GOSFORD HIGH SCHOOL

2019

Trial HSC Examination

Mathematics Extension 2

General Instructions

Total Marks – 100

All questions may be attempted

Section I (10 Marks)

Answer questions 1-10 on the Multiple Choice answer sheet provided.

Questions 1-10 are of equal values

Section II (90 Marks)

For Questions 11-16, start a new answer booklet for each question.

Questions 11-16 are of equal values

- Writing time – 3 Hours
- Write using black pen.
- NESA approved calculators and templates maybe used.
- Leave your answers in the simplest exact form, unless otherwise stated.
- Marks may be deducted for careless or badly arranged work.
- All necessary working should be shown.
- A Reference Sheet is provided.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.

Candidate Number ----------------------------------

Section I – Multiple Choice (10 marks)

Attempt Questions 1 - 10

Use multiple-choice answer sheet for Questions 1 – 10

1) Find $\int \frac{1}{x^2+2x+2}dx$

(A) $\sin^{-1}(x+1)+C$

(B) $\tan^{-1}(2x+1)+C$

(C) $\cos^{-1}(2x-1)+C$

(D) $\tan^{-1}(x+1)+C$

2) What is $z=-\sqrt{2}+\sqrt{2}i$ in modulus-argument form?

(A) $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(B) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

(C) $2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(D) $2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

3) The fifth roots of 1 are

(A) $1, cis\frac{3\pi}{5}, cis\frac{5\pi}{5}, cis\frac{6\pi}{5}, cis\frac{9\pi}{5}$

(B) $1,\ cis\frac{3\pi}{5}, cis\frac{5\pi}{5}, cis\frac{6\pi}{5}, cis\frac{8\pi}{5}$

(C) $1, cis\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis\frac{6\pi}{5}, cis\frac{8\pi}{5}$

(D) $1, cis\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis\frac{6\pi}{5}, cis\frac{9\pi}{5}$

4) Ten people, consisting of 5 boys, 3 teachers and 2 girls sit around a circular table. In how many ways can they be seated if the girls must sit together, but not next to any teacher?

(A) 28800

(B) 14400

(C) 30

(D) 34900

5) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the values of $\alpha^2 + \beta^2 + \gamma^2$?

(A) -4

(B) -2

(C) 6

(D) 8

6) Given $x^3 + y^3 = 1, \frac{dy}{dx} = ?$

(A) $-\left(\frac{x}{y}\right)^2$

(B) $-\left(\frac{x}{y}\right)^3$

(C) $\left(\frac{y}{x}\right)^2$

(D) $3x^2 + 2y^2$

7) Find the centre of the ellipse $3x^2 + 24x + y^2 + 36 = 0$

(A) $(4,2)$

(B) $(4,0)$

(C) $(-6,0)$

(D) $(-4,0)$

8) Suppose that $f(x)$ is a non zero odd function. Which of the functions below is also odd?

(A) $f(x^2)\cos x$

(B) $f(f(x))$

(C) $f(x^3)\sin x$

(D) $f(x^2) - f(x)$

9) A solid has its base in the xy plane being the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

Cross-sections perpendicular to the major axis are squares. Find the area of

The cross-section at $x=k$, where k is a constant.

(A) $16-\frac{7k^2}{4}$

(B) $36-\frac{9k^2}{4}$

(C) $34-\frac{9k^3}{7}$

(D) $18+\frac{9k^2}{4}$

10) A particle is moving along a straight line so that its displacement is $x=1$, its velocity is $v=2$, and its acceleration is $a=4$.

Which is a possible equation describing the motion of the particle?

(A) $v=2\sin(x-1)+2$

(B) $v=2+4\log_e x$

(C) $v^2=4(x^2-2)$

(D) $v=x^2+2x+4$

---------------------- End of Section I ------------------

Section II – Written Response (90 marks)

Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet.

Questions 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}dx$ **2**

(b) Find $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ **3**

(c) Sketch, showing critical points the graph of $y = x^2 - |x|$ **2**

(d) Complex numbers $z = \frac{a}{1+i}$ and $w = \frac{b}{1+2i}$ **2**

where a and b are real, such that $z + w = 1$. Find a and b

(e) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to the tangent at P.

(i) Show that $ST = \frac{|e\cos\theta - 1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$ **3**

(ii) Hence prove $ST \cdot S'T' = b^2$ **3**

Questions 12 (15 marks) Use a SEPARATE writing booklet.

(a) On an Argand Diagram, sketch the locus of $3 \geq \operatorname{Re} Z \geq 0$ and $3 \geq \operatorname{Im} Z \geq 1$ **1**

(b)

(i) Sketch the graph of $y = x^3 - 12x$, showing all essential features. **2**

(ii) Use this graph to find the set of values of the real number k for which the equation $x^3 - 12x + k = 0$ has exactly one real root. **2**

(c) Express the polynomial $x^3 - 4x^2 + 6x - 4$ as a product of a linear factor and a quadratic factor. **2**

(d) Given that $\omega = cis\dfrac{2\pi}{7}$

(i) Write down the modulus and argument of ω^4 and ω^5. **1**

(ii) Plot the points represented by $\omega, \omega^4, \omega^5$ on an argand diagram and prove they form the vertices of an isosceles triangle. **2**

(iii) Find the value of $(\omega + \omega^6)(\omega^2 + \omega^5) + (\omega^2 + \omega^5)(\omega^3 + \omega^4) + (\omega^3 + \omega^4)(\omega + \omega^6)$ **2**

(e) If $p(x) = 4x^3 + 15x^2 + 12x - 4$ has a double zero, find all the zeros and factorise $P(x)$ fully over the real numbers. **3**

Questions 13 (15 marks) Use a SEPARATE writing booklet.

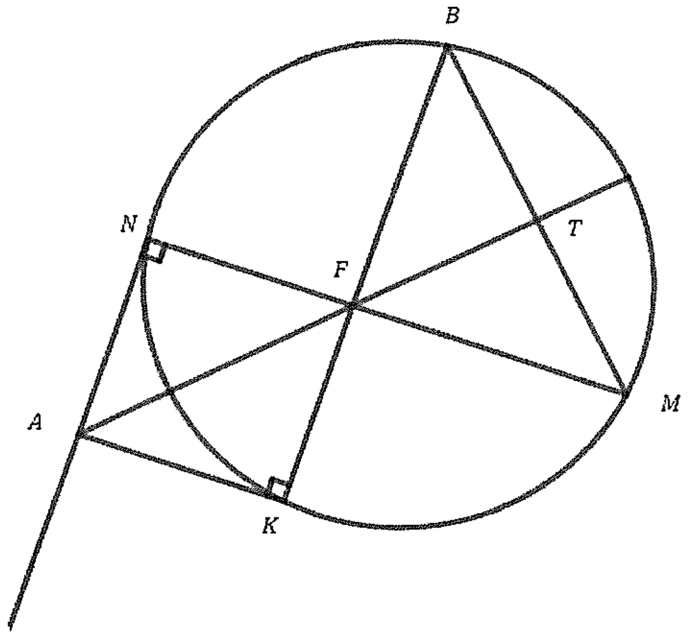
(a) Let $I_n = \int_1^e (\ln t)^n \, dt$ for $n \geq 0$,

(i) Show that $I_n = e - nI_{n-1}$ for $n = 1,2,3,.....$ **2**

(ii) Hence or otherwise, find the exact value of I_4 **2**

(b) Sketch on the Argand Diagram the locus $|z-2| = |z+2i|$ **2**

(c) A circle has two chords *KB* and *MN* intersecting at *F*. Perpendiculars are drawn to these chords at *K* and at *N* intersecting at *A*. *AF* produced meets *MB* at *T*. Prove that *AF* is perpendicular to *MB*. **4**



(d) In how many ways can 8 people sit at a square table, 2 people to a side ? **2**

(e) A round-robin tennis tournament consists of each player playing every other player exactly once. How many matches will be held during a *n*- person round-robin tennis tournament when $n > 2$? **3**

Questions 14 (15 marks) Use a SEPARATE writing booklet

(a) A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex $(0,0)$ and the line $x = a$, about the x axis. What is the volume of this solid using the method of cylindrical shells? **2**

(b) A particle of unit mass is moving horizontally in a straight line. It is initially at the origin and is moving with velocity $Ums^{-1}\left(U > 0\right)$. The particle is moving against a resistance $v^2 + v^3$ where v *is* the velocity. After T seconds the particle is X metres from the origin and is moving with velocity $\frac{1}{2}U\ ms^{-1}$.

(i) Show that $\ddot{x} = -\left(v^2 + v^3\right)$ **1**

(ii) Show that $X = \ln\left(\frac{2+U}{1+U}\right)$ **2**

(iii) Show that $t = \frac{1}{v} - \frac{1}{U} + \ln\left|\frac{v(1+U)}{(1+v)U}\right|$ **3**

(c) Prove that

(i) $\cot\frac{\alpha}{2} - \cot\alpha = \operatorname{cosec}\alpha$ **2**

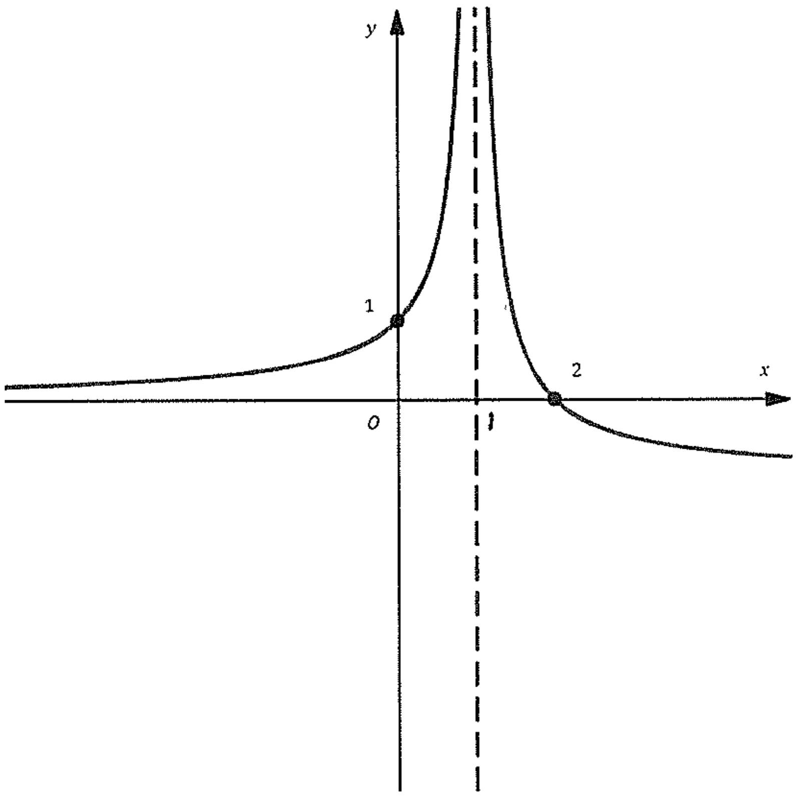
(ii) Hence find a simplified expression for $\sum_{k=1}^{n} \operatorname{cosec}\left(2^k\alpha\right)$ **3**

(d) If α, β and γ are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, find

The equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

Questions 15 (15 marks) Use a SEPARATE writing booklet

(a)



The graph of $y = f(x)$ is shown above. On separate number planes, sketch :

(i) $y = f(x+2)$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(b) Solve $\cos 3x + 3\sin 2x = 3\cos x$ for $0 \le x \le 2\pi$ **3**

(c) Given that $\sin^{-1}x, \cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute,

Show that: $\sin\left(\sin^{-1}x - \cos^{-1}x\right) = 2x^2 - 1$ **2**

(d) Express $\dfrac{3x^2 - 6x + 10}{(x-4)(x^2+1)}$ as a sum of partial fractions **2**

(e) Show that if $y = mx + k$ is a tangent to the rectangular hyperbola

$xy = c^2$, then $k^2 + 4mc^2 = 0$ **2**

Questions 16 (15 marks) Use a SEPARATE writing booklet

(a) A point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola is given by $x + t^2 y = 2ct$. **2**

(ii) Prove that the area bounded by the tangent and the asymptotes of the rectangular hyperbola is a constant. **2**

(b) Consider the complex numbers z and w where $z = \cos\frac{\pi}{k} + i\sin\frac{\pi}{k}$,

and $w = z^n$, where $k = 1,2,3.....$, and $n = 1,2,3.....$,

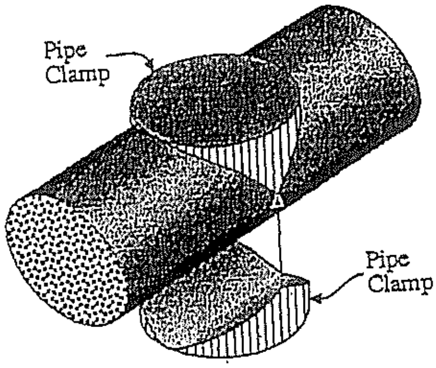
(i) Explain geometrically how w is obtained from z. **2**

(ii) Show that for n even, the real part of w is given by

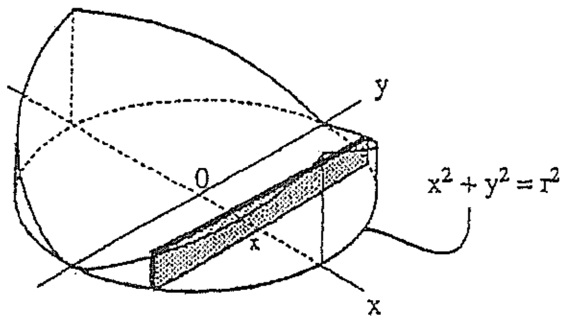
$$\sum_{r=0}^{\frac{n}{2}} \binom{n}{2r} (-1)^r \cos^{n-2r}\frac{\pi}{k}\sin^{2r}\frac{\pi}{k}$$ **3**

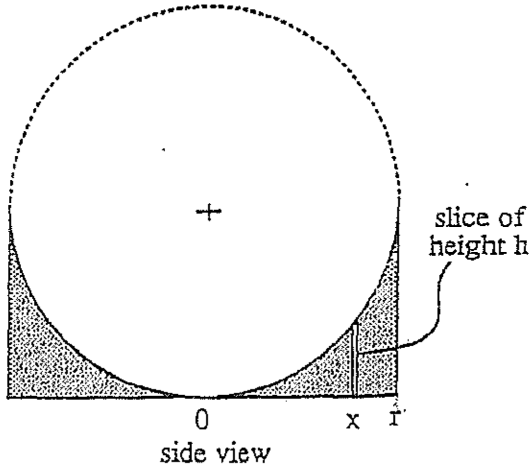
(c) A pipe-clamp is made of two identical pieces. Each piece has a circular base of radius r units and the other face is curved so as to fit flush against the pipe held between the two pieces.

The pipe also has a radius of r units.



A vertical slice, of thickness Δx, taken x units from the centre of the base is in the shape of a rectangle with one side in the circular base and of height necessary to reach the cylindrical pipe as shown in the diagram below:





(i) Show that the height of the slice taken x units from O is given by

$$h = r - \sqrt{r^2 - x^2}$$ **2**

(ii) Show that the volume ΔV, of such a slice is given by

$$\Delta V = \left[2r\sqrt{r^2 - x^2} - 2\left(r^2 - x^2\right)\right]\Delta x$$ **2**

(iii) Hence find by integration, the volume of ONE piece of the pipe-clamp **2**

END OF TASK

1) $\int \frac{1}{x^2+2x+2}\,dx$

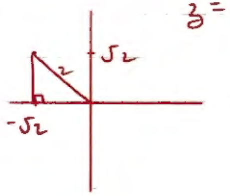
$= \int \frac{1}{x^2+2x+1+1}\,dx$

$= \int \frac{1}{1+(1+x)^2}\,dx$

$= \tan^{-1}(1+x) + c$ (D)

2) $z = -\sqrt{2} + \sqrt{2}i$

$z = 2\,\text{cis}\,\frac{3\pi}{4}$ (D)



3) $z^5 = 1$

$\therefore \cos 5\theta = 1$

$5\theta = 0 + 2k\pi$

$\theta = \frac{2k\pi}{5}$ $k = 0, \pm 1, \pm 2$ (C)

4)

choose boys / seat rest / change boys / change girls

$^5C_2 \times 6! \times 2 \times 2$

= (A)

OR ④ [GG] ⑤ $1 \times 5 \times 4 \times 6! \times 2$

5) $x^3 + 2x - 1 = 0$

$\Sigma\alpha = 0$

$\Sigma\alpha\beta = 2$

$\alpha\beta\gamma = 1$

$(\alpha+\beta+\gamma)^2 = \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2$

$= \Sigma\alpha^2 + 2(\Sigma\alpha\beta)$

$\therefore \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2(\Sigma\alpha\beta)$

$= 0^2 - 2(2)$

$= -4$ (A)

6) $x^3 + y^3 = 1$

$3x^2 + 3y^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-3x^2}{3y^2}$

$= -\frac{x^2}{y^2}$ (A)

7) $3x^2 + 24x + y^2 + 36 = 0$

$3(x^2 + 8x + 16) + y^2 = -36 + 48$

$3(x+4)^2 + y^2 = 12$

$\frac{(x+4)^2}{4} + \frac{y^2}{12} = 1$

$\therefore$ centre $(-4, 0)$ (D)

8) $f(x)$ is odd

A) $f(x^2)\cos x$

n | e → n

B) $f(f(x))$

$f(f(-x)) = f[-f(x)]$

$= -f[f(x)]$

$\therefore$ odd (B)

and

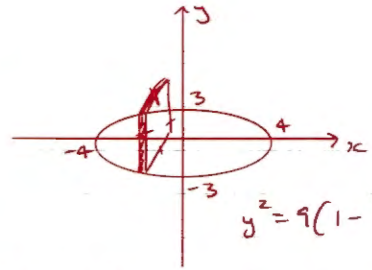
$f[f(x)] + f[f(-x)] = 0$

C) $f(x^3)\sin x$ → even

o | o

D) $f(x^2) - f(x)$ neither

9)



$y^2 = 9\left(1 - \frac{x^2}{16}\right)$

Area $= (2y)^2$

$= 4y^2$

$= 4\left(9\left(1 - \frac{k^2}{16}\right)\right)$

$= 36 - \frac{9k^2}{4}$ (B)

10) $x = 1, v = 2, a = 4$

A) $v = 2\sin(x-1) + 2$

$\frac{1}{2}v^2 = \frac{1}{2}(4\sin^2(x-1) + 8\sin(x-1) + 4)$

$\ddot{x} = 4\sin(x-1)\cos(x-1) + 4\cos(x-1)$

$x = 1$ $\ddot{x} = 4$ ✓

(A)

11/(a) $\displaystyle\int_0^{\frac{1}{\sqrt2}} \frac{x}{\sqrt{1-x^4}}\,dx$

Let $u=x^2$

when $x=\frac{1}{\sqrt2}$ $\quad u=\frac{1}{2}$

when $x=0$ $\quad u=0$

$u=x^2$

$\frac{du}{dx}=2x$

$dx=\frac{du}{2x}$

$\therefore \displaystyle\frac{1}{2}\int_0^{\frac12}\frac{1}{\sqrt{1-u^2}}\,du$

$=\frac{1}{2}\left[\sin^{-1}u\right]_0^{\frac12}$

$=\frac{1}{2}\left[\frac{\pi}{6}-0\right]$

$=\frac{\pi}{12}$

(b) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x}$

Let $t=\tan\frac{x}{2}$

$dx=\frac{2}{1+t^2}\,dt$

when $x=\frac{\pi}{2}$ $\quad t=1$

when $x=0$ $\quad t=0$

$=\displaystyle\int_0^1\frac{2}{2+\frac{1-t^2}{(1+t^2)}}\,\frac{dt}{(1+t^2)}$

$=\displaystyle\int_0^1\frac{2\,dt}{2+2t^2+1-t^2}$

$=\displaystyle\int_0^1\frac{2}{t^2+3}\,dt$

$=2\displaystyle\int_0^1\frac{1}{t^2+(\sqrt3)^2}\,dt$

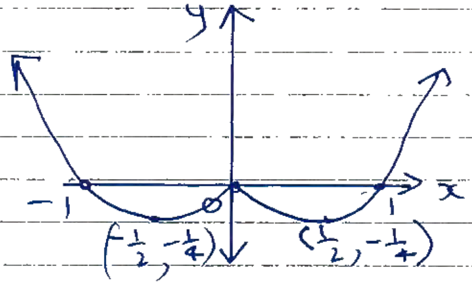
$=2\left[\frac{1}{\sqrt3}\tan^{-1}\frac{t}{\sqrt3}\right]_0^1$

$=\frac{2}{\sqrt3}\left[\frac{\pi}{6}-0\right]$

$=\frac{\pi}{3\sqrt3}$

(c) For $x\geq 0$ $\quad y=x^2-x$

For $x\leq 0$ $\quad y=x^2+x$



(d) $z=\frac{a}{1+i}\times\frac{1-i}{1-i}$ $\qquad w=\frac{b}{1+2i}\times\frac{1-2i}{1-2i}$

$=\frac{a-ai}{2}$ $\qquad =\frac{b-2bi}{5}$

$\therefore 5(a-ai)+2(b-2bi)=10$ (which is real)

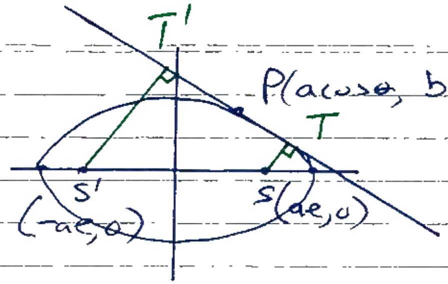
$5a-5ai+2b-4bi=10$

$\therefore 5a+2b=10$, $-5a-4b=0$ } Add $\quad -5a-4b=0$

$-2b=10$

$b=-5$ $\qquad a=4$

(e)



Equation of tangent at P

$\Rightarrow \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

$\therefore \quad d_{(ST)} = \dfrac{\left|ae.\frac{\cos\theta}{a} + 0.\frac{\sin\theta}{b} - 1\right|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$

$= \dfrac{|e\cos\theta - 1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$ (as required)

(ii) $ST' = \dfrac{|-e\cos\theta - 1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$

$ST.ST' = \dfrac{|e\cos\theta - 1|.|-1|.|e\cos\theta + 1|}{\left(\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}\right)^2}$

$= \dfrac{1 - e^2\cos^2\theta}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$

Using $e^2 = 1 - \frac{b^2}{a^2}$

$= \dfrac{1 - \left[\frac{a^2 - b^2}{a^2}\right]\cos^2\theta}{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}$

$= \dfrac{\frac{a^2 - [a^2 - b^2]\cos^2\theta}{a^2}}{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}$

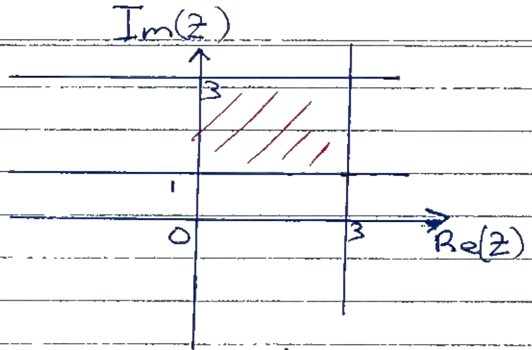
$= \dfrac{a^2 - a^2\cos^2\theta + b\cos^2\theta}{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{b^2}}$

$= \dfrac{(a^2(1-\cos^2\theta) + b^2\cos^2\theta)}{(b\cos^2\theta + a^2\sin^2\theta)} \cdot b^2$

$= b^2$

Q12

(a)



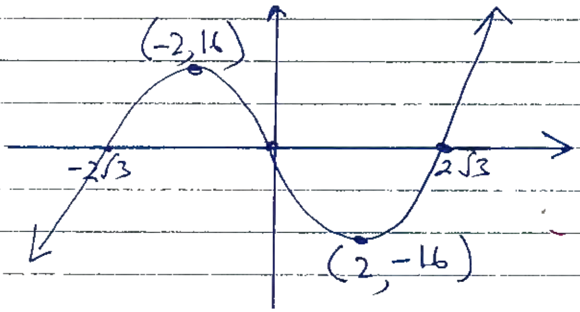
(b)(i) $y = x(x^2-12)$
$= x(x-2\sqrt{3})(x+2\sqrt{3})$

$\frac{dy}{dx} = 3x^2 - 12$

Let $\frac{dy}{dx} = 0$

$= 3(x^2-4)$

$= 3(x-2)(x+2)$



(ii) From graph
$K > 16$ OR $K < -16$

(c) $P(2) = 0$ $\therefore$ $x-2$ is a factor

$P(x) = (x-2)(x^2+bx+2)$

Now $bx^2 - 2x^2 = -4x^2$

$b - 2 = -4$

$b = -2$ $\therefore$ $P(x) = (x-2)(x^2-2x+2)$

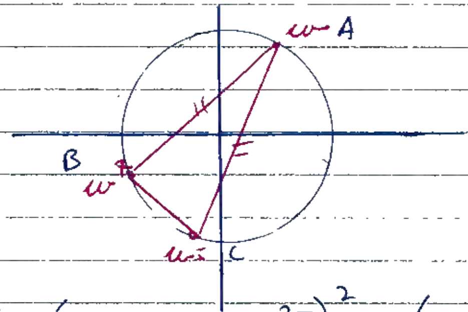
(d) $w = \text{cis}\,\frac{2\pi}{7}$

(i) $w^4 = \text{cis}\,\frac{8\pi}{7}$ $\quad w^5 = \text{cis}\,\frac{10\pi}{7}$ (By De Moivres)

$\arg w^4 = \frac{8\pi}{7}$ $\quad |w^4| = 1$

$\arg w^5 = \frac{10\pi}{7}$ $\quad |w^5| = 1$

(ii)



$(AB)^2 = \left(\cos\frac{8\pi}{7} - \cos\frac{2\pi}{7}\right)^2 + \left(\sin\frac{8\pi}{7} - \sin\frac{2\pi}{7}\right)^2$

$= \cos^2\frac{8\pi}{7} - 2\cos\frac{8\pi}{7}\cos\frac{2\pi}{7} + \cos^2\frac{2\pi}{7} + \cos^2\frac{2\pi}{7}$
$+ \sin^2\frac{8\pi}{7} - 2\sin\frac{8\pi}{7}\sin\frac{2\pi}{7} + \sin^2\frac{2\pi}{7}$

$= 2 - 2\left(\cos\frac{8\pi}{7}\cos\frac{2\pi}{7} + \sin\frac{8\pi}{7}\sin\frac{2\pi}{7}\right)$

$= 2 - 2\left(\cos\frac{6\pi}{7}\right)$

Similarly $\therefore (AC)^2 = 2 - 2\left(\cos\frac{8\pi}{7}\right)$ $\quad$ but $\cos\frac{8\pi}{7} = \cos\frac{6\pi}{7}$

$= 2 - 2\left(\cos\frac{6\pi}{7}\right)$

$\therefore AB = AC$ $\therefore$ ΔABC is isosceles

(iii) $(\omega^{7}+\omega^{6}+\omega^{8}+\omega^{11}) + (\omega^{5}+\omega^{6}+\omega^{8}+\omega^{9})$
$+ (\omega^{4}+\omega^{9}+\omega^{5}+\omega^{10})$ ①

Now $\omega^{7} = \text{cis}\ \frac{14\pi}{7}$
$= \text{cis}\ 2\pi$
$= 1$

① becomes
$(\omega^{3}+\omega^{6}+\omega+\omega^{4}) + (\omega^{5}+\omega^{6}+\omega+\omega^{2})$
$+ (\omega^{4}+\omega^{2}+\omega^{5}+\omega^{3})$

$= 2(\omega+\omega^{2}+\omega^{3}+\omega^{4}+\omega^{5}+\omega^{6})$

$= 2 \times S_6$ $\quad a=\omega \quad r=\omega \quad n=6$

$= 2 \times \frac{\omega(\omega^{6}-1)}{\omega - 1}$

$= 2 \times \left[\frac{\omega^{7}-\omega}{\omega-1}\right]$

$= 2 \times \frac{(1-\omega)}{-(1-\omega)}$

$= 2 \times -1$

$= -2$

(e) $p(x) = 4x^{3}+15x^{2}+12x-4$

$p'(x) = 12x^{2}+30x+12$

$p(-2)=0$ and $p'(-2)=0$

$\therefore 4x^{3}+15x^{2}+12x-4 \equiv (x+2)^{2}(4x+b)$

$\therefore b=-1$

$4x^{3}+15x^{2}+12x-4 \equiv (x^{2}+4x+4)(4x-1)$

Check $= 4x^{3}-x^{2}+16x^{2}-4x-4+16x$
$= 4x^{3}+15x^{2}+12x-4$ ✓

$\therefore (x+2)^{2}(4x-1)$

13(a) $I_n = \int_1^e (\ln t)^n \, dt \quad n \ge 0$

(i) $u = (\ln t)^n \qquad v = t$

$\frac{du}{dt} = \frac{n}{t}[\ln t]^{n-1} \qquad \frac{dv}{dt} = 1$

$I_n = [t \ln t]_1^e - n\int_1^e (\ln t)^{n-1} \, dt$

$= e - nI_{n-1}$

(ii) $I_4 = e - 4I_3$

$= e - 4[e - 3I_2]$

$= e - 4e + 12[e - 2I_1]$

$= -3e + 12e - 24[e - I_0]$

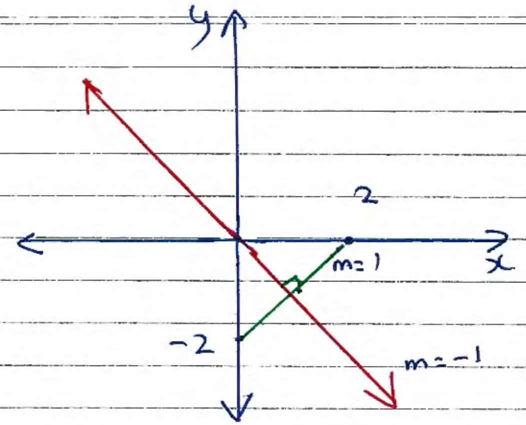
$= -15e + 24\int_1^e 1 \, dt$

$= -15e + 24[t]_1^e$

$= -15e + 24[e - 1]$

$= 9e - 24$

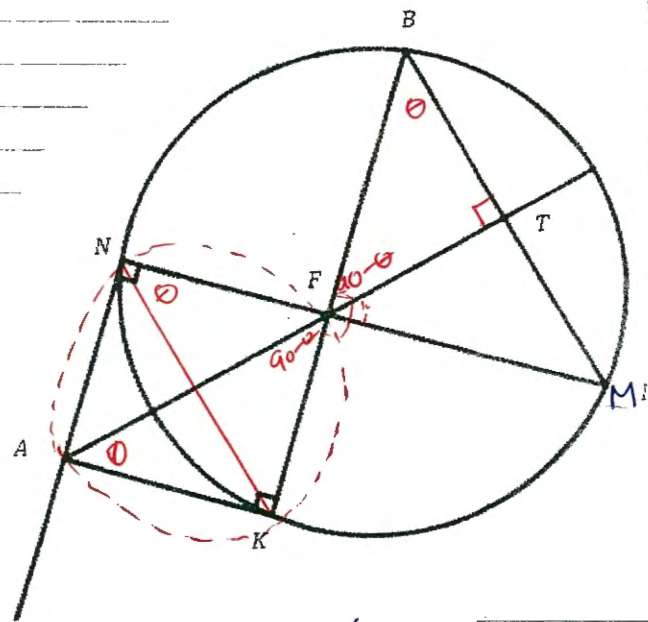
(b)



midpoint is $(1, -1)$

$\therefore$ locus is $y = -x$ including origin

(c)



$\angle ANF + \angle FKA = 180°$

$\therefore ANFK$ is cyclic (opposite $\angle$ supplementary)

Let $\angle MBF = \theta$

$\therefore \angle MBF = \angle FNK$ ($\angle$'s on circumference stand on arc KM)

$\angle FNK = \angle FAK = \theta$ ($\angle$'s on circumference stand arc FK)

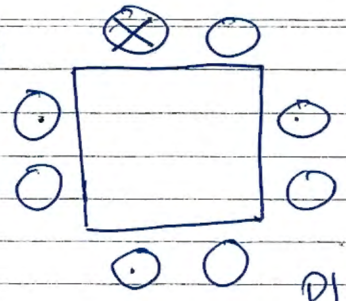
$\therefore \angle AFK = 90 - \theta$ ($\angle$ sum $\triangle AKF$)

$\angle AFK = \angle BFT = 90 - \theta$ (vertically opposite $\angle$'s $=$)

$\angle BTF = 90°$ ($\angle$ sum $\triangle BFT$)

$\therefore BM \perp AF$

(d)(i)



Placing one person they can sit either on the LHS or RHS

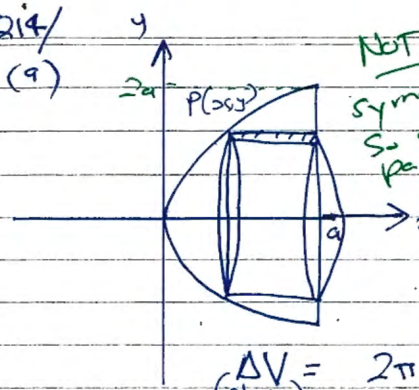
$\therefore 2!$ ways

Place other people $= 7!$

$\therefore 2! \times 7! = 10{,}080$ ways

(ii) $\dfrac{n(n-1)}{2}$

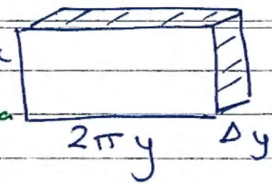
Q14/ (a)



NOTE symmetrical over x axis

So only have partitions from $y=0 \Rightarrow y=2a$

around x axis



$y^2 = 4ax$

$x = \dfrac{y^2}{4a}$

$\Delta V_{(\text{shell})} = 2\pi y (a-x)\Delta y$

$= 2\pi y \left(a - \dfrac{y^2}{4a}\right)\Delta y$

when $x=0$, $y=0$

when $x=a$, $y=\pm 2a$

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{2a} 2\pi \left(ay - \frac{y^3}{4a}\right)$$

$$V = 2\pi \int_0^{2a} ay - \frac{y^3}{4a} \, dy$$

$$= 2\pi \left[\frac{ay^2}{2} - \frac{y^4}{16a}\right]_0^{2a}$$

$$= 2\pi \left[\frac{4a^3}{2} - \frac{16a^4}{16a} - (0-0)\right]$$

$$= 2\pi \left[2a^3 - a^3\right]$$

$$= 2\pi a^3$$

Q14

(b) (i) $m\ddot{x} = -m(v^2+v^3)$

Divide by m

$\ddot{x} = -(v^2+v^3)$

(ii) $v\frac{dv}{dx} = -(v^2+v^3)$ relating x and v

$\frac{dv}{dx} = -(v+v^2)$

$\frac{dx}{dv} = \frac{-1}{v+v^2}$

By partial fractions $\frac{-1}{v+v^2} = \frac{a}{v} + \frac{b}{1+v}$

$\therefore -1 = a(1+v) + bv$

Let $v=0$ Let $v=-1$

$-1 = a$ $-1=-b$

$b = 1$

$\frac{dx}{dv} = \frac{1}{1+v} - \frac{1}{v}$

$x = \ln|1+v| - \ln v + C$

when $t=0$ $x=0$ and when $t=T$, $x=X$

$v=U$ $v=\frac{U}{2}$

$\therefore X = \int_U^{\frac{U}{2}} \frac{1}{1+v} - \frac{1}{v}\, dv$

$= \left[\ln\left|\frac{1+v}{v}\right|\right]_U^{\frac{U}{2}}$

$= \ln\left|\frac{1+\frac{U}{2}}{\frac{U}{2}}\right| - \ln\left|\frac{1+U}{U}\right|$

$= \ln\left|\frac{1+\frac{U}{2}}{\frac{U}{2}} \times \frac{U}{1+U}\right|$

$= \ln\left|\frac{2+U}{2} \times \frac{2}{U} \times \frac{U}{1+U}\right|$

$= \ln\left|\frac{2+U}{1+U}\right|$ as required.

(iii) Relating t and v

$\frac{dv}{dt} = -(v^2+v^3)$

$\frac{dt}{dv} = \frac{-1}{v^2+v^3}$

$= \frac{-1}{v^2(1+v)}$

By partial fractions

$\frac{-1}{v^2(1+v)} = \frac{a}{v} + \frac{b}{v^2} + \frac{c}{1+v}$

$-1 = av(1+v) + b(1+v) + cv^2$

Let $v=-1$ Let $v=0$

$-1 = c$ $b=-1$

Let $v=1$

$-1 = 2a - 2 - 1$

$2a = 2$

$a = 1$

$\therefore \frac{dt}{dv} = \frac{1}{v} - \frac{1}{v^2} - \frac{1}{(1+v)}$

$\therefore t = \ln|V| + \frac{1}{V} - \ln|1+V| + C$

when $t=0$ $V=U$

$\therefore 0 = \ln\left|\frac{U}{1+U}\right| + \frac{1}{U} + C$

$C = \ln\left|\frac{1+U}{U}\right| - \frac{1}{U}$

$\therefore t = \ln\left|\frac{V}{1+V}\right| + \frac{1}{V} + \ln\left|\frac{1+U}{U}\right| - \frac{1}{U}$

$= \frac{1}{V} - \frac{1}{U} + \ln\left|\frac{V(1+U)}{U(1+V)}\right|$

(c)(i) LHS $= \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} - \frac{\cos\alpha}{\sin\alpha}$

Or by t method

Let $t = \tan\frac{\alpha}{2}$

LHS $= \frac{1}{t} - \left(\frac{1-t^2}{2t}\right)$

$= \frac{2-1+t^2}{2t}$

$= \frac{1+t^2}{2t}$

$= \operatorname{cosec}\alpha$

$= \frac{\cos\frac{\alpha}{2}\sin\alpha - \sin\frac{\alpha}{2}\cos\alpha}{\sin\frac{\alpha}{2}\sin\alpha}$

$= \frac{\sin\left(\alpha - \frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\sin\alpha}$

$= \frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\sin\alpha}$

$= \operatorname{cosec}\alpha$ as required

(ii) $\sum_{k=1}^{n} \operatorname{cosec}(2^k\alpha) = \operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha + \operatorname{cosec}8\alpha + \dots + \operatorname{cosec}2^n\alpha$

$= (\cot\alpha - \cot 2\alpha) + (\cot 2\alpha - \cot 4\alpha) + (\cot 4\alpha - \cot 8\alpha) + \dots + (\cot 2^{n-1}\alpha - \cot 2^n\alpha)$

$= \cot\alpha - \cot 2^n\alpha$

(d) Replace x with $\frac{1}{x}$

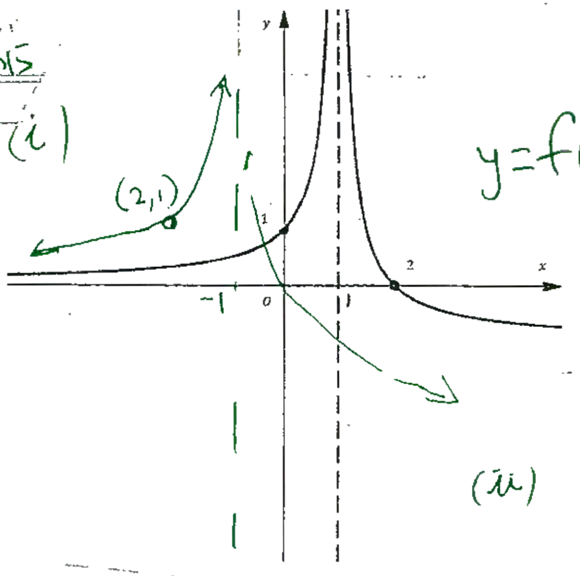
$\frac{1}{x^3} + \frac{4}{x^2} - \frac{3}{x} + 1 = 0$

multiply by x^3

$x^3 - 3x^2 + 4x + 1 = 0$

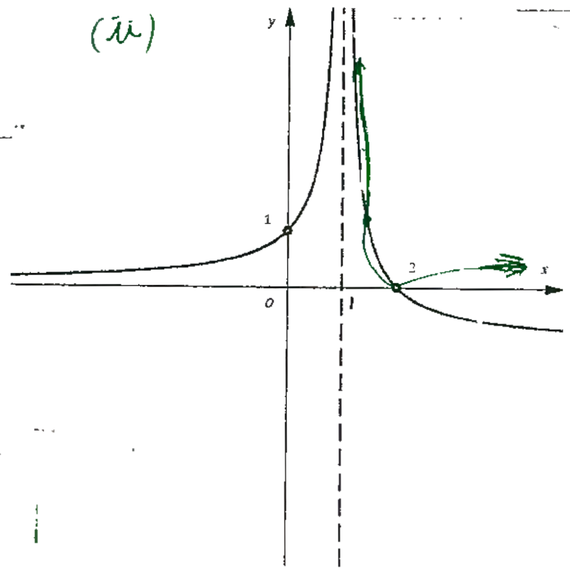
Q15

(a)(i)

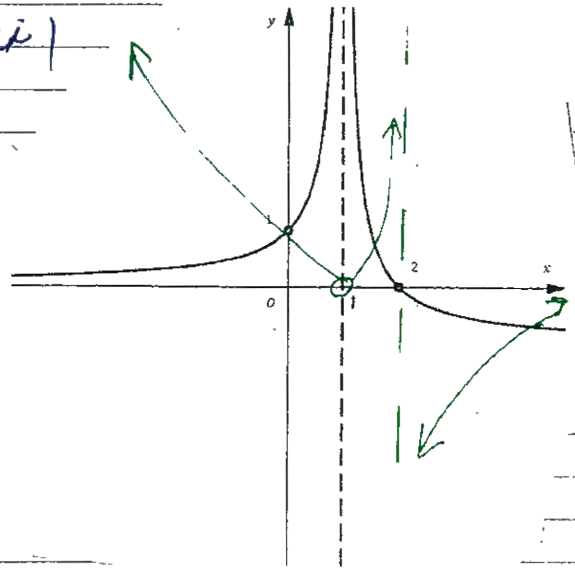


$y = f(x+2)$

(ii)



(iii)



(b) $\cos 3x + 3\sin 2x = 3\cos x$

$4\cos^3 x - 3\cos x + 6\sin x\cos x - 3\cos x = 0$

$4\cos^3 x - 6\cos x + 6\sin x\cos x = 0$

$2\cos x(2\cos^2 x - 3 + 3\sin x) = 0$

$\therefore 2\cos x = 0$ OR $2(1-\sin^2 x) - 3 + 3\sin x = 0$

$\cos x = 0$ $\quad 2 - 2\sin^2 x - 3 + 3\sin x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad 2\sin^2 x - 3\sin x + 1 = 0$

$2\sin x \quad -1$

$\sin x \quad -1$

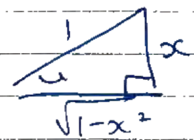
$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ $\quad \sin x = 1$

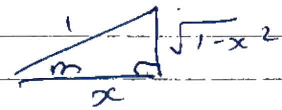
$x = \frac{\pi}{6}, \frac{5\pi}{6}$ $\quad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$

(c) Let $u = \sin^{-1} x$



Let $m = \cos^{-1} x$



$\sin(\sin^{-1} x - \cos^{-1} x)$

$= \sin(u - m)$

$= \sin u \cos m - \cos u \sin m$

$= x \cdot x - (\sqrt{1-x^2} \cdot \sqrt{1-x^2})$

$= x^2 - (1 - x^2)$

$= 2x^2 - 1$ (as required)

(d) $3x^2-6x+10=a(x^2+1)+(bx+c)(x-4)$

Let $x=4$

$34=17a$

$a=2$

Let $x=i$

$-3-6i+10 = 7-6i$

$(bi+c)(i-4) = -b-4bi+ci-4c$

$\therefore 7-6i = (-b-4c)+(c-4b)i$

$-b-4c=7$ $\qquad$ $c-4b=-6$

$-4b-16c=28$
$-4b+c=-6$ } subtract

$-17c=34$

$c=-2$

$-2-4b=-6$

$-4b=-4$

$b=1$

$$\therefore \frac{3x^2-6x+10}{(x-4)(x^2+1)} = \frac{2}{x-4}+\frac{x-2}{x^2+1}$$

(e) $y=mx+k$ $\qquad$ $xy=c^2$

$\therefore x(mx+k)=c^2$

$mx^2+kx-c^2=0$

If tangent $\Delta=0$

$\therefore K^2-4(m)(-c^2)=0$

$\therefore K^2+4mc^2=0$

Q16 (i) $x=ct$ $\qquad$ $y=ct^{-1}$

$\frac{dx}{dt}=c$ $\qquad$ $\frac{dy}{dt}=\frac{-c}{t^2}$

$\frac{dt}{dx}=\frac{1}{c}$

$\therefore \frac{dy}{dx}=\frac{-c}{t^2}\times\frac{1}{c}$

$=\frac{-1}{t^2}$

$y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$

$t^2y-ct=-x+ct$

$x+t^2y=2ct$

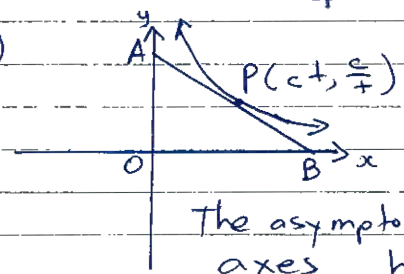
Alternatively $y=c^2x^{-1}$

$\frac{dy}{dx}=\frac{-c^2}{x^2}$ $\quad$ at $x=ct$

$m_T=-\frac{c^2}{c^2t^2}$

$=-\frac{1}{t^2}$

(ii)



The asymptotes are the x & y axes, hence find the y intercept A and x intercept B

$x+t^2y=2ct$

At A, $x=0$ $\quad t^2y=2ct$

$y=\frac{2ct}{t^2}$

$y=\frac{2c}{t}$

At B, $y = 0$ $x = 2ct$

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 2c\cancel{t} \times \frac{2c}{\cancel{t}}$$

$$= 2c^2$$

(which is a constant)

(b) (i) $z = \text{cis}\,\frac{\pi}{K}$

$w = z^n = \text{cis}\,\frac{n\pi}{K}$ (by De Moivres)

$\therefore$ w is obtained by rotating z anticlockwise by $\frac{\pi}{K}$ n times. (or simply $\frac{n\pi}{K}$)

(1)

(ii) $w = z^n$

$$= \left(\cos\frac{\pi}{K} + i\sin\frac{\pi}{K}\right)^n$$

$$= {}^nC_0 \cos^n\frac{\pi}{K} + {}^nC_1 \cos^{n-1}\frac{\pi}{K}\left(i\sin\frac{\pi}{K}\right) + {}^nC_2 \cos^{n-2}\frac{\pi}{K}\left(i\sin\frac{\pi}{K}\right)^2$$

$$+ {}^nC_4 \cos^{n-4}\frac{\pi}{K}\left(i\sin\frac{\pi}{K}\right)^4 + \ldots + {}^nC_n\left(i\sin\frac{\pi}{K}\right)^n$$

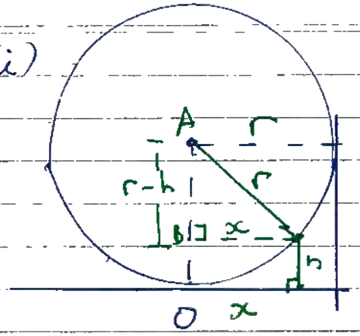
Each even power of n, yields $\left(i\sin\frac{\pi}{K}\right)^{\text{even}}$ which is real.

$$\therefore \text{Re}(w) = {}^nC_0 \cos^n\frac{\pi}{K} + {}^nC_2 \cos^{n-2}\frac{\pi}{K}\, i^2 \sin^2\frac{\pi}{K} +$$

$${}^nC_4 \cos^{n-4}\frac{\pi}{K}\, i^4 \sin^4\frac{\pi}{K} + \ldots (-1)^{\frac{n}{2}}\, {}^nC_n \sin^n\frac{\pi}{K}$$

$$= \sum_{r=0}^{\frac{n}{2}} \binom{n}{2r} (-1)^r \cos^{n-2r}\frac{\pi}{K} \sin^{2r}\frac{\pi}{K}$$

(c) (i)



In ΔABC

$(r-h)^2 + x^2 = r^2$ (By Py thag)

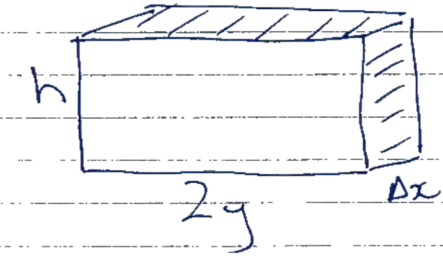
$$(r-h)^2 = r^2 - x^2$$

$$r - h = \sqrt{r^2 - x^2}$$

$$r - \sqrt{r^2 - x^2} = h$$

(as required)

(ii)



$$x^2 + y^2 = r^2$$

$$y = \sqrt{r^2 - x^2}$$ as $y > 0$

$$\therefore \Delta V = 2yh\,\Delta x$$

$$= 2\sqrt{r^2 - x^2}\left(r - \sqrt{r^2 - x^2}\right)\Delta x$$

$$= 2r\sqrt{r^2 - x^2} - 2(r^2 - x^2)\,\Delta x$$

(as required)

(iii) $V = 2\int_{-r}^{r} r\sqrt{r^2 - x^2} - (r^2 - x^2)\,dx$

semicircle

$$= 4r\int_0^r \sqrt{r^2 - x^2} - 4\int_0^r r^2 - x^2\,dx$$

$$= 4r\,\frac{\pi r^2}{4} - 4\left[r^2 x - \frac{x^3}{3}\right]_0^r$$

$$= \pi r^3 - \frac{8r^3}{3}$$