NAME ______________________________

GOSFORD HIGH SCHOOL

Trial Higher School Certificate

2016

MATHEMATICS

EXTENSION 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A Reference Sheet is provided
- For Questions 11 – 16, start each question in a new booklet
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Section I – 10 marks

- Allow approximately 15 minutes
- Use the attached multiple choice answer sheet to answer questions 1 to 10

Section II – 90 marks

- Allow approximately 2h 45mins
- Start each question from 11 to 16 in a new booklet
- Use the *graph template* provided to answer Question 13(d)

Section I

10 Marks Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for questions 1-10

1 The ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ has foci $S(0,3)$ and $S(0,-3)$.
What are the equations of its' directrices?

(A) $y = \pm\frac{25}{3}$

(B) $x = \pm\frac{25}{3}$

(C) $y = \pm\frac{16}{3}$

(D) $x = \pm\frac{16}{3}$

2 If ω is one of the complex roots of $z^3 - 1 = 0$, then the value of $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$ is?

(A) -1

(B) 2

(C) 0

(D) 1

3 Given that $z = 3\left(cos\frac{\pi}{6} + isin\,\frac{\pi}{6}\right)$, what is the value of $(\bar{z})^3$?

(A) $z = 9\left(cos\frac{\pi}{2} - isin\,\frac{\pi}{2}\right)$

(B) $z = 9\left(cos\frac{\pi}{2} + isin\,\frac{\pi}{2}\right)$

(C) $z = 27\left(cos\frac{\pi}{2} - isin\,\frac{\pi}{2}\right)$

(D) $z = 27\left(cos\frac{\pi}{2} + isin\,\frac{\pi}{2}\right)$

4 Which expression gives the gradient of the normal to the curve $x^3 + xy + y^2 = 7$ at any point on the curve?

(A) $\frac{-3x^2 - y}{x + 2y}$

(B) $\frac{x + 2y}{3x^2 + y}$

(C) $\frac{3x^2 + y}{x + 2y}$

(D) $\frac{-x - 2y}{3x^2 + y}$

5 A five-digit number is formed from the numerals 5, 6, 7, 8 and 9. How many numerals can be formed, with repetitions NOT allowed, that would be less than 89 765?

(A) $5 \times 4!$

(B) $5! - 4!$

(C) $5! - 4! - 1$

(D) $4! \times 3! \times 2!$

6 $\int \frac{dx}{x^2 - 4x + 13} =$

(A) $\frac{1}{9} tan^{-1} \frac{x-2}{9} + c$

(B) $\frac{1}{9} tan^{-1} \frac{x-2}{3} + c$

(C) $\frac{1}{3} tan^{-1} \frac{x-2}{9} + c$

(D) $\frac{1}{3} tan^{-1} \frac{x-2}{3} + c$

7 The area enclosed by the curve $y = 3x^2 - x^3$, the x-axis between $x = 0$ and $x = 3$ is rotated about the y axis. Using cylindrical shells, the volume generated is given by?

(A) $2\pi \int_0^1 y^3(3 - y)dy$

(B) $\pi \int_0^1 y^4(3 - y)^2 dy$

(C) $\pi \int_0^3 x^4(3 - x)^2 dx$

(D) $2\pi \int_0^3 x^3(3 - x)\, dx$

8 The polynomial $x^4 - 5x^3 + 5x^2 + cx + d$, where c and d are real constants, has two distinct real roots and one of the other roots is $3 + 2i$. The sum of the two real roots is ?

(A) 5

(B) -1

(C) 2

(D) -11

9 The equation $x^3 + 3x^2 + 2x - 1 = 0$ has roots α, β and γ.

Which equation has roots $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$?

(A) $x^3 + 4x^2 - 12x + 8 = 0$

(B) $x^3 - 4x^2 - 12x - 8 = 0$

(C) $8x^3 - 12x^2 - 4x + 1 = 0$

(D) $8x^3 + 12x^2 + 4x - 1 = 0$

10 The function $f(x)$ is given by $f(x) = \frac{1-x}{1-\sqrt{x}}$ for $x \neq 1$, and $f(1) = k$ for some constant k.

If $f(x)$ is continuous at $x = 1$, what is the value of k?

(A) 0

(B) 1

(C) 2

(D) 4

Section II

90 Marks

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Answer the questions in the writing books provided. Use a separate writing book for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet**

(a) Consider the complex numbers $\omega = -1 + \sqrt{3}\,i$ and $z = \sqrt{3} + 2i$

(i) Evaluate $\omega\bar{z}$ (1)

(ii) Evaluate $|\omega|$ (1)

(iii) Find the value of $arg\ (\omega)$ (1)

(iv) Find the value of ω^5 (1)

(v) Evaluate $\frac{\omega}{z}$ (2)

(b) Find the values of A, B and C such that:

$$\frac{6x^2+17x+15}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}$$ (3)

(c) Sketch the region in the Argand diagram where $-\frac{\pi}{6} \leq arg\ z \leq \frac{\pi}{3}$ and $z\bar{z} \leq 4$ (3)

(d) Use logarithms, implicit differentiation and the product rule to find the derivative of $y = x^x$ (3)

End of Question 11

Question 12 **(15 marks)** **Use a separate writing booklet**

(a) Find $\int cos^3 x\, dx$ (2)

(b) Use integration by parts to find $\int xe^{-2x} dx$ (3)

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+cosx}$ using the substitution $t = tan\frac{x}{2}$ (3)

(d) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (2)

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{cosx-sinx}{1+sinxcosx} dx$ (3)

(e) $OABC$ is a rectangle on the Argand diagram, where O is the Origin.
If A represents the complex number $1+2i$, find the complex numbers represented by B and C, given that the side OC is twice the length of OA and that the Argument of C is negative. (2)

End of Question 12

Question 13 **(15 marks)** **Use a separate writing booklet**

(a) The equation $x^3 + px + q = 0$ has roots α , β and γ.
Show that $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$. (3)

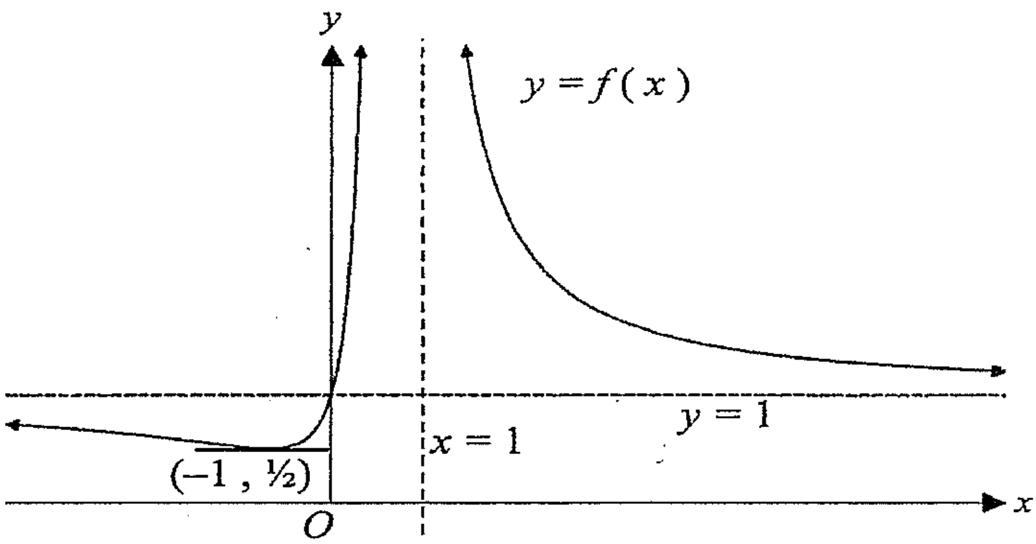
(b) Solve the polynomial equation $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, given that the equation has an integral double root. (3)

(c) (i) Find the nature and the coordinates of the stationary points on the curve $y = x + 1 - \frac{4}{(x-2)^2}$. (2)

(ii) Given that the curve passes through the point (3,0) , sketch the curve $y = x + 1 - \frac{4}{(x-2)^2}$, showing clearly the turning point(s) and asymptotes. (2)

(d)

REMEMBER TO USE THE PROVIDED TEMPLATES FOR THIS QUESTION



The diagram shows the graph of the function $y = f(x)$,
where $f(x) \to +\infty$ as $x \to 1$ from below or above, $f(x) \to 1$ as $x \to \pm\infty$ and the curve has a minimum turning point at $\left(-1, \frac{1}{2}\right)$.

On separate diagrams, sketch the following curves showing the important features

(i) $y = f(x - 1)$ (2)

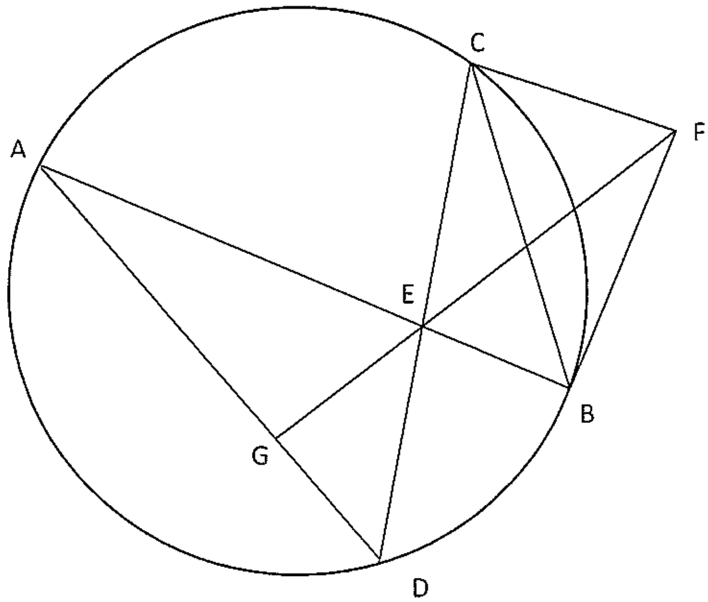
(ii) $y = \frac{1}{f(x)}$ (3)

End of Question 13

Question 14 (15 marks) **Use a separate writing booklet**

(a) In the following figure AB and CD are two chords of the circle.
AB and CD intersect at E.
F is a point such that ABF and DCF are right angles.
FE produced meets the chord AD at G.

NOT TO SCALE



Copy the diagram into your writing booklet.

(i) State why ECFB is a cyclic quadrilateral (1)

(ii) Prove AFBG is a cyclic quadrilateral (3)

(iii) Hence, or otherwise prove that FG is perpendicular to AD (1)

(b) Use the method of cylindrical shells find the volume generated when the area bounded by the curve $y = tan^{-1}x$, the x axis, and the line $x = 1$ is rotated about the y axis (5)

(c) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated about the line $y = c$, where $c > b$.
Show that the volume generated is $2\pi^2 abc$ cubic units (5)

End of Question 14

Question 15 **(15 marks)** **Use a separate writing booklet**

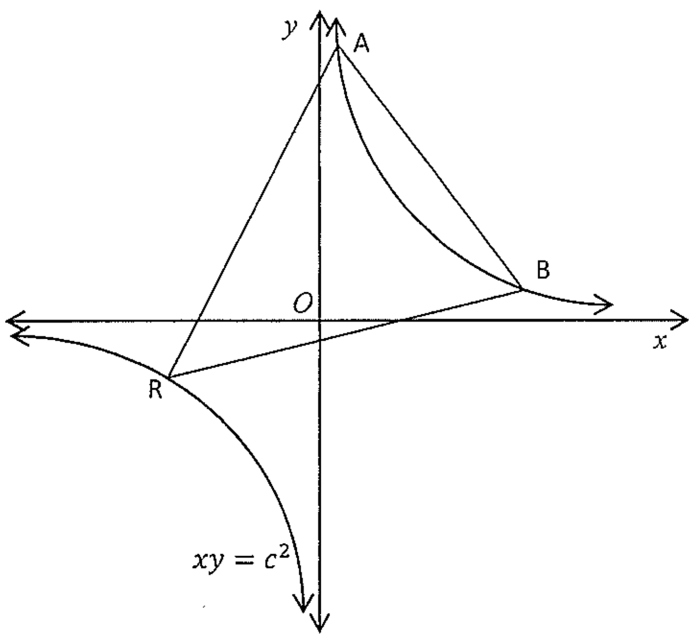
(a) A particle P of mass m kg is projected vertically upwards in a medium where the resistance to the motion has magnitude $\frac{1}{20}mv^2$ when the speed of the particle is $v\ ms^{-1}$.
The terminal velocity of the particle is $V_T\ ms^{-1}$, and the speed of projection is $kV_T\ ms^{-1}$ for some constant $k > 0$

(i) The upward acceleration of the particle is given by the equation $\ddot{x} = -\left(g + \frac{1}{20}v^2\right)$, where g is the gravitational constant. During this upward journey the height of P above the point of projection is x metres. With consideration to the downward acceleration of the particle in the same medium, show that $\ddot{x} = -\frac{1}{20}\left({V_T}^2 + v^2\right)$ (2)

(ii) If H metres is the greatest height reached by the particle,
show that $H = 10ln(1 + k^2)$ (3)

(iii) Find the value of k if the particle has 80% of its terminal velocity on return to its projection point. (5)

(b)



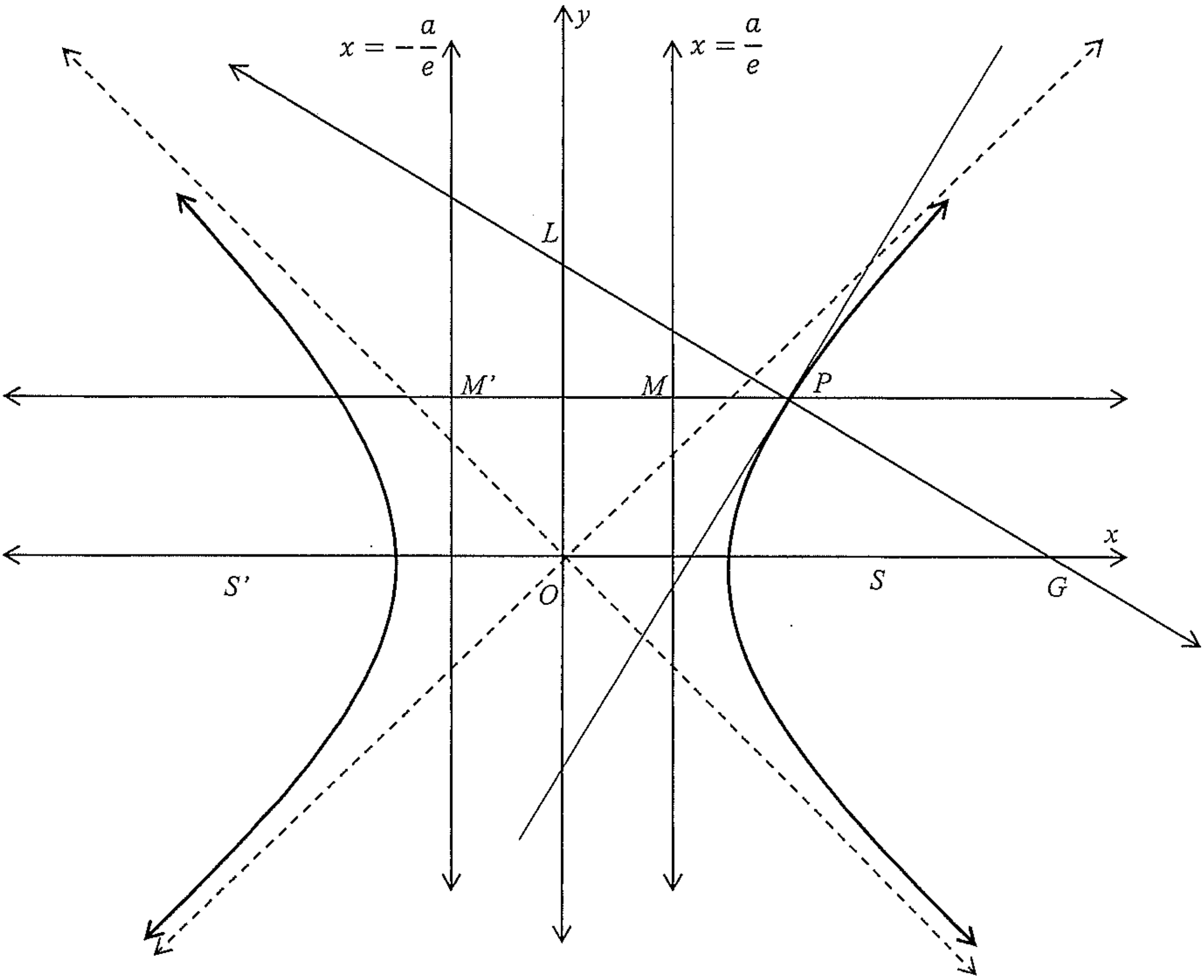
(i) Show that the gradient of the chord joining the points $A\left(ct_1, \frac{c}{t_1}\right)$ and $B\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is given by $\frac{-1}{t_1 t_2}$. (1)

(ii) Hence, given that the point $R\left(ct_3, \frac{c}{t_3}\right)$ also lies on the hyperbola in part (i), show that the orthocentre of the triangle (i.e. the point of intersection of the altitudes of the triangle) also lies on the hyperbola. (4)

End of Question 15

Question 16 (15 marks) **Use a separate writing booklet**

(a)



The variable point $P(a\sec\theta, b\tan\theta)$, where θ is acute, lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The line PMM' is drawn parallel to the x axis and meets the two directrices at M and M'. The hyperbola has eccentricity e and foci S and S'.

(i) Using the focus-directrix definition of the hyperbola, or otherwise, show that $PS = a(e\sec\theta - 1)$ and hence that $S'P - SP = 2a$. (3)

(ii) The normal at P to the hyperbola has equation $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ and meets the transverse axis at G.

Show that $SG : SP = e : 1$ (3)

(iii) This normal meets the conjugate axis in L.
Show that the midpoint (K) of GL lies on the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = \frac{(a^2+b^2)^2}{4a^2b^2}$. (4)

Question 16 continues on the next page

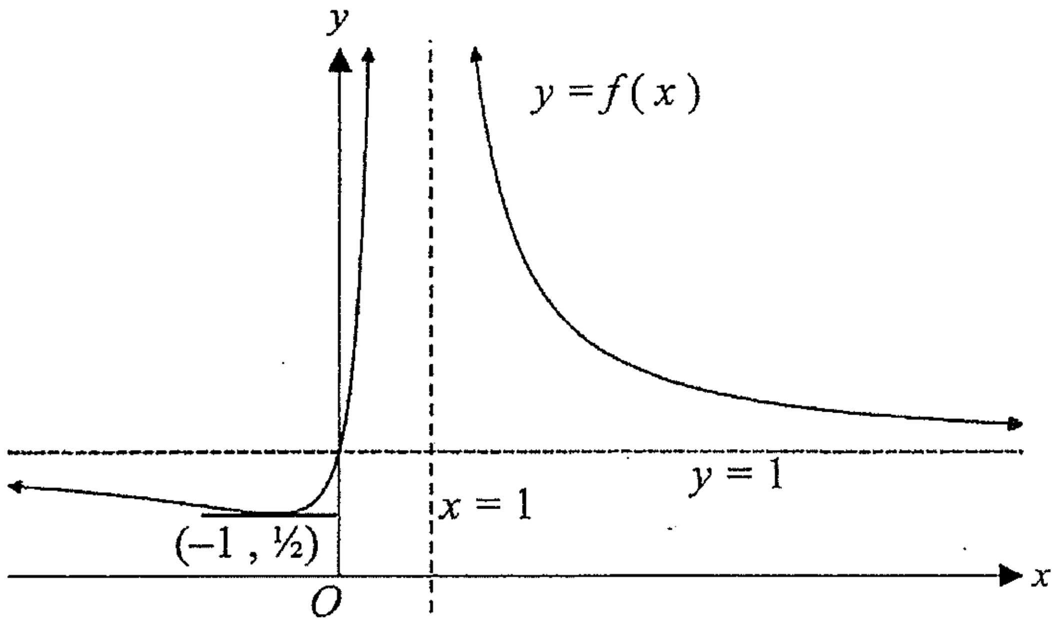
(b) If $U_n = \int_0^{\pi} \frac{\cos nx \, dx}{5-4\cos x}$, where n is a non-negative integer

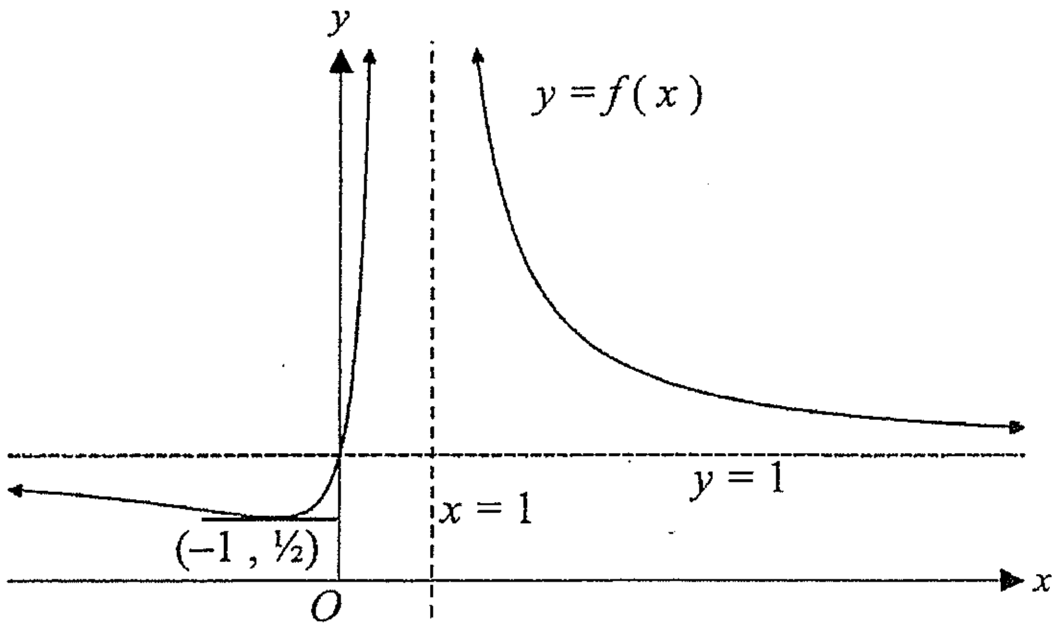
(i) show that $U_{n+1} + U_{n-1} - \frac{5U_n}{2} = 0$ (4)

(ii) given $U_0 = \frac{\pi}{3}$ and $U_1 = \frac{\pi}{6}$, evaluate U_2 (1)

End of Examination

Templates for Question 13 (d) NUMBER ____________________





GHS. Trial HSC - 2016
Mathematics Extension 2
Solutions

(1) $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

using $b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$e = \dfrac{3}{5}$

Directrices are $y = \pm \dfrac{a}{e}$

ie $y = \pm \dfrac{25}{3}$

Answer (A)

(2) $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2} = \dfrac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$

$= \dfrac{1}{1+\omega^2+\omega+\omega^3}$ using $1+\omega+\omega^2=0$

$= \dfrac{1}{1}$ since $\omega^3 = 1$ & $1+\omega+\omega^2 = 0$

$= 1$

Answer (D)

(3) $z = \operatorname{cis}\dfrac{\pi}{6}$

$\bar{z} = 3\operatorname{cis}\left(\dfrac{-\pi}{6}\right)$

$(\bar{z})^3 = 3^3\operatorname{cis}\left(\dfrac{-3\pi}{6}\right)$

$= 27\operatorname{cis}\left(\dfrac{-\pi}{2}\right)$

$= 27\left[\cos\dfrac{\pi}{2} - i\sin\dfrac{\pi}{2}\right]$

Answer (C)

(4). $x^3 + xy + y^2 = 7$

$\therefore 3x^2 + y\times 1 + x\times\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$

$\therefore \dfrac{dy}{dx} = \dfrac{-(3x^2+y)}{x+2y}$

$\therefore$ Grad. of Norm is $\dfrac{x+2y}{3x^2+y}$

Answer (B)

(5) Five digit N°s $= 5!$

Five digit N°s $> 90000 = 1\times 4!$

$= 4!$

$\therefore$ Possible numbers $= 5! - 4! - 1$

(noting that 89765 is the largest number less than 90000 and needs to be eliminated)

Answer (C)

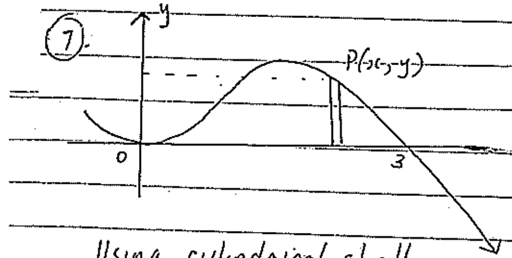
(6) $\displaystyle\int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9}$

$\displaystyle = \int \frac{dx}{(x-2)^2+9}$

$= \dfrac{1}{3}\tan^{-1}\left(\dfrac{x-2}{3}\right) + c$

Answer (D)

(7).



Using cylindrical shells

$\displaystyle V = \int_0^3 2\pi x\cdot y\, dx.$

$\displaystyle = 2\pi\int_0^3 x(3x^2 - x^3)\,dx.$

$\displaystyle = 2\pi\int_0^3 x^3(3-x)\,dx$

Answer (D)

(8) Let real roots be α and β

Since polynomial is Real the complex roots must occur in conjugate pairs

$\therefore 3-2i$ is also a root

Sum of roots $= -\dfrac{b}{a}$

$\therefore \alpha + \beta + 3 + 2i + 3 - 2i = 5$

$\therefore \alpha + \beta = -1$

Answer (B)

(9) Since α is a root then

$\alpha^3 + 3\alpha^2 + 2\alpha - 1 = 0$

We need $x = \dfrac{2}{\alpha}$ to be a root

ie $\alpha = \dfrac{2}{x}$

$\therefore \left(\dfrac{2}{x}\right)^3 + 3\left(\dfrac{2}{x}\right)^2 + 2\left(\dfrac{2}{x}\right) - 1 = 0$

$\dfrac{8}{x^3} + \dfrac{12}{x^2} + \dfrac{4}{x} - 1 = 0$

ie $8 + 12x + 4x^2 - x^3 = 0$

i.e. $x^3 + 4x^2 - 12x - 8 = 0$

Answer (B)

(10) Since continuous at $x=1$

$\displaystyle\lim_{x\to 1}[f(x)] = k$

i.e. $\displaystyle\lim_{x\to 1}\left[\frac{1-x}{1-\sqrt{x}}\right] = k$

$\displaystyle\lim_{x\to 1}\left[\frac{1-x}{1-\sqrt{x}}\times\frac{1+\sqrt{x}}{1+\sqrt{x}}\right] = k$

$\displaystyle\lim_{x\to 1}\left[\frac{(1-x)(1+\sqrt{x})}{1-x}\right] = k$

$\therefore \displaystyle\lim_{x\to 1}\left[1+\sqrt{x}\right] = k.$

ie $1+1 = k$

$k = 2$

Answer (C)

Question 11

(a)(i) $\omega \bar{z} = (-1+\sqrt{3}i)(\sqrt{3}-2i)$

$= -\sqrt{3} + 2i + 3i + 2\sqrt{3}$

$= \sqrt{3} + 5i$

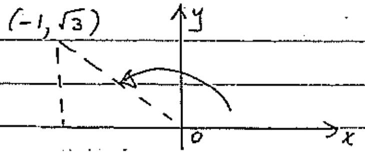
(ii) $|\omega| = \sqrt{(-1)^2 + (\sqrt{3})^2}$

$= 2$

(iii) $\arg(\omega) = \pi - \tan^{-1}(\sqrt{3})$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$



(iv) $w^5 = \left(2\,\mathrm{cis}\,\frac{2\pi}{3}\right)^5$

$= 2^5\,\mathrm{cis}\left(\frac{10\pi}{3}\right)$

$= 32\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

$= 32\left[\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right]$

$= 32\left[-\frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right]$

$= -16(1 + i\sqrt{3})$

(v) $\frac{w}{z} = \frac{-1+i\sqrt{3}}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$

$= \frac{-\sqrt{3}+2i+3i+2\sqrt{3}}{7}$

$= \frac{\sqrt{3}+5i}{7}$

b) $6x^2+17x+15 \equiv$

$A(x+2)(x-3) + Bx(x-3) + Cx(x+2)$

true for $x = 0$

$\therefore\ 15 = -6A \Rightarrow A = -\frac{5}{2}$

true for $x = 3$

$\therefore\ 120 = 15c$

$c = 8$

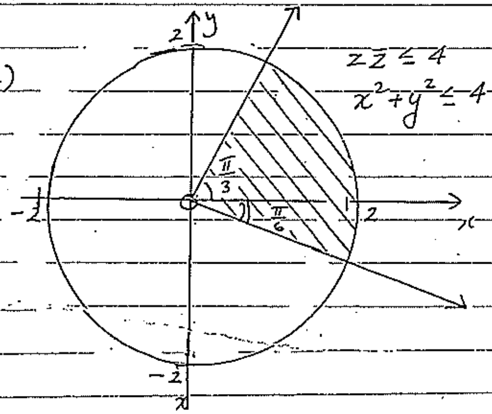
true for $x = -2$

$5 = 10B$

$B = \frac{1}{2}$

$\therefore\ A = -\frac{5}{2},\ B = \frac{1}{2},\ C = 8$

c)



d) $y = x^x$

$\log_x y = x$

$\therefore\ \frac{\log_e y}{\log_e x} = x$

$\log_e y = x\log_e x$

$\therefore\ \frac{1}{y}\frac{dy}{dx} = \log_e x \times 1 + x \times \frac{1}{x}$

$\frac{dy}{dx} = y\left[1 + \log_e x\right]$

$= x^x(1 + \log_e x)$

Question 12

a) $\int \cos^3 x\,dx = \int \cos x\cdot(1-\sin^2 x)\,dx$

$= \int \cos x - \int \cos x\cdot\sin^2 x\,dx$

$= \sin x - \frac{\sin^3 x}{3} + C$

b) $\int x e^{-2x}\,dx = \int x\cdot\frac{d}{dx}\left[-\frac{1}{2}e^{-2x}\right]dx$

$= \left[-\frac{1}{2}e^{-2x}\times x\right] - \int 1\times\left(-\frac{1}{2}e^{-2x}\right)dx$

$= -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$

$= -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$

c) $t = \tan\left(\frac{x}{2}\right) \longrightarrow x = 2\tan^{-1}t \rightarrow \frac{dx}{dt} = \frac{2}{1+t^2}$

when $x = \frac{\pi}{2} \rightarrow t = 1$ when $x = 0,\ t = 0$

$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{(1-t^2)}{(1+t^2)}} \times \frac{2\,dt}{(1+t^2)}$

$= \int_0^1 \frac{2}{2(1+t^2)+(1-t^2)}\,dt$

$= \int_0^1 \frac{2}{3+t^2}\,dt$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}}\left[\frac{\pi}{6} - 0\right]$$

$$= \frac{\pi}{3\sqrt{3}}$$

d) (i) Consider $\int_0^a f(a-x)\,dx$

Let $u = a - x \rightarrow \frac{du}{dx} = -1 \rightarrow dx = -du$

when $x = a \rightarrow u = 0$

when $x = 0 \rightarrow u = a$

$$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(u)\cdot(-1)\,du$$

$$= \int_0^a f(u)\,du$$

$$= \int_0^a f(x)\,dx$$

(ii)
$$\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right) - \sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cdot\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\cos x\sin x}\,dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx$$

$$\therefore 2\int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx = 0$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{1+\sin x\cos x}\,dx = 0$$

e) $\overrightarrow{OC} = 2 \times \overrightarrow{OA} \times (-i)$

$= 2(1+2i)\times(-i)$

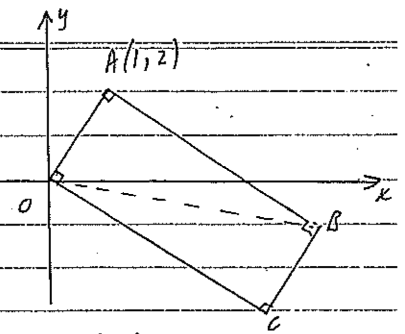
$= 4 - 2i$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= \overrightarrow{OC} + \overrightarrow{OA}$

$= 4-2i + 1+2i$

$= 5 + 0i$



Mod arg ok

Question 13

Since $x^3 + px + q = 0$

a) $\alpha^3 + p\alpha + q = 0$

$\beta^3 + p\cdot\beta + q = 0 \Rightarrow$

$\gamma^3 + p\gamma + q = 0$

$\therefore \alpha^3+\beta^3+\gamma^3 = -p(\alpha+\beta+\gamma) - 3q$

$= -p\times 0 - 3q$ (Sum of Roots $= 0$)

$= -3q$

Now $\alpha\beta\gamma = -q$ (Product of roots)

$\therefore \alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma$

b) Let $P(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$

$P'(x) = 4x^3 - 18x^2 + 18x + 4$

$P'(2) = 32 - 72 + 36 + 4 = 0$

$P(2) = 16 - 48 + 36 + 8 - 12 = 0 = P'(2)$

$\therefore$ Double root at $x = 2$

$\therefore (x-2)^2 = (x^2 - 4x + 4)$ is a factor

$\therefore P(x) = (x^2-4x+4)(x^2+mx-3)$

to find m consider term in x on both sides

$\therefore 4x = 12x + 4ax$

$x = (3+a)x$

$1 = 3+a \longrightarrow a = -2$

$\therefore P(x) = (x-2)^2(x^2-2x-3)$ 1

$= (x-2)^2(x-3)(x+1)$

$\therefore x = 2, 3, -1$ 1

c) (i) $y = x+1-4(x-2)^{-2}$

$\frac{dy}{dx} = 1+8(x-2)^{-3}$ $\qquad \frac{d^2y}{dx^2} = -24(x-2)^{-4}$

$= \frac{-24}{(x-2)^4} < 0$

For stat. pts. $\frac{dy}{dx} = 0$

$\therefore \frac{1}{(x-2)^3} = -\frac{1}{8}$ $\qquad$ Alway concave down except at $x=2$

$x-2 = -2$ 1

$x = 0$ 1

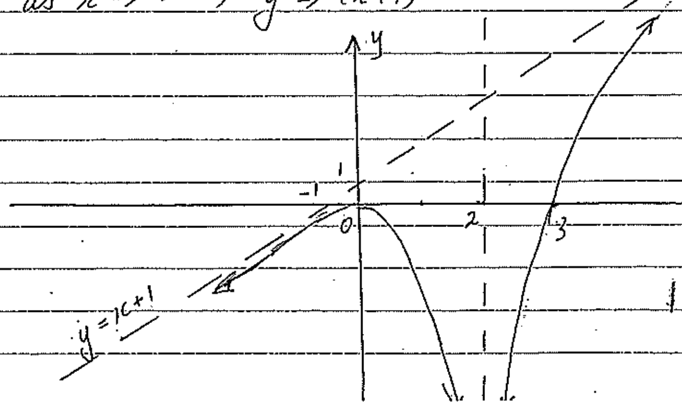
$\therefore (0,0)$ is a maximum turning point.

(ii) as $x \to -\infty$, $y \to (x+1)^-$ $\qquad$ as $x \to 2^-$, $y \to -\infty$

as $x \to 2^+$, $y \to -\infty$.

as $x \to \infty$, $y \to (x+1)^+$

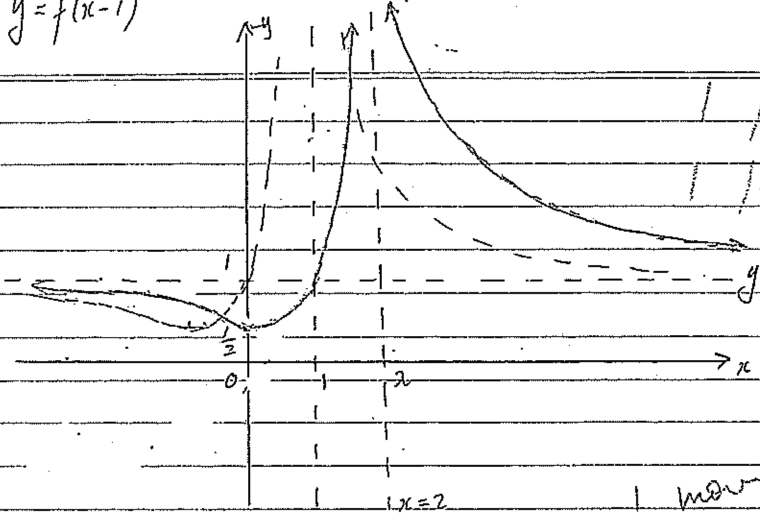
2 asympt



1 for good attempt

1 for correct answer.

d) i) $y = f(x-1)$



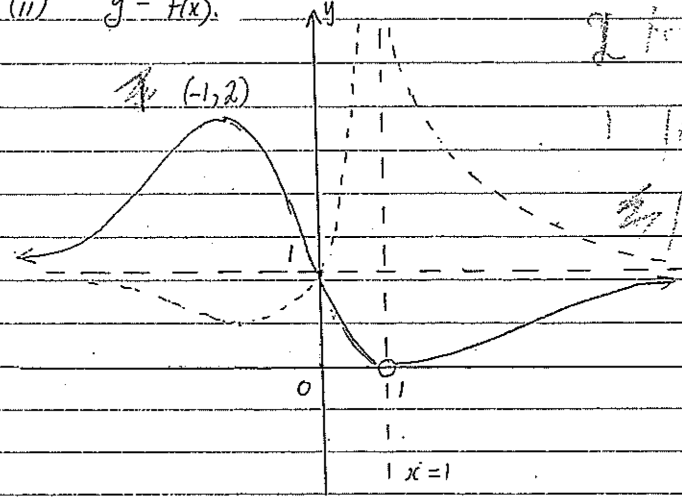
1 for shape

1 for min

1 moving to right without...

1 marking

(ii) $y = \frac{1}{f(x)}$



2 for shape L + R

1 for max

for discontinuity at x=1

Question 14

a) (i) $E\hat{C}F + E\hat{B}F = 180°$

$\therefore$ ECFB is a cyclic quadrilateral

(a pair of opposite angles are supplementary 1

(ii) $\hat{BFG} = \hat{BCE}$ (angles at the circumference of circle ECFB standing on the same chord ED are equal)

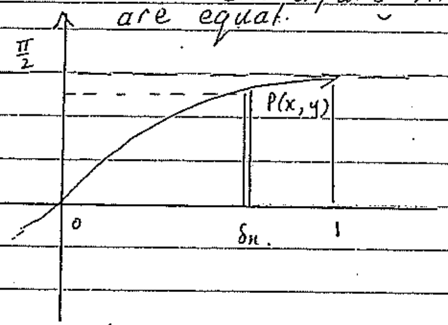
$\hat{DAB} = \hat{DCB}$ (angles at the circumference of circle ACBD standing on the same arc DB are equal)

$\therefore \hat{BFG} = \hat{DAB}$ ($\hat{BCE}$ is the same angle as $\hat{DCB}$)

$\therefore$ A, F, B and G are con-cyclic as the interval GB subtends a pair of equal angles on the same side of it. ie at A and F.

(iii) $\hat{AGF} = \hat{ABT} = 90°$ (angles at the circumference of circle AFBG standing on the same chord/arc AF are equal.

b)



$\delta V = 2\pi x y \, \delta x$

$$V = 2\pi \int_0^1 x \tan^{-1} x \, dx.$$

$$V = 2\pi \int_0^1 \tan^{-1} x \cdot \frac{d}{dx}\left[\frac{x^2}{2}\right] dx.$$

$$= 2\pi \left[\left[\frac{x^2}{2} \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx \right]$$

$$= 2\pi \left[\frac{1}{2} \times \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2} \, dx. \right]$$

$$= 2\pi \left[\frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2} \, dx \right]$$

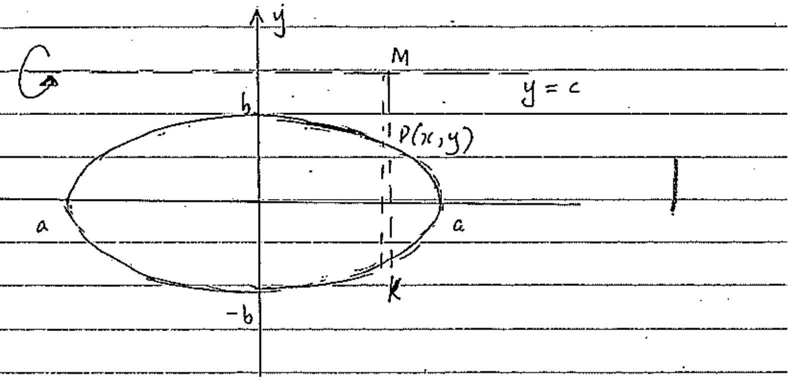
$$= 2\pi \left[\frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^1 \right]$$

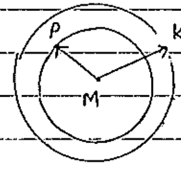
$$V = 2\pi \left[\frac{\pi}{8} - \frac{1}{2}\left[1 - \frac{\pi}{4}\right] \right]$$

$$= 2\pi \left[\frac{\pi}{4} - \frac{1}{2} \right]$$

$$= \pi \left(\frac{\pi}{2} - 1\right) \text{ cubic units}$$

(c)





$$\delta V = \pi \left[(c+y)^2 - (c-y)^2 \right] \delta x$$

$$= \pi \left[c^2 + 2cy + y^2 - c^2 + 2cy - y^2 \right] \delta x.$$

$$= \pi (4cy) \, \delta x.$$

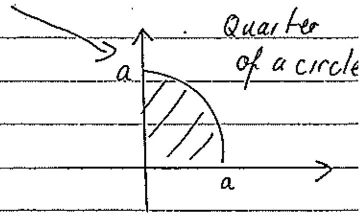
$$V = 4\pi c \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2} \, dx. \quad \text{since } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$= \frac{4\pi bc}{a} \times 2 \int_0^a \sqrt{a^2 - x^2} \, dx$$

$$= \frac{8\pi bc}{a} \times \frac{1}{4}\pi a^2$$

$$= 2\pi^2 abc$$



Question 15

a) (i) $\ddot{x} = -\left(\frac{1}{20}v^2 + g\right)$ for Upwards Motion

For Downwards Motion $F_D = mg - \frac{1}{20}mv^2$

$$m\ddot{x}_D = m\left(g - \frac{1}{20}v^2\right)$$

$$\ddot{x}_D = g - \frac{1}{20}v^2$$

when $\ddot{x}_D = 0$, $v = V_T$

$$\therefore 0 = g - \frac{1}{20}V_T^2$$

$$\text{ie } g = \frac{1}{20}V_T^2$$

Now $\ddot{x} = -\left(\frac{1}{20}v^2 + \frac{1}{20}V_T^2\right)$

$$\ddot{x} = -\frac{1}{20}\left(V_T^2 + v^2\right)$$

(ii) $v\frac{dv}{dx} = -\frac{1}{20}\left(V_T^2 + v^2\right)$

$$-20\frac{dv}{dx} = \frac{V_T^2 + v^2}{v}$$

$$-\frac{1}{20}\frac{dx}{dv} = \frac{v}{V_T^2 + v^2}$$

$$-\frac{x}{20} = \frac{1}{2}\ln\left(V_T^2 + v^2\right) + c$$

$$-\frac{x}{10} = \ln\left(V_T^2 + v^2\right) + c$$

when $x = 0$, $v = kV_T$

$$\therefore 0 = \ln\left[V_T^2 + k^2V_T^2\right] + c$$

$$\therefore -\frac{x}{10} = \ln\left[\frac{V_T^2 + v^2}{(k^2+1)V_T^2}\right]$$

when $x = H$, $v = 0$

$$\therefore H = 10\ln\left[\frac{V_T^2(k^2+1)}{V_T^2}\right]$$

$$H = 10\ln\left(1 + k^2\right) \quad \ldots\ldots (1)$$

(iii) For return $\ddot{x} = \frac{1}{20}V_T^2 - \frac{1}{20}v^2$

$$= -\frac{1}{20}\left(v^2 - V_T^2\right)$$

$$\therefore v\frac{dv}{dx} = -\frac{1}{20}\left(v^2 - V_T^2\right)$$

$$\frac{dv}{dx} = -\frac{1}{20}\left(\frac{v^2 - V_T^2}{v}\right)$$

$$-\frac{1}{20}\frac{dx}{dv} = \left(\frac{v}{v^2 - V_T^2}\right)$$

$$-\frac{x}{10} = \ln\left(v^2 - V_T^2\right) + c$$

when $x = 0$, $v = 0$

$$0 = \ln\left(-V_T^2\right) + c$$

$$\therefore -\frac{x}{10} = \ln\left[\frac{v_2^2 - V_T^2}{-V_T^2}\right]$$

$\therefore x = 10 \ln \left[\dfrac{V_T}{V_T^2 - v^2} \right]$

when $x = H$, $v = \dfrac{4V_T}{5}$

$\therefore H = 10 \ln \left[\dfrac{V_T^2}{V_T^2 - \frac{16V_T^2}{25}} \right]$

$H = 10 \ln \left[\dfrac{25}{25 - 16} \right]$

$H = 10 \ln \left(\dfrac{25}{9} \right)$

$\therefore \ln (1 + k^2) = \ln \left(\dfrac{25}{9} \right)$ using (1)

$1 + k^2 = \dfrac{25}{9}$

$k^2 = \dfrac{16}{9}$

$k = \dfrac{4}{3}$ since $k > 0$

b) (i) $M_{AB} = \dfrac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}$

$= \dfrac{\frac{t_1 - t_2}{t_1 t_2}}{t_2 - t_1}$

$= -\dfrac{1}{t_1 t_2}$

(ii) Equation of altitude AB to R

$y - \dfrac{c}{t_3} = t_1 t_2 (x - ct_3)$

$yt_3 - c = t_1 t_2 t_3 (x - ct_3)$ .... (1)

Similarly altitude RB to A is

$yt_1 - c = t_1 t_2 t_3 (x - ct_1)$ .... (2)

(1) − (2) $y(t_3 - t_1) = -ct_1 t_2 t_3^2 + ct_1^2 t_2 t_3$

$y = \dfrac{-ct_1 t_2 t_3 (t_3 - t_1)}{t_3 - t_1}$

$y = -ct_1 t_2 t_3$

$\therefore -ct_1 t_2 t_3^2 - c = t_1 t_2 t_3 (x - ct_3)$ using (1)

$\therefore -c = t_1 t_2 t_3 x$

$x = \dfrac{-c}{t_1 t_2 t_3}$

$\therefore$ Altitudes intersect at $\left[\dfrac{-c}{t_1 t_2 t_3}, -ct_1 t_2 t_3 \right]$

Hyperbola has equation $xy = c^2$

$\left(\dfrac{-c}{t_1 t_2 t_3} \right) \left(-ct_1 t_2 t_3 \right) = c^2$

$\therefore$ Pt of intersection lies on the hyperbola.

Question 16 a) (i) $\frac{PS}{PM} = e$

$\therefore PS = e\,PM$
$= e\left(a\sec\theta - \frac{a}{e}\right)$
$= a(e\sec\theta - 1)$

Similarly $\frac{PS'}{PM'} = e \longrightarrow PS' = e\,PM'$
$= e\left(a\sec\theta + \frac{a}{e}\right)$
$= a(e\sec\theta + 1)$

$\therefore S'P - SP = a(e\sec\theta + 1) - a(e\sec\theta - 1)$
$= 2a$

(ii) at G, $y = 0$ $\therefore \frac{ax}{\sec\theta} = a^2 + b^2$
$x = \frac{(a^2+b^2)\sec\theta}{a}$

But since $b^2 = a^2(e^2-1) \longrightarrow a^2 + b^2 = a^2e^2$

$\therefore x = \frac{a^2e^2\sec\theta}{a} = ae^2\sec\theta$

$\therefore G\left[ae^2\sec\theta, 0\right]$

$SG : SP = \frac{ae^2\sec\theta - ae}{a(\sec\theta - 1)} = \frac{ae(e\sec\theta - 1)}{a(\sec\theta - 1)}$

$= e$ (or $e:1$)

(iii) for L, $x = 0$, $\frac{by}{\tan\theta} = a^2 + b^2$
$y = \frac{a^2e^2\tan\theta}{b}$

$\therefore L\left[0, \frac{a^2e^2\tan\theta}{b}\right]$

$\therefore K\left[\frac{ae^2\sec\theta}{2}, \frac{a^2e^2\tan\theta}{2b}\right]$

Now $\frac{x^2}{b^2} - \frac{y^2}{a^2} = \frac{1}{b^2}\cdot\frac{a^2e^4\sec^2\theta}{4} - \frac{1}{a^2}\cdot\frac{a^4e^4\tan^2\theta}{4b^2}$

$= \frac{1}{4b^2}\cdot a^2e^4\left[\sec^2\theta - \tan^2\theta\right]$

$= \frac{a^2e^4}{4b^2}\cdot 1$

$= \frac{a^4e^4}{4a^2b^2}$

$= \frac{(a^2+b^2)^2}{4a^2b^2}$

$\therefore$ K satisfies the given hyperbola

b) $U_n = \int_0^{\pi} \frac{\cos nx}{5-4\cos x}\, dx$

$$U_{n+1} + U_{n-1} - \frac{5}{2}U_n = \int_0^{\pi} \frac{\cos(n+1)x}{5-4\cos x} + \int_0^{\pi} \frac{\cos(n-1)x}{5-4\cos x} - \frac{5}{2}\int \frac{\cos nx}{5-4\cos x}$$

$$R.H.S = 2\int_0^{\pi} \frac{\cos nx \cos x - \sin nx \sin x + \cos nx \cos x + \sin nx \sin x}{2(5-4\cos x)} - 5\int_0^{\pi} \frac{\cos nx}{2(5-4\cos x)}$$

$$= \int_0^{\pi} \frac{4\cos nx \cos x - 5\cos nx}{2(5-4\cos x)}\, dx.$$

$$= \int_0^{\pi} \frac{-\cos nx\,(5-4\cos x)}{2(5-4\cos x)}\, dx = \int_0^{\pi} \frac{-\cos nx}{2}\, dx$$

$$= -\frac{1}{2n}\Big[\sin nx\Big]_0^{\pi}$$

$$= -\frac{1}{2n}(0)$$

$$= 0$$

(ii) $U_0 = \frac{\pi}{3}$, $U_1 = \frac{\pi}{6}$

$$U_2 + U_0 - \frac{5U_1}{2} = 0$$

$$\therefore U_2 = \frac{5}{2} \times \frac{\pi}{6} - \frac{\pi}{3}$$

$$U_2 = \frac{\pi}{12}$$