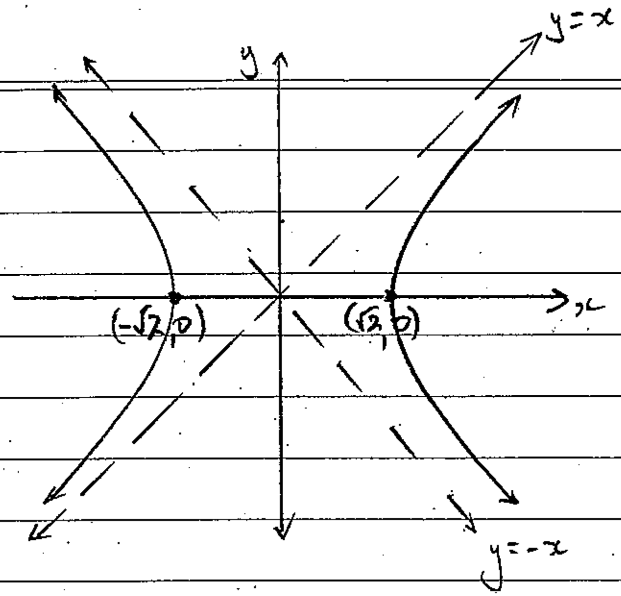
$$a.\sqrt{2} = 2$$

$$a = \sqrt{2}$$

1

Hence the eqn is $\frac{x^2}{2} - \frac{y^2}{2} = 1.$

(ii)



(iii) If $\frac{x^2}{2} - \frac{y^2}{2} = 1$

$$x^2 - y^2 = 2$$

$$\therefore \quad 2x - 2y \cdot y' = 0$$

$$y' = \frac{2x}{2y}$$

$$y' = \frac{x}{y}$$

at $P(\sqrt{2}\sec\theta, \sqrt{2}\tan\theta)$ $y' = \frac{\cancel{\sqrt{2}}\sec\theta}{\cancel{\sqrt{2}}\tan\theta}$

$$= \frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin\theta}{\cancel{\cos\theta}}}$$

$$= \frac{1}{\sin\theta}$$

$\therefore$ the gradient of the normal is $-\sin\theta$

Hence the eqn is

$$y - \sqrt{2}\tan\theta = -\sin\theta\,(x - \sqrt{2}\sec\theta)$$

$$y - \sqrt{2}\tan\theta = -\sin\theta\, x + \sqrt{2}\sin\theta\sec\theta$$

$$y - \sqrt{2}\tan\theta = -\sin\theta\, x + \sqrt{2}\tan\theta$$

$$\sin\theta\, x + y = 2\sqrt{2}\tan\theta$$

$\div$ throughout by $\cos\theta$

$$\frac{\sin\theta}{\cos\theta}x + \frac{y}{\cos\theta} = 2\sqrt{2}\,\frac{\tan\theta}{\cos\theta}$$

$\tan\theta . x + \sec\theta . y = 2\sqrt{2} \tan\theta \sec\theta$

3

ie. $x \tan\theta + y \sec\theta = 2\sqrt{2} \sec\theta \tan\theta$

(iv) When $y = 0$, $x \cancel{\tan\theta} = 2\sqrt{2} \sec\theta \cancel{\tan\theta}$

$\therefore x = 2\sqrt{2} \sec\theta$

Q is $(2\sqrt{2}\sec\theta, 0)$

When $x = 0$, $y \cancel{\sec\theta} = 2\sqrt{2} \cancel{\sec\theta} \tan\theta$

$\therefore y = 2\sqrt{2} \tan\theta$

P is $(0, 2\sqrt{2}\tan\theta)$

$\therefore$ T is $(2\sqrt{2}\sec\theta, 2\sqrt{2}\tan\theta)$

4

If $X = 2\sqrt{2}\sec\theta$, $\sec\theta = \dfrac{X}{2\sqrt{2}}$

If $Y = 2\sqrt{2}\tan\theta$, $\tan\theta = \dfrac{Y}{2\sqrt{2}}$

Now $1 + \tan^2\theta = \sec^2\theta$

$\therefore 1 + \dfrac{Y^2}{8} = \dfrac{X^2}{8}$

ie $\dfrac{X^2}{8} - \dfrac{Y^2}{8} = 1$

$a = 2\sqrt{2}$ $e = \sqrt{2}$

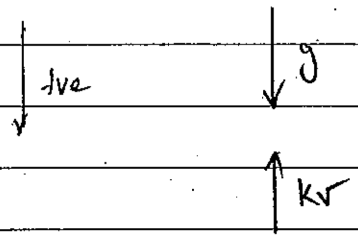
$\therefore ae = 4$

$\dfrac{a}{e} = \dfrac{2\sqrt{2}}{\sqrt{2}} = 2$

Thus the locus of T is a rectangular hyperbola with vertices $(\pm 2\sqrt{2}, 0)$, foci $(\pm 4, 0)$ and directrices $x = \pm 2$.

QUESTION 7

a) (i)



$F = ma, \quad m = 1$

$\therefore \ddot{y} = g - kv$ 1

i.e. $\frac{dv}{dt} = g - kv$

When $t=0$, $v=0$ since it is instantaneously at rest at max. height

(ii) $\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g-kv}$

$t = \int -\frac{1}{k} \cdot \frac{-k}{g-kv}\, dv$

$\therefore t = -\frac{1}{k} \ln(g-kv) + c$

If $t=0$, $v=0$

$0 = -\frac{1}{k} \ln(g) + c$

$c = \frac{1}{k} \ln g$.

$\therefore t = -\frac{1}{k} \ln(g-kv) + \frac{1}{k} \ln g$

$t = \frac{1}{k} \ln\left(\frac{g}{g-kv}\right)$.

$kt = \ln\left(\frac{g}{g-kv}\right)$

$e^{kt} = \frac{g}{g-kv}$ 3

$e^{kt}(g-kv) = g$

$g - kv = \frac{g}{e^{kt}}$

$kv = g - ge^{-kt}$

$v = \frac{g}{k}\left(1 - e^{-kt}\right), \quad t \geq 0$

(iii) $\frac{dv}{dt} = v\frac{dv}{dy}$

$\therefore v\frac{dv}{dy} = g - kv$

$\frac{dv}{dy} = \frac{g-kv}{v}$

$\frac{dy}{dv} = \frac{v}{g-kv}$

$= -\frac{1}{k}\left(\frac{-kv}{g-kv}\right)$

$= -\frac{1}{k}\left(\frac{g-kv-g}{g-kv}\right)$

$= -\frac{1}{k}\left(1 - \frac{g}{g-kv}\right)$

$y = \int -\frac{1}{k}\left(1 - \frac{g}{g-kv}\right) dv$

$y = -\frac{1}{k}\int 1 + \frac{g}{k} \cdot \frac{-k}{g-kv}\, dv$ 3

$y = -\frac{1}{k}\left(v + \frac{g}{k}\ln(g-kv)\right) + C_1$

$-ky = v + \frac{g}{k}\ln(g-kv) + C_2$

If $v=0$, $y=0$

$0 = 0 + \frac{g}{k}\ln(g-0) + C_2$

$C_2 = -\frac{g}{k}\ln g$

$\therefore -ky = v + \frac{g}{k}\ln(g-kv) - \frac{g}{k}\ln g$

$-ky = v + \frac{g}{k}\ln\left(\frac{g-kv}{g}\right)$

ie $\frac{g}{k}\ln\left(\frac{g-kv}{g}\right) + v = -ky$

(iv) From (iii)

$$v+ky = -\frac{g}{k}\ln\left(\frac{g-kv}{g}\right)$$

From (ii) $v = \frac{g}{k}\left(1-e^{-kt}\right)$

$$\therefore v+ky = -\frac{g}{k}\ln\left[\frac{g-k\cdot\frac{g}{k}\left(1-e^{-kt}\right)}{g}\right]$$

$$\frac{v+ky}{g} = -\frac{1}{k}\ln\left[\frac{g}{g}-\frac{g}{g}\left(1-e^{-kt}\right)\right]$$

$$\frac{v+ky}{g} = -\frac{1}{k}\ln\left(1-1+e^{-kt}\right)$$

$$\frac{v+ky}{g} = -\frac{1}{k}\ln\left(e^{-kt}\right)$$

$$\frac{v+ky}{g} = -\frac{1}{k}\times -kt$$

(2)

$$\frac{v+ky}{g} = t$$

ie $t = \frac{v+ky}{g}$

(v) When the particle returns to ground level
$y = h$

$$\therefore t = \frac{V+kh}{g}$$

$$= \frac{V}{g} + \frac{k}{g}\cdot\frac{1}{k}\left[u-\frac{g}{k}\ln\left(\frac{g+ku}{g}\right)\right]$$

$$= \frac{V}{g} + \frac{u}{g} - \frac{1}{k}\ln\left(\frac{g+ku}{g}\right)$$

∴ the total time T is given by

$$T = \cancel{\frac{1}{k}\log_e\left(\frac{g+kU}{g}\right)} + \frac{V}{g} + \frac{U}{g} \cancel{- \frac{1}{k}\ln\left(\frac{g+kU}{g}\right)}$$

$$T = \frac{V+U}{g}$$ 1

b) $I_n = \int_0^1 (x^2-1)^n \, dx$

(i) $I_0 = \int_0^1 1 \, dx$ 1

$= [x]_0^1$

$= 1.$

(ii) $I_n = \int_0^1 (x^2-1)^n \, dx$ $\quad u = (x^2-1)^n \quad v' = 1$

$u' = n(x^2-1)^{n-1} \cdot 2x \quad v = x$

$\therefore I_n = \left[x(x^2-1)^n\right]_0^1 - 2n\int_0^1 x^2(x^2-1)^{n-1} \, dx$

$= 0 - 2n\int_0^1 \left[(x^2-1)+1\right](x^2-1)^{n-1} \, dx$

$= 0 - 2n\int_0^1 (x^2-1)^n + (x^2-1)^{n-1} \, dx$

$= 0 - 2n(I_n + I_{n-1})$

$I_n + 2nI_n = -2nI_{n-1}$ 3

$I_n(2n+1) = -2nI_{n-1}$

$I_n = \frac{-2n}{2n+1} I_{n-1}$.

(iii) $I_4 = -\frac{8}{9} I_3$

$= -\frac{8}{9} \times -\frac{6}{7} I_2$

$= -\frac{8}{9} \times -\frac{6}{7} \times -\frac{4}{5} \times I_1$ 1

$= -\frac{8}{9} \times -\frac{6}{7} \times -\frac{4}{5} \times -\frac{2}{3} \times I_0$

$= \frac{128}{315}$

QUESTION 8.

a) (i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

But $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$
$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

RHS $= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

Equating real & imaginary parts

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$
$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

÷ throughout by $\cos^4\theta$

$$\tan 4\theta = \frac{\dfrac{4\cancel{\cos^3\theta}\sin\theta}{\cancel{\cos^4\theta}} - \dfrac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos^4\theta}\cos^3\theta}}{1\dfrac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \dfrac{6\cancel{\cos^2\theta}\sin^2\theta}{\cancel{\cos^4\theta}\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

If $t = \tan\theta$

4

$$\tan 4\theta = \frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

(ii) If $\tan 4\theta = 1$

$$1 = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

$$1 - 6t^2 + t^4 = 4t - 4t^3$$

ie $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$

Now when $\tan 4\theta = 1$

$$4\theta = n\pi + \frac{\pi}{4}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$4\theta = \frac{4n\pi + \pi}{4}$$

$$= \pi\left(\frac{4n+1}{4}\right)$$

3

$\therefore \theta = \dfrac{\pi(4n+1)}{16}$, $n = 0, \pm 1, -2$ give 4 distinct values

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{-3\pi}{16}, \frac{-7\pi}{16}$$

$$\therefore t = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{-3\pi}{16}, \tan\frac{-7\pi}{16}$$

b) (i) Horizontally

$$\ddot{x} = 0 \qquad\qquad \ddot{y} = -g$$

$$\dot{x} = V\cos\alpha \qquad\qquad \dot{y} = -gt + V\sin\alpha$$

$$x = V\cos\alpha \cdot t \qquad\qquad y = -\frac{1}{2}gt^2 + V\sin\alpha \cdot t$$

If $y = 0$,

$$-\frac{1}{2}gt^2 + V\sin\alpha \cdot t = 0$$

$$t\left(-\frac{1}{2}gt + V\sin\alpha\right) = 0$$

$$t = 0 \quad \text{or} \quad \frac{2V\sin\alpha}{g}$$

When $t = \dfrac{2V\sin\alpha}{g}$

$$x = V\cos\alpha \cdot \frac{2V\sin\alpha}{g}$$

$\therefore \quad x = \dfrac{V^2 \sin 2\alpha}{g}$

Max range occurs when $\sin 2\alpha = 1$

4

ie $2\alpha = \frac{\pi}{2}$

$\alpha = \pi/4$

$\therefore$ Max range is $\dfrac{V^2}{g}$

(ii) Let the new velocity be W

$x = W\cos\alpha . t \quad , \quad y = -\frac{1}{2}gt^2 + W\sin\alpha . t$

When $\alpha = \pi/4$, $x = \dfrac{V^2}{g}$

$\therefore x = W \cos \frac{\pi}{4} . t$

So $\dfrac{V^2}{g} = W . \dfrac{1}{\sqrt{2}} . t$

$\therefore \quad t = \dfrac{V^2\sqrt{2}}{gW}$

When $t = \dfrac{V^2\sqrt{2}}{gW}$

$y = -\frac{1}{2}gt^2 + W\sin\alpha . t$

$= \dfrac{-\cancel{g}}{\cancel{2}} . \dfrac{V^4 . \cancel{2}}{g^{\cancel{2}} W^2} + \dfrac{\cancel{W}}{\cancel{\sqrt{2}}} . \dfrac{V^2 \cancel{\sqrt{2}}}{g\cancel{W}}$

$\therefore \quad h = \dfrac{-V^4}{gW^2} + \dfrac{V^2}{g}$