GOSFORD HIGH SCHOOL

2018 HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

- **General Instructions**
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 14

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. An ellipse has Cartesian equation $\frac{x^2}{4}+\frac{y^2}{2}=1$.

   What is the parametric equation of the ellipse?

   (A) $x=2\cos\theta,\ y=\sqrt{2}\sin\theta$

   (B) $x=4\cos\theta,\ y=2\sin\theta$

   (C) $x=\sqrt{2}\sin\theta,\ y=2\cos\theta$

   (D) $x=2\sin\theta,\ y=4\cos\theta$

2. What is the square root of $12-16i$?

   (A) $\pm(2-4i)$

   (B) $\pm(2\sqrt{3}-4i)$

   (C) $\pm(4-2i)$

   (D) $\pm(4-2\sqrt{3}i)$

3. The region bounded by the curve $y = x^2$, the x-axis, $x = 0$ and $x = 2$ is rotated around the line $x = 2$.

Which of the following gives the volume of the solid formed?

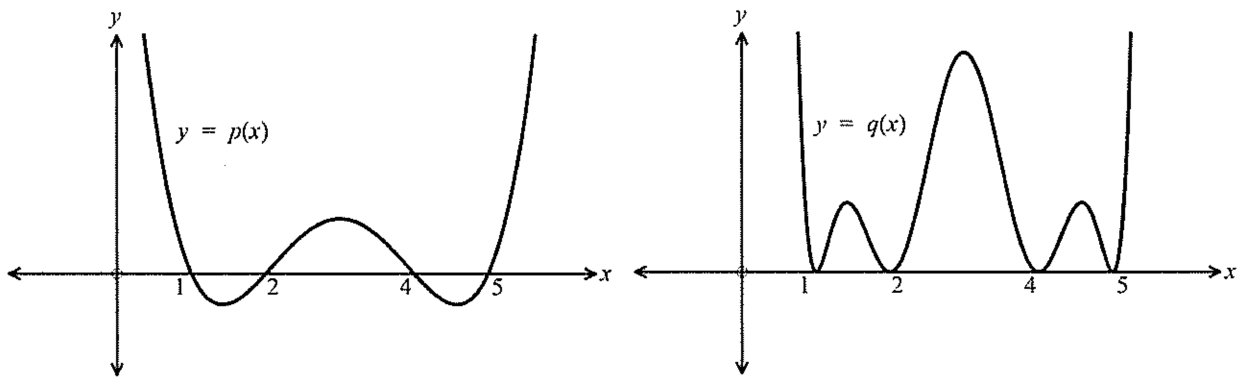
(A) $V = \pi \int_0^2 (2-x)x^2 \, dx$

(B) $V = \pi \int_0^4 (2-x)x^2 \, dx$

(C) $V = 2\pi \int_0^2 (2-x)x^2 \, dx$

(D) $V = 2\pi \int_0^2 x^2 (2-x)^2 \, dx$

4. The graphs of two functions, $y = p(x)$ and $y = q(x)$ are drawn below.



Which of the following describes the relationship between the two functions?

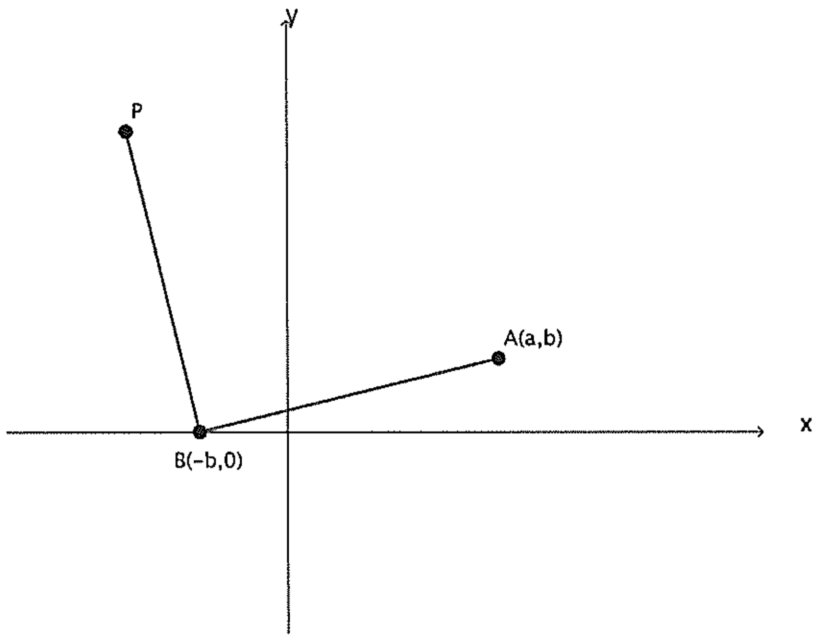
(A) $q(x) = \frac{1}{p(x)}$

(B) $q(x) = [p(x)]^2$

(C) $p(x) = \frac{1}{q(x)}$

(D) $p(x) = [q(x)]^2$

5.



The Argand diagram above shows the point $A(a,b)$ representing the complex number $z = a + ib$, where a and b are real. B is the point $(-b,0)$.

P is a point such that $PB = 2 \times AB$ and $\angle ABP = 90^\circ$.

Which of the following complex numbers does P represent?

(A) $-2b + i(2a)$

(B) $-b + ai$

(C) $-2b + i(2a + 2b)$

(D) $-3b + i(2a + 2b)$

6. Given $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$, which of the following is the value of $\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4}$?

(A) e^5

(B) $e^5 + 1$

(C) e^6

(D) $e^6 + 1$

7. What is the value of the constant B such that $P(x)=(x-\alpha)^2 Q(x)+Ax+B$?

(A) $B=P(\alpha)$

(B) $B=P'(\alpha)$

(C) $B=P(\alpha)-\alpha P'(\alpha)$

(D) $B=P'(\alpha)-\alpha P(\alpha)$

8. Solve the inequality $\frac{x+1}{x-3}\leq\frac{x+3}{x-2}$

(A) $x<2$ and $x>3$

(B) $x<2$ and $3<x\leq 7$

(C) $2<x<3$

(D) $2<x<3$ and $x\geq 7$

9. What is the value of the constant k such that the function, $f(x)$, is continuous at x=0,

where $f(x)$ is defined by:

$$f(x)=\frac{\sqrt{x+1}-1}{x} \text{ for } x\neq 0$$

and $f(0)=k$, at x=0

(A) $k=-1$

(B) $k=0$

(C) $k=\frac{1}{2}$

(D) $k=1$

10. $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$, where $p \neq 0$ and $q \neq 0$, are two points on the rectangular hyperbola $xy = c^2$.

What is the condition for the tangent to the hyperbola at P to be perpendicular to the line OQ?

(A) $|pq| = 1$

(B) $p^2q = 1$

(C) $pq^2 = 1$

(D) $p^2 - q^2 = 1$

Section II

90 marks

Attempt Questions 11 – 16.

Allow about 2 hours and 45 minutes for this section.

Answer each question in a **SEPARATE** writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 writing booklet.

(a) If $z = 2 + i$ and $w = 4 - i$ find in the form $a + ib$, where a and b are real, the values of:

(i) $\overline{2z - w}$ **1**

(ii) $\dfrac{w}{z}$ **2**

(b) Find $\displaystyle\int \frac{e^{3x} + 8}{e^{x} + 2}\,dx$ **2**

(c) The equation $x^3 - 5x^2 + 3x - 2 = 0$ has roots α, β and γ. **3**

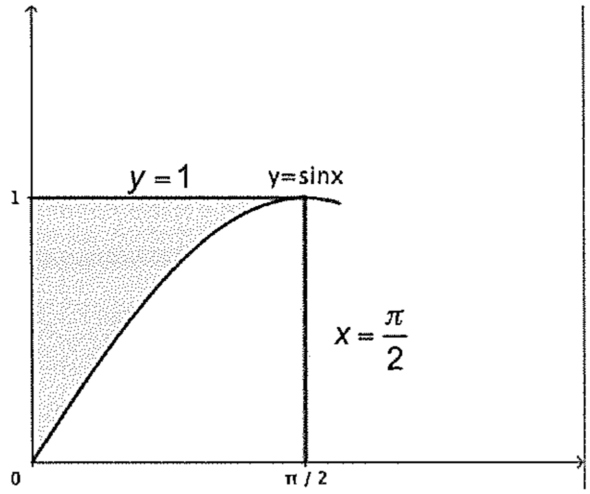
Find a cubic equation with integer coefficients that has roots α^2, β^2 and γ^2.

(d) Use the substitution $u = \cos 2x$ to find $\int \cos^2 2x \sin^3 2x \, dx$. **3**

Question 11 continues on page 8

Question 11 continued

(e)



In the diagram, the area above the curve $y = \sin x$, between the lines $x = 0$ and $x = \frac{\pi}{2}$, is rotated about the line $y = 1$.

(i) Use discs formed by slicing perpendicular to the line $y = 1$ to show that the solid formed is given by $V = \pi \int_0^{\frac{\pi}{2}} \left(1 - 2\sin x + \sin^2 x\right) dx$. **1**

(ii) Find the value of V in simplest exact form. **3**

End of Question 11

Question 12 (15 marks) Use the Question 12 writing booklet.

(a) z is a complex number such that $\left|z-2\sqrt{2}(1+i)\right|=2$.

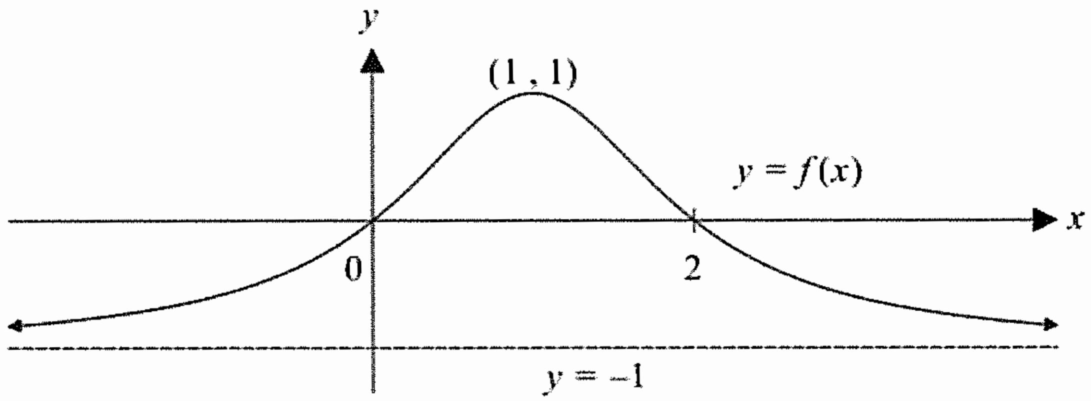
(i) Sketch the locus of the point P representing the complex number z in an Argand diagram. **2**

(ii) Q is the point on the locus where z has its smallest principal argument. Find the value of the complex number represented by Q in mod-arg form. **2**

(b) The hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ (where a and b are positive) has a focus at the point $\left(3\sqrt{2},0\right)$ and the line $y=\dfrac{2x}{\sqrt{5}}$ is an asymptote. **3**

Find the values of a and b.

(c)



The diagram shows the graph of the function $f(x)=\dfrac{x(2-x)}{x^2-2x+2}$.

On separate diagrams sketch the graphs of the following curves, clearly showing the intercepts on the axes and the equations of any asymptotes.

(i) $y=-f(|x|)$ **2**

Question 12 continues on page 10

Question 12 continued

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \log_e f(x)$ **2**

(iv) $y = f'(x)$ **2**

End of Question 12

Question 13 (15 marks) Use the Question 13 writing booklet.

(a) The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 - 2xy}{x^2 - y^2}$ **2**

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. **3**

(b) Let α, β, γ be the non-zero roots of the equation $x^3 + rx + s = 0$.

(i) Find, in terms of s, the simplified value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

(ii) If $x^3 + rx + s = 0$ has a double root, show that $x = -\dfrac{3s}{2r}$ **2**

(c) A solid has an elliptical base with equation $4x^2 + 25y^2 = 100$. Each cross section perpendicular to the x-axis is a right angled isosceles triangle with the hypotenuse in the base of the solid.

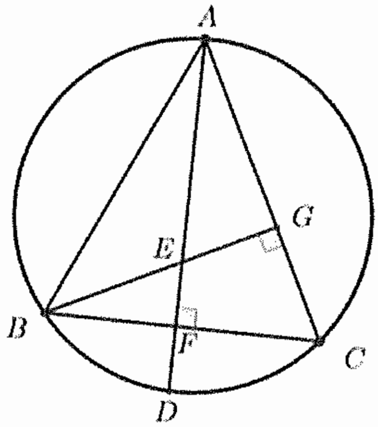
(i) Draw a diagram representing the elliptical base, showing all intercepts with the axes. **2**

(ii) Find the volume of the solid. **4**

End of Question 13

Question 14 (15 marks) Use the Question 14 writing booklet.

(a)



The diagram above shows triangle ABC inscribed in a circle.

G is the point on AC such that BG is perpendicular to AC and F is the point on BC such that AF is perpendicular to BC.

AF and BG meet at E.

AF produced meets the circle at D.

(i) Explain why $ABFG$ is a cyclic quadrilateral. **1**

(ii) Prove that $DF = EF$. **3**

(b) (i) Show that $\dfrac{1}{(2t+1)(t+2)} = \dfrac{2}{3(2t+1)} - \dfrac{1}{3(t+2)}$ **2**

(ii) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate in simplest exact form $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x}$. **3**

Question 14 continues on page 13

Question 14 continued

(c) $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ are points on the rectangular hyperbola $xy = 9$.

(i) Show that the equation of the chord PQ is $x + pqy = 3(p+q)$ **2**

(ii) Find the co-ordinates of N, the midpoint of PQ. **1**

(iii) If the chord PQ is a tangent to the parabola $y^2 = 3x$, prove that the locus of N is $3x = -8y^2$. **3**

End of Question 14

Question 15 (15 marks) Use the Question 15 writing booklet.

(a) Initially a speedboat is travelling at a speed of 15 ms^{-1} in a straight line across a lake. At time t seconds later, the speedboat has a velocity v ms^{-1} and the engine is producing a constant force of 600 Newtons.

The speedboat experiences a resistance force of magnitude 90v Newtons.

The mass of the speedboat plus passengers is 450 kg.

Assume the water in the lake is still.

(i) Show that $\frac{dv}{dt} = -\frac{3v-20}{15}$ **1**

(ii) Find an expression for v in terms of t. **3**

(iii) Find the time taken for the speed of the speedboat to reduce to 10 ms^{-1}. **2**

(b) Given that $I_n = \int_0^3 x^n\sqrt{9-x^2}\,dx$, $n = 0, 1, 2, \ldots$

(i) Show that $(n+2)I_n = 9(n-1)I_{n-2}$, $n = 2, 3, 4, \ldots$ **3**

(ii) Find the value of I_4 **2**

(c) Given that $w+x+y+z=\pi$:

(i) Show that $\sin z = \sin(w+x)\cos y + \cos(w+x)\sin y$. **1**

(ii) Hence show that $\sin w \sin y + \sin x \sin z = \sin(w+x)\sin(x+y)$. **3**

End of Question 15

Question 16 (15 marks) Use the Question 16 writing booklet.

(a)

(i) Use de Moivre's theorem to show that $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$. **3**

(ii) Explain why $\sin^2\dfrac{\pi}{7}$ is a root of the equation $64x^3 - 112x^2 + 56x - 7 = 0$ and write down the two other roots in trigonometric form. **2**

(iii) Hence show that the value of $\operatorname{cosec}^2\dfrac{\pi}{7} + \operatorname{cosec}^2\dfrac{2\pi}{7} + \operatorname{cosec}^2\dfrac{3\pi}{7} =$ **1**

(b)

(i) Using the binomial theorem, write down the expansion of $(1+i)^{2m}$, where $i = \sqrt{-1}$ and m is a positive integer. **1**

(ii) Hence show that **2**

$$1 - \binom{2m}{2} + \binom{2m}{4} - \binom{2m}{6} + \ldots + (-1)^m \binom{2m}{2m} = 2^m \cos\left(\frac{1}{2}\pi m\right),$$ where m is a positive integer.

(c) A particle of mass m kg is projected vertically upwards with speed U ms^{-1}.

At time t seconds the particle has vertical height x metres above the point of projection, speed v ms^{-1} and acceleration a ms^{-2}.

The particle moves vertically under gravity in a medium where the resistance to motion has magnitude $\dfrac{mv^2}{g}$ Newtons, where g ms^{-2} is the acceleration due to gravity.

(i) Show that $a = -\dfrac{1}{g}\left(g^2 + v^2\right)$. **1**

(ii) Show that $v = g\left(\dfrac{U - g\tan t}{g + U\tan t}\right)$ and find the time taken for the particle to reach its greatest height. **3**

(iii) Express x in terms of t. **2**

End of Paper

EXT 2 TRIAL 2018

m-c

1) $\frac{x^2}{4}+\frac{y^2}{2}=1$

$x=2\cos\theta$ (A)

$y=\sqrt{2}\sin\theta$

(ie $\frac{4\cos^2\theta}{4}+\frac{2\sin^2\theta}{2}=1$)

2) $12-16i$

$(x+iy)^2=12-16i$

$x^2-y^2=12$

$2xy=-16 \rightarrow y=\frac{-8}{x}$

$x^2-\frac{64}{x^2}=12$

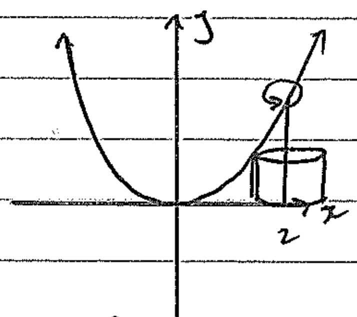
$x^4-12x^2-64=0$

$(x^2-16)(x^2+4)=0$

$x=\pm 4$

$x=4 \quad y=-2 \quad 4-2i$ (C)

$x=-4 \quad y=2 \quad -4+2i$

3) 

$V=2\pi\int_0^2 (2-x)x^2\,dx$ (C)

4) (B)

5) $\overrightarrow{BA}=\overrightarrow{BO}+\overrightarrow{OA}$

$=b+a+bi$

$\overrightarrow{BP}=(\overrightarrow{BA})2i$

$=2i(a+b)-2b$

$\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{BP}$

$=-b-2b+2i(a+b)$

$=-3b+2i(a+b)$ (D)

6) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Find $\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4}$

let $\frac{x+4}{x-1}=1+\frac{1}{y}$

$\therefore \frac{1}{y}=\frac{x+4-x+1}{x-1}$

$\frac{1}{y}=\frac{5}{x-1}$

$\therefore x=5y+1$

$\lim_{x\to\infty}\left(\frac{x+4}{x-1}\right)^{x+4}$

$=\lim_{y\to\infty}\left(\frac{5y+5}{5y}\right)^{5y+5}$

$=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{5y+5}$

$=\lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^5\left(1+\frac{1}{y}\right)^5$

$=e^5(1)$

$=e^5$ (A)

Q11

a) $z = 2+i \quad w = 4-i$

i) $2z - w = 4 + 2i - 4 + i$

$= 3i$

$\overline{2z - w} = -3i$ ①

ii) $\frac{w}{z} = \frac{4-i}{2+i} \times \frac{2-i}{2-i}$ 1

$= \frac{8 - 6i - 1}{4+1}$

$= \frac{7}{5} - \frac{6}{5} i$ 1 ②

b) $\int \frac{e^{3x} + 8}{e^{x} + 2} \, dx$

$= \int \frac{(e^{x})^3 + 2^3}{e^{x} + 2}$

$= \int \frac{(e^{x} + 2)(e^{2x} - 2e^{x} + 4)dx}{e^{x} + 2}$ 1

$= \int e^{2x} - 2e^{x} + 4 \, dx$

$= \frac{1}{2} e^{2x} - 2e^{x} + 4x + c$ 1

②

c) $x^3 - 5x^2 + 3x - 2 = 0$

$x = \alpha^2 \quad \alpha = \sqrt{x}$

$(\sqrt{x})^3 - 5(\sqrt{x})^2 + 3\sqrt{x} - 2 = 0$ 1

$x\sqrt{x} + 3\sqrt{x} = 5x + 2$

$(x\sqrt{x} + 3\sqrt{x})^2 = (5x + 2)^2$ 1

$x^3 + 6x^2 + 9x = 25x^2 + 20x + 4$

$x^3 - 19x^2 - 11x - 4 = 0$ 1

③

d) $\int \cos^2 2x \sin^3 2x \, dx$

$u = \cos 2x$

$\frac{du}{dx} = -2\sin 2x$

$dx = \frac{du}{-2\sin 2x}$ 1

$I = \int u^2 \cdot -\frac{1}{2}(1 - u^2) \, du$ 1

$= -\frac{1}{2} \int u^2 - u^4 \, du$

$= -\frac{1}{2}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + c$

$= \frac{\cos^5 2x}{10} - \frac{\cos^3 2x}{6} + c$ 1

③

e) i) $\delta V = \pi (1-y)^2 \delta x$ ①

$V = \int_0^{\pi/2} \pi (1 - \sin x)^2 \, dx$

$= \pi \int_0^{\pi/2} (1 - 2\sin x + \sin^2 x) \, dx$

ii) $V = \pi \int_0^{\pi/2} (1 - 2\sin x + (\frac{1}{2} - \frac{1}{2}\cos 2x$ 1

$= \pi \int_0^{\pi/2} \left(\frac{3}{2} - 2\sin x - \frac{1}{2}\cos 2x\right)$

$= \pi \left[\frac{3x}{2} + 2\cos x - \frac{\sin 2x}{4}\right]$ 1

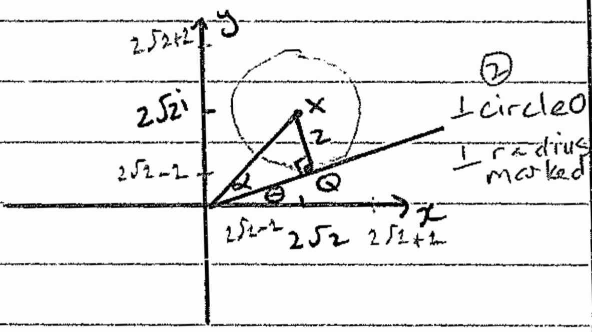
$= \pi\left(\left(\frac{3\pi}{4} + 0\right) - (2)\right)$

$= \frac{3\pi^2}{4} - 2\pi$ 1 ③

Q12

b) $|z-(2\sqrt{2}+2\sqrt{2}i)|=2$

i)



(2) 1 circle
1 radius marked

ii) $OX=\sqrt{8+8}=4$

$\sin\alpha=\frac{1}{2}\quad\therefore\ \alpha=\frac{\pi}{6}$

& $\tan(\alpha+\theta)=1$

$\therefore\quad \alpha+\theta=\frac{\pi}{4}$

$\therefore\quad \theta=\frac{\pi}{4}-\frac{\pi}{6}$

$\theta=\frac{\pi}{12}$ 1

$OQ=\sqrt{16-4}$

$=\sqrt{12}=2\sqrt{3}$ 1

$\therefore\ Q:\ 2\sqrt{3}\operatorname{cis}\frac{\pi}{12}$ (2)

c) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$S(3\sqrt{2},0)$ asympt $y=\frac{2x}{\sqrt{5}}$

$\therefore\ \frac{b}{a}=\frac{2}{\sqrt{5}}\qquad ae=3\sqrt{2}$ 1

$b=\frac{2a}{\sqrt{5}}\qquad e=\frac{3\sqrt{2}}{a}$

$b^2=a^2(e^2-1)$

$\frac{4a^2}{5}=a^2\left(\frac{18}{a^2}-1\right)$ 1

$\frac{4a^2}{5}=18-a^2$

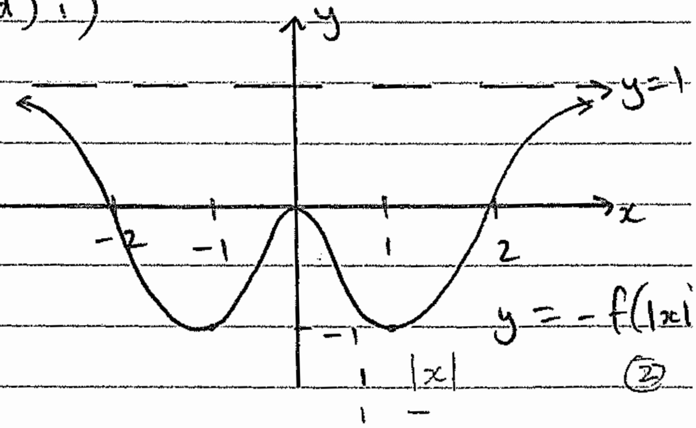
$9a^2=90$

$a^2=10$

$a=\sqrt{10}$

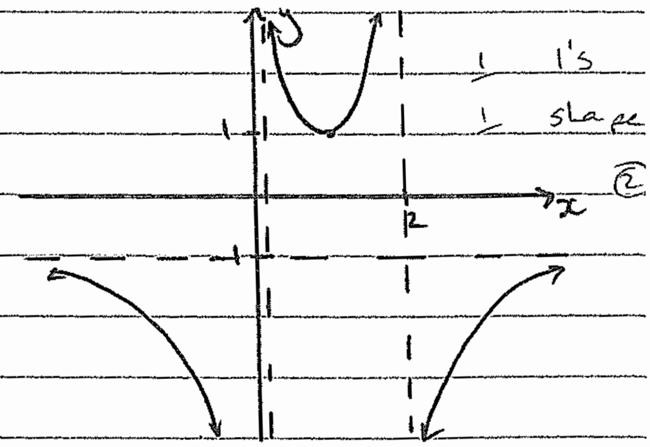
$b=2\sqrt{2}$ 1 (3)

d) i)



(2)

ii)



1 1's
1 shape
(2)

Q13

a) $x^3 - 3x^2y + y^3 = 3$

i) $3x^2 - y.6x - 3x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3x^2 - 3y^2) = 3x(x-2y)$

$\frac{dy}{dx} = \frac{3x(x-2y)}{3(x^2-y^2)}$

$= \frac{x^2-2xy}{x^2-y^2}$ 1 has some $\frac{d}{dy}\frac{dy}{dx}$ 1 answer

(2)

ii) parallel to x-axis

$\frac{dy}{dx} = 0$

$\therefore x(x-2y) = 0$

$\left(\begin{array}{l} x=0 \\ y=\sqrt[3]{3} \end{array}\right)$ 1 $x = 2y$ 1

$8y^3 - 12y^3 + y^3 = 3$

$-3y^3 = 3$

$y^3 = -1$ $y = -1$

$x = -2$ 1

$\therefore$ pts parallel to x-axis

$(0, \sqrt[3]{3}), (-2, -1)$ (3)

b) $x^3 + rx + s = 0$

i) $x^3 = -rx - s$ let $x = \alpha, \beta, \gamma$

$\alpha^3 = -r\alpha - s$

$\beta^3 = -r\beta - s$ 1

$\gamma^3 = -r\gamma - s$

$\Sigma\alpha^3 = -r(\Sigma\alpha) - 3s$

$\Sigma\alpha = 0$ (2)

$\therefore \alpha^3 + \beta^3 + \gamma^3 = -3s$ 1

ii) $P(x) = x^3 + rx + s$

$P'(x) = 3x^2 + r$

if double root

$P(x) = P'(x) = 0$

$\therefore x^3 + rx + s = 3x^2 + r = 0$

let $x^2 = -\frac{r}{3}$

$\therefore x\left(-\frac{r}{3}\right) + rx + s = 0$

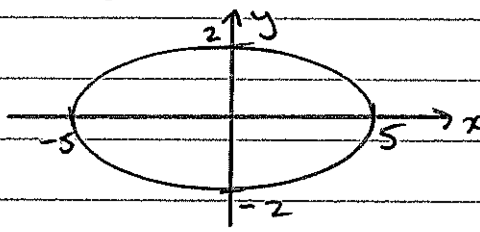
$x\left(r - \frac{r}{3}\right) = -s$

$x\left(\frac{2r}{3}\right) = -s$

$x = -\frac{3s}{2r}$ 1   (2)

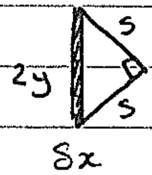
c) $4x^2 + 25y^2 = 100$

i) $\frac{x^2}{25} + \frac{y^2}{4} = 1$ (2)



1 intercept, 1 shape

ii)



$2s^2 = 4y^2$ 1

$s^2 = 2y^2$

$A = \frac{1}{2}s^2$

$= \frac{1}{2}.2y^2$

$= y^2$ 1

$y^2 = 4 - \frac{4}{25}x^2$

$V = 2\int_0^5 4 - \frac{4}{25}x^2 \, dx$ 1

$= 2\left[4x - \frac{4x^3}{75}\right]_0^5$

$= 8\left[\left(5 - \frac{125}{75}\right) - 0\right]$ (4)

$= \frac{80}{3} u^3$ 1

Q14

a) i) $\angle AFB = \angle AGB = 90$

(given $BG \perp AC$, $AF \perp BC$)

$\therefore$ ABFG is cyclic quad

(= $\angle$'s at circumference from same arc AB) (1)

ii) $\angle BFE = \angle BFD = 90$

($AF \perp BC$, given)

$\angle FBG = \angle FAG$ 1

(= $\angle$'s at circumference from same arc FG)

$\angle FAG = \angle DAC$

$\angle DAC = \angle DBC$

(= $\angle$'s at circumference from same arc DC)

$\therefore \angle DBC = \angle DBF$

$\angle DBF = \angle FBE$

$\therefore$

In $\triangle BFE$ & $\triangle BFD$

BF is common side

$\therefore \triangle BFE \equiv \triangle BFD$ (SAA) 1

$\therefore DF = EF$ 1

(corresponding sides in congruent $\triangle$'s) (3)

b) i) $\dfrac{1}{(2t+1)(t+2)} = \dfrac{a}{2t+1} + \dfrac{b}{t+2}$

$\therefore a(t+2) + b(2t+1) = 1$ 1

equate coeffs.

$a + 2b = 0 \qquad a = -2b$

$2a + b = 1$

$\therefore -4b + b = 1$

$-3b = 1$

$b = -\frac{1}{3}$ 1

$a = \frac{2}{3}$ (2)

$\therefore \dfrac{1}{(2t+1)(t+2)} = \dfrac{2}{3(2t+1)} - \dfrac{1}{3(t+2)}$

ii) $\displaystyle\int_0^{\pi/2} \frac{dx}{4+5\sin x}$

$t = \tan\frac{x}{2}$

$x = 2\tan^{-1}t$

$\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$

$x = \frac{\pi}{2} \quad t = 1$

$x = 0 \quad t = 0$

$$I = \int_0^1 \frac{2}{1+t^2} \frac{dt}{4+5\left(\frac{2t}{1+t^2}\right)} \quad 1$$

$$= \int_0^1 \frac{2\,dt}{4+4t^2+10t}$$

$$= \int_0^1 \frac{dt}{2t^2+5t+2}$$

$$= \int_0^1 \frac{dt}{(2t+1)(t+2)}$$

$$= \int_0^1 \frac{2}{3(2t+1)} - \frac{1}{3(t+2)}\,dt$$

$$= \frac{1}{3}\Big[\ln(2t+1) - \ln(t+2)\Big]_0^1 \quad 1$$

$$= \frac{1}{3}\left[\ln\frac{2t+1}{t+2}\right]_0^1$$

$$= \frac{1}{3}\left(\ln\frac{3}{3} - \ln\frac{1}{2}\right]$$

$$= \frac{1}{3}\ln 2 \quad 1$$

(3)

Q15

a) at $t=0$ $v=15$

i) $\square \rightarrow 600N$

$\overleftarrow{90v}$ $m=450$

$450a = 600 - 90v$

$a = \frac{20-3v}{15}$

$\frac{dv}{dt} = -\frac{3v-20}{15}$ ①

ii) $\frac{dt}{dv} = \frac{-15}{3v-20}$

$t = -5\ln(3v-20) + c$ 1

at $t=0$ $v=15$

$\therefore 0 = -5\ln 25 + c$

$c = 5\ln 25$

$\therefore t = -5\ln(3v-20) + 5\ln 25$

$t = 5\ln\frac{25}{3v-20}$ 1

$e^{\frac{t}{5}} = \frac{25}{3v-20}$

$3v-20 = \frac{25}{e^{t/5}}$

$v = \frac{25+20e^{t/5}}{3e^{t/5}}$ 1 ③

iii) find t at $v=10$

$10 = \frac{25+20e^{t/5}}{3e^{t/5}}$

$30e^{t/5} - 20e^{t/5} = 25$

$10e^{t/5} = 25$ 1

$e^{t/5} = 5/2$

$\frac{t}{5} = \ln(\frac{5}{2})$

$t = 5\ln(\frac{5}{2})$

$t \doteq 4.6s$ 1 ②

b) $I_n = \int_0^3 x^n\sqrt{9-x^2}\,dx$

i) $= \int_0^3 x^{n-1}.x\sqrt{9-x^2}\,dx$ 1

$u = x^{n-1}$ $v' = x\sqrt{9-x^2}$

$u' = (n-1)x^{n-2}$ $v = -\frac{1}{2}(9-x^2)^{3/2}.\frac{2}{3}$

$v = \frac{-1}{3}(9-x^2)^{3/2}$

$I_n = \frac{-1}{3}\left[x^{n-1}(9-x^2)^{3/2}\right]_0^3$

$+\frac{1}{3}\int_0^3 (9-x^2)^{3/2}(n-1)x^{n-2}\,dx$ 1

$= \frac{n-1}{3}\int_0^3 x^{n-2}\sqrt{9-x^2}\,(9-x^2)\,dx$

$= \frac{n-1}{3}\int_0^3 9x^{n-2}\sqrt{9-x^2} - x^n\sqrt{9-x^2}\,dx$

$= \frac{n-1}{3}(9I_{n-2} - I_n)$

$I_n(1+\frac{n-1}{3}) = 3(n-1)I_{n-2}$

$I_n(\frac{3+n-1}{3}) = 3(n-1)I_{n-2}$

$(n+2)I_n = 9(n-1)I_{n-2}$

ii) $I_4 = \frac{9}{6}(3)I_2$

$= \frac{9}{2}(\frac{9}{4}I_0)$

$= \frac{81}{8}I_0$ 1 ③

Q16

a) i) $(cis\theta)^7 = cis 7\theta$

& $(cis\theta)^7$

$= \binom{7}{0}c^7 + \binom{7}{1}isc^6 - \binom{7}{2}s^2c^5$

$- \binom{7}{3}is^3c^4 + \binom{7}{4}s^4c^3$

$+ \binom{7}{5}is^5c^2 - \binom{7}{6}s^6c$

$- \binom{7}{7}is^7$

equate Im parts

$\therefore \sin 7\theta$

$= 7\sin\theta\cos^6\theta - 35\sin^3\theta\cos^4\theta$

$+ 21\sin^5\theta\cos^2\theta - \sin^7\theta$

$(\div \sin\theta)$

$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\sin^2\theta\cos^4\theta$

$+ 21\sin^4\theta\cos^2\theta - \sin^6\theta$

$= 7(1-\sin^2\theta)^3 - 35\sin^2\theta(1-\sin^2\theta)^2$

$+ 21\sin^4\theta(1-\sin^2\theta) - \sin^6\theta$

$= 7(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)$

$- 35s^2(1-2s^2+s^4)$ 1 progress

$+ 21s^4 - 21s^6 - s^6$

$= 7 - 21s^2 + 21s^4 - 7s^6$

$- 35s^2 + 70s^4 - 35s^6$

$+ 21s^4 - 21s^6 - s^6$

$= 7 - 56\sin^2\theta + 112\sin^4\theta$

$- 64\sin^6\theta$ 1 (3)

ii) Let $x = \sin^2\frac{\pi}{7}$

$\therefore 0 = 7 - 56x + 112x^2 - 64x^3$

ie $64x^3 - 112x^2 + 56x - 7 = 0$

N.B $\sin\theta \neq 0$

& $\sin 7\theta = 0$

$\therefore 7\theta = \pi k$ 1

$\theta = \frac{\pi k}{7}$ $k = 1, 2, 3$

$\therefore$ roots are $\sin^2\frac{\pi}{7}, \sin^2\frac{2\pi}{7}, \sin^2\frac{3\pi}{7}$ 1 (2)

iii) $\text{cosec}^2\frac{\pi}{7} + \text{cosec}^2\frac{2\pi}{7} + \text{cosec}^2$

$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\Sigma\alpha\beta}{\alpha\beta\gamma}$

$= \frac{\frac{56}{64}}{\frac{7}{64}}$

$= 8$ (1)

b) i) $(1+i)^{2m}$

$= (\sqrt{2}\, cis\frac{\pi}{4})^{2m}$

By Binomial theorem $(1+i)^{2m}$

$= \binom{2m}{0} + \binom{2m}{1}i - \binom{2m}{2} - \binom{2m}{3}i$

$\ldots + \binom{2m}{2m}(-1)^m$ (1)

ii) Equate real parts 1

$\therefore 1 - \binom{2m}{2} + \binom{2m}{4} + \ldots \binom{2m}{2m}(-1)^m$

$= 2^m\cos\frac{\pi m}{2}$ 1 (2)

$(\sqrt{2}\, cis\frac{\pi}{4})^{2m}$