THE HILLS GRAMMAR SCHOOL

STRIVE FOR EXCELLENCE

THE HILLS GRAMMAR SCHOOL

TRIAL HSC
1999

MATHEMATICS

4 UNIT (ADDITIONAL)

TIME ALLOWED: 3 Hours (plus 5 minutes reading time)

Teacher Responsible: Mr D Price

INSTRUCTIONS:

1. Attempt all eight questions.
2. All questions are of equal value.
3. All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
4. Standard Integrals are printed on the last page.
5. Board approved calculators and templates may be used.
6. Start each question on a new page and hand up your papers in one bundle with your name clearly marked on each page.

Question 1

Marks

(a) Let $z_1 = -1 + 3i$ and $z_2 = 1 + i$. 6

(i) Find in the form $a + i\,b$, where a and b are real, the numbers $z_1 z_2$ and $\frac{z_1}{z_2}$.

(ii) (α) Describe geometrically the locus of z on the Argand plane such that

$$|z - z_2| = |z - z_1|$$

(β) Sketch the locus of z on the Argand plane such that

$$\arg\left(\frac{z - z_2}{z - z_1}\right) = \frac{\pi}{2}$$

(b) Consider the equation $z^5 + 1 = 0$. 5

(i) If $w \neq 1$ is a complex root of this equation, prove that $\overline{w}$ is also a root.

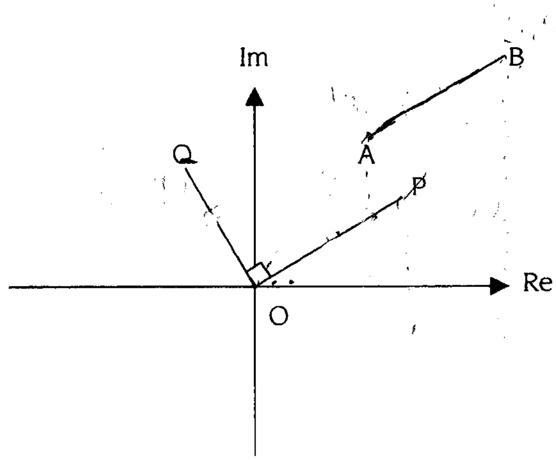
(ii) Find all the five roots of this equation and plot them on an Argand diagram.

(iii) The points representing the roots in your diagrams of (b)(ii) are joined to form a regular pentagon.

Show that the side length of this pentagon is given by $2 \sin \frac{\pi}{5}$.

Question 1 continued **Marks**

(c) 4



In the above Argand diagram, $AB = OP = OQ$, $OP \parallel AB$ and $OP \perp OQ$.
If A represents the complex number $3 + 5i$ and B represents $9 + 8i$ then find the complex number represented by the point:

(i) P

(ii) Q.

Question 2

(a) Find $\int \frac{e^x\,dx}{\sqrt{1 - e^{2x}}}$ 2

(b) Evaluate: 6

(i) $\int_0^1 \tan^{-1} x \; dx$

(ii) $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$

Question 2 continued

(c) (i) Find the values of A, B and C such that 4

$$\frac{3-x}{(1+2x^2)(1+6x)} \equiv \frac{Ax+B}{1+2x^2} + \frac{C}{1+6x}$$

(ii) Hence show that

$$\int_0^2 \frac{3-x}{(1+2x^2)(1+6x)}\,dx = \frac{1}{2}\ln\frac{13}{3}$$

(d) Evaluate $\int_0^{\frac{\pi}{2}} \sin x \cos 2x \; dx$. 3

Question 3

(a) Consider the function $f(x) = \dfrac{e^x - 1}{e^x + 1}$ 10

(i) Show that $f(x)$ is an odd function.

(ii) Show that the function is always increasing.

(iii) Find $f'(0)$.

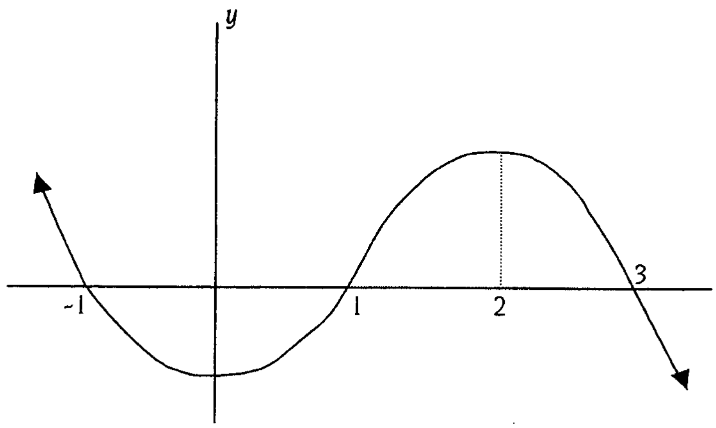
(iv) Discuss the behaviour of $f(x)$ as $x \to \pm\infty$.

(v) Sketch the graph of $y = f(x)$.

(vi) Use your graph to find the values of k for which $\dfrac{e^x - 1}{e^x + 1} = kx$ has three real solutions.

Question 3 continued **Marks**

(b) 5



The graph of the function $y = g(x)$ is sketched above. On a separate number plane diagram, sketch the graphs of:

(i) $y = g(x+1)$

(ii) $y = g(1-x)$

(iii) $|y| = g(|x|)$

Question 4 8

(a) The hyperbola H has equation $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

(i) Sketch H, showing the co-ordinates of its foci and the equation of its directrices and asymptotes.

(ii) $P\,(4\sec\theta\,,\,3\tan\theta\,)$ is a point on H. Perpendiculars from P to the asymptotes meet these lines in M and N. Prove that $PM.PN$ is independent of the position of P.

Question 4 continued | **Marks**

(b) Let $C_1 = x^2 + 3y^2 - 9$ and $C_2 = 2x^2 + y^2 - 3$ and let λ be a real number. 7

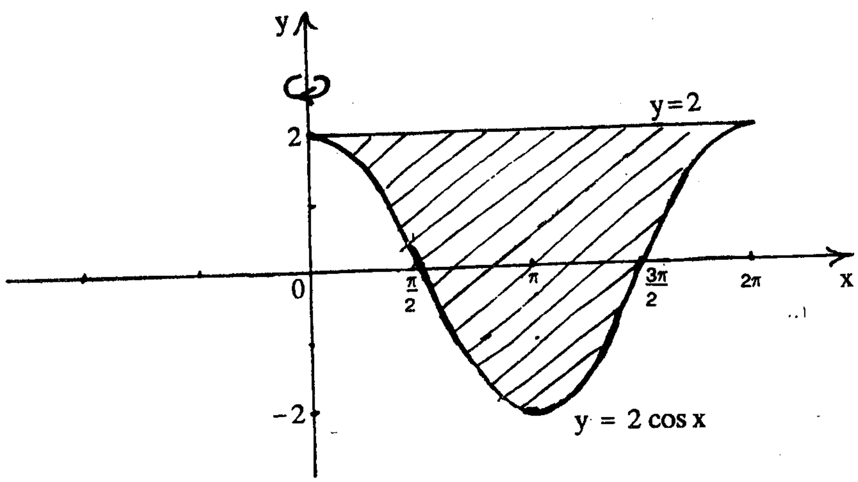
(i) Explain why $C_1 + \lambda C_2 = 0$ is the equation of a curve through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.

(ii) Sketch the curve $C_1 + \lambda C_2 = 0$ when $\lambda = 1$. Indicate on your diagram the positions (with co-ordinates) of the foci and the equations of the directrices.

(iii) Find the equation of the circle which passes through the points of intersection of $C_1 = 0$ and $C_2 = 0$.

Question 5

(a) 8



The shape of the interior of a cake pan is obtained by rotating the region bounded by the curve $y = 2\cos x$ for $0 \le x \le 2\pi$ and the line $y = 2$ through 360° about the y-axis. Use the method of cylindrical shells to show that the volume of the cake pan is given by

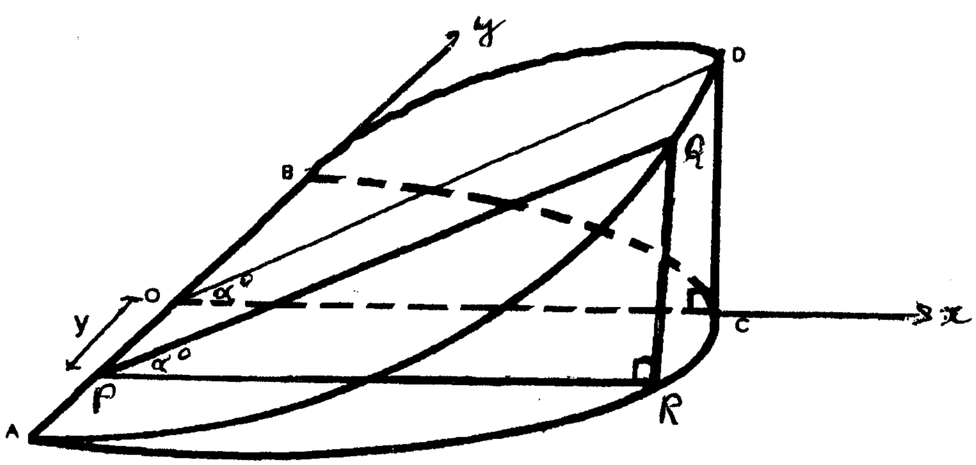
$$4\pi \int_0^{2\pi} x(1 - \cos x)\,dx$$

Hence find this volume.

Question 5 continued **Marks**

(a) The solid S below is a wedge with the following characteristics: 7

- the base of S is half an ellipse with minor axis $AB = 2b$ and semi-major axis $OC = a$.
- cross-sections taken perpendicular to the base and minor axis AB are right triangles (see diagram where a typical cross-section PQR is shown).
- the angle between the two flat surfaces of the wedge is $\alpha\,°$



(i) The cross-section PQR meets the y-axis in y.

Show that the area of triangle PQR is $\dfrac{a^2}{2b^2}(b^2 - y^2)\tan\alpha\,°$.

(ii) Hence, or otherwise, find the volume of S.

Question 6 | **Marks**

(a) A particle of mass m kilograms is given an initial speed of u metres per second and it subsequently moves in a straight line. The only force acting on this particle is a resistive one whose magnitude is $mkv^{\frac{3}{2}}$ Newtons where $k > 0$ is a constant and v metres per second is its speed when it has travelled a distance of x metres. 10

(i) Draw a clear, neat diagram showing all this information.

(ii) Show that $\ddot{x} = -kv^{\frac{3}{2}}$.

(iii) Find v as a function of x.

(iv) Find v as a function of time, t seconds.

(v) Does the particle finally come to rest? Briefly discuss.

(b) A body of unit mass falls under gravity through a resisting medium. The body falls from rest. The resistance to its motion is $\frac{1}{100}v^2$ Newtons where v metres per second in the speed of the body when it has fallen a distance of x metres. 5

(i) Show that the equation of motion of the body is $\ddot{x} = g - \frac{1}{100}v^2$, where g is the magnitude of the acceleration due to gravity.

[Note: Draw a diagram!]

(ii) Show that the terminal speed, V_T, is given by $V_T = 10\sqrt{g}$.

(iii) Show that $V^2 = V_T{}^2(1 - e^{\frac{-x}{50}})$.

Question 7

(a) **(i)** Show that $\dfrac{x^2}{1-x^2} = -1 + \dfrac{1}{1-x^2}$. 6

(ii) Let $I_n = \int_0^1 (1-x^2)^n \, dx$ where n is an integer and $n \geq 0$.

(α) Show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$.

(β) Hence evaluate I_4.

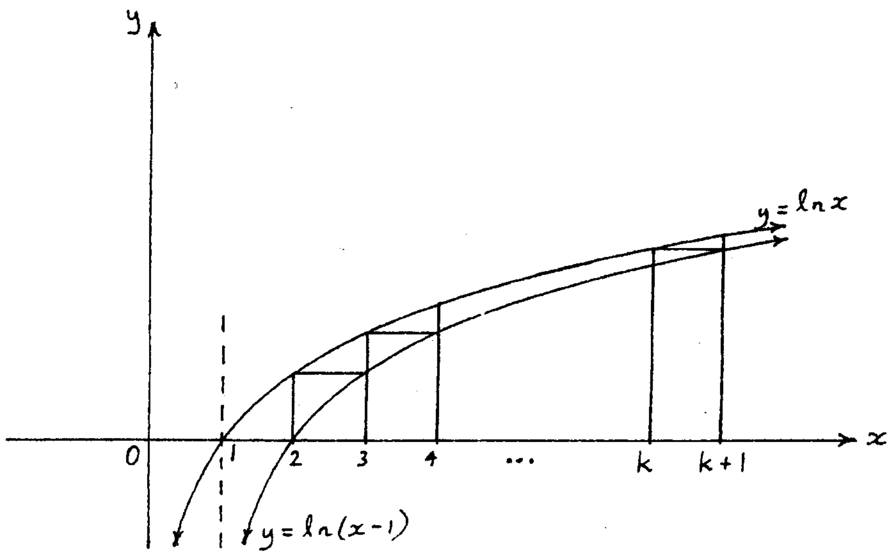
Question 7 continued **Marks**

(b) **(i)** Show that for $x > 0$ 9

$$\int \ln x dx = x \ln x - x$$

(ii) The continuous functions $f(x)$, $g(x)$ are such that $f(x) < g(x)$ for $a \leq x \leq b$. Explain why $\int_a^b f(x)dx < \int_a^b g(x)dx$.

(iii) The diagram below shows a sketch of the curves $y = lnx$ and $y = ln(x-1)$. Also, $(k-1)$ rectangles are constructed, as shown, between $x = 2$ and $x = k + 1$ where $k \geq 2$.



(α) Show that the sum of the areas of the $(k-1)$ rectangles is $ln(k!)$.

(β) Evaluate $\int_2^{k+1} \ln(x-1)dx$ and $\int_2^{k+1} \ln x\ dx$.

(γ) Hence, or otherwise, show that

$$k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1} \text{ for } k \geq 2.$$

Question 8 **Marks**

(a) Let $[x]$ be the largest integer less than or equal to x. For example, $[1 \cdot 3]=1$ and $[-1 \cdot 3]=-2$. 4

(i) Draw the graph of $y=[x]$ for $-1 \leq x \leq 2$.

(ii) Evaluate $\int_{-1}^{2} [x]dx$.

(b) Consider the curve C in the x-y plane defined by $\sqrt{|x|} + \sqrt{y} = 1$. 5

(i) Write down the domain for C.

(ii) For $x > 0$, show that $\frac{dy}{dx} < 0$.

(iii) Sketch a graph of C, paying close attention to the gradient of the curve at $x = 0$.

(c) You may take, without proof, that for any real numbers $a > 0$ and $c > 0$, it follows that $a + c \geq 2\sqrt{ac}$. 6

(i) Prove that for any real $x > 0$ that $x + \frac{1}{x} \geq 2$.

(ii) The real numbers $a > 0$, $b > 0$, $c > 0$ are such that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression.

(α) Show that $b = \frac{2ac}{a+c}$.

(β) By using the result in (b)(i), or otherwise, show that

$$\frac{\sqrt{ac}}{b} \geq 1.$$

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

4 unit.

1999 Trial HSC

a)

(i) $z_1 z_2$

$= (-1+3i)(1+i)$

$= -1-3+i(-1+3)$

$= -4+2i$ 1

ii) $\dfrac{z_1}{z_2}$

$= \dfrac{-1+3i}{1+i}$

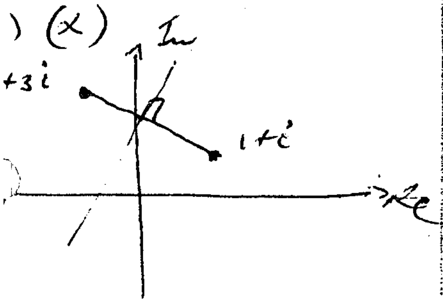
$= \dfrac{(-1+3i)(1-i)}{2}$

$= \dfrac{-1+3+i(3+1)}{2}$

$= \dfrac{2+4i}{2}$

$= 1+2i$ 1

) (α)

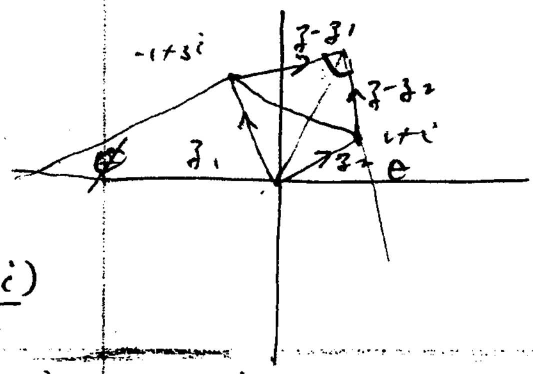


perp bisector of the joining $-1+3i$ to $1+i$ 2

(β)

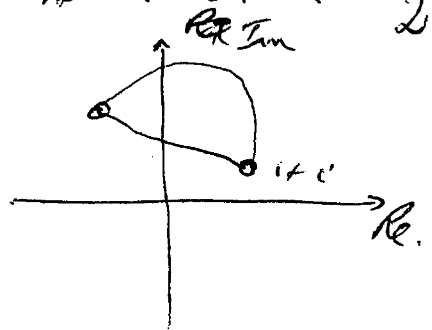
$\arg\left(\dfrac{z-z_2}{z-z_1}\right) = \dfrac{\pi}{2}$

$\arg(z-z_2) - \arg(z-z_1) = \pi/2$



$\theta - \phi = \pi/2$

∴ locus is a semi-circle 2



(b)

(i) as ω is a root ⟹

$\omega^5 + 1 = 0$

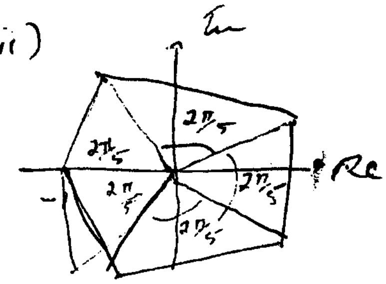
$\overline{\omega^5+1} = \overline{0} = 0$

$\overline{\omega^5} + \overline{1} = 0$

$(\overline{\omega})^5 + 1 = 0$

⟹ $\overline{\omega}$ is a root. 1

(ii)



$z^5 = -1$

$z = -1$

$\text{cis}\left(\pm \frac{\pi}{5}\right)$

$\text{cis}\left(\pm \frac{3\pi}{5}\right)$ 2

(iii)

by cosine rule

$(\text{side length})^2$

$= 1^2 + 1^2 - 2 \times 1 \times 1 \cos\frac{2\pi}{5}$

$= 2\left(1 - \cos\frac{2\pi}{5}\right)$ How?

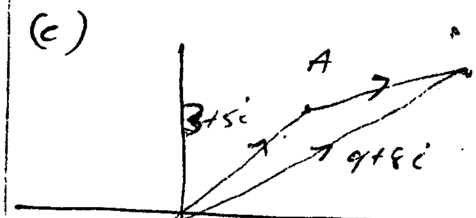
$= 2 \times 2\sin^2\frac{\pi}{5}$

side length

$= \sqrt{4\sin^2\pi/5}$ 2

$= 2\sin\pi/5$

(c)



$\overrightarrow{AB} = (9+8i) - (3+5i)$

$= 6+3i$

)
$P = \overrightarrow{AB}$
$= 6 + 3i$ 2

)
$Q = i \times P$
$= i \times (6 + 3i)$
$= -3 + 6i$ 2

)
$= \int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$

put $u = e^x$
$du = e^x\,dx$

$= \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u + c$

$= \sin^{-1}(e^x) + c$ 2

) (i) $\int_0^1 \tan^{-1} x\,dx$

$= \int_0^1 \tan^{-1} x \frac{d(x)}{dx}\,dx$

$= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$\frac{\pi}{4} - \left[\frac{1}{2} \ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ 3

(ii) $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$= \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_{-1}^{1}$

$= \frac{1}{2}\tan^{-1} 1 = \frac{\pi}{8}$ 3

(c)
$(Ax+B)(1+6x) + c(1+2x^2) \equiv 3-x$

$x = -\frac{1}{6}$
$c\left(1 + \frac{2}{36}\right) = 3 + \frac{1}{6}$

$\frac{19}{18} c = \frac{19}{6}$

$c = 3$

coefft of x^2
$6A + 2c = 0$
$3A + c = 0$
$3A = -3$
$A = -1$

put $x = 0$
$B + c = 3$
$B + 3 = 3$

$B = 0$ 2

(ii)
$\int_0^2 \frac{3-x}{(1+2x^2)(1+6x)}\,dx$

$= \int_0^2 \frac{-x}{1+2x^2}\,dx + \int_0^2 \frac{3}{1+6x}\,dx$

$= \left[-\frac{1}{4}\ln(1+2x^2)\right]_0^2 + \frac{1}{2}\left[\ln(1+6x)\right]_0^2$

$= -\frac{1}{4}\ln 9 + \frac{1}{4}\ln 1 + \frac{1}{2}\ln 13 - \frac{1}{2}\ln 1$

$= \frac{1}{2}\ln 13 - \frac{1}{2}\ln 9^{\frac{1}{2}}$

$= \frac{1}{2}\ln \frac{13}{3}$ 2

(d) $\int_0^{\pi/2} \sin x \cos 2x\,dx$

$= \int_0^{\pi/2} \sin x\left[2\cos^2 x - 1\right]dx$

$= 2\int_0^{\pi/2} \sin x \cos^2 x\,dx - \int_0^{\pi/2} \sin x\,dx$

$\left[-\frac{2}{3}\cos^3 x\right]_0^{\pi/2} + \left[\cos x\right]_0^{\pi/2}$

$\frac{2}{3} - 0) + (0 - 1)$

$-\frac{1}{3}$ 3

)

$) \; f(x) = \dfrac{e^x - 1}{e^x + 1}$

$(-x) = \dfrac{e^{-x} - 1}{e^{-x} + 1} \times \dfrac{e^x}{e^x}$

$= \dfrac{1 - e^x}{1 + e^x}$

$= -\dfrac{e^x - 1}{e^x + 1}$

$= -f(x)$ 1

$f'(x) = \dfrac{e^x(e^x+1) - (e^x-1)e^x}{(e^x+1)^2}$

$\dfrac{2e^x}{(e^x+1)^2}$

$e^x > 0$

$(e^x+1)^2 > 1$

$f'(x) > 0$

$\Rightarrow f(x)$ is always increasing 2

(iii) $f'(0) = \dfrac{2e^0}{(e^0+1)^2}$

1 $= \dfrac{2}{2^2} = \dfrac{1}{2}$

(iv)

$f(x) = \dfrac{e^x - 1}{e^x + 1}$

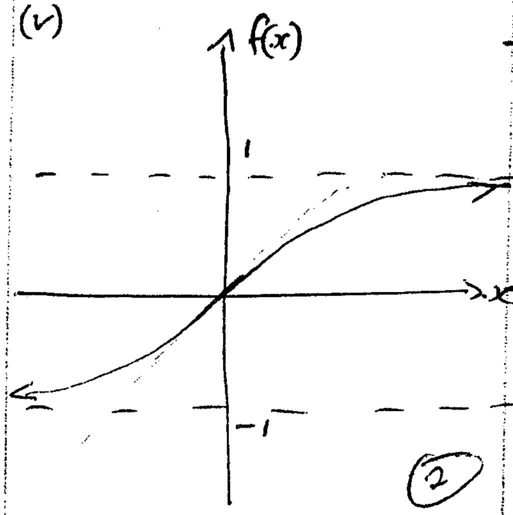
$= \dfrac{1 - e^{-x}}{1 + e^{-x}}$

$\longrightarrow 1^-$ as $x \longrightarrow \infty$

$f(x) = \dfrac{e^x - 1}{e^x + 1}$ 2

$\longrightarrow -1^+$ as $x \longrightarrow -\infty$

(v)



2

(vi) need to solve

$y = f(x)$

$y = kx$

Inflexion pt at $(0,0)$ where $f'(0) = \frac{1}{2}$

This gives maximum tangent slope —

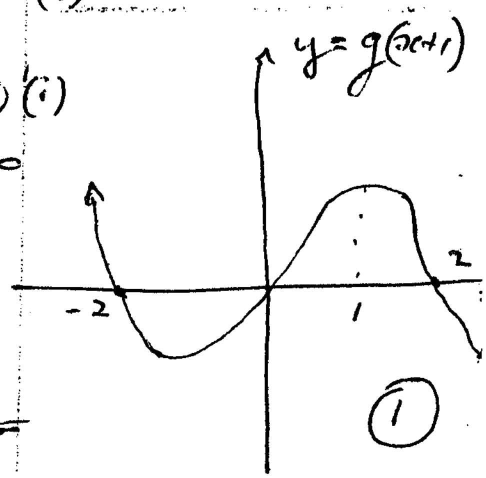
$y = \frac{1}{2}x$ cuts $y = f(x)$ in $(0,0)$ only.

equation will have three solutions if and only if

$0 < k < \frac{1}{2}$ 2

(b)

(i)

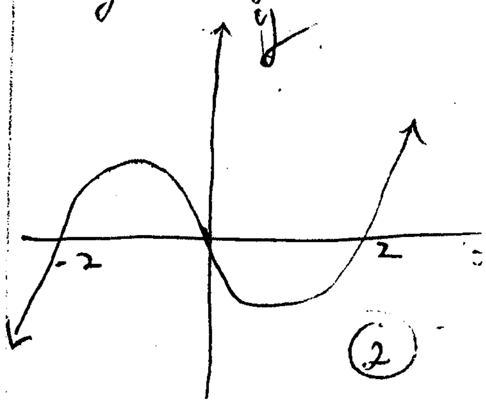


1

(ii)

$g(x) \xrightarrow{\text{1 left}} g(x+1)$

$\xrightarrow[\text{in y-axis}]{\text{reflect}} g(-x+1)$



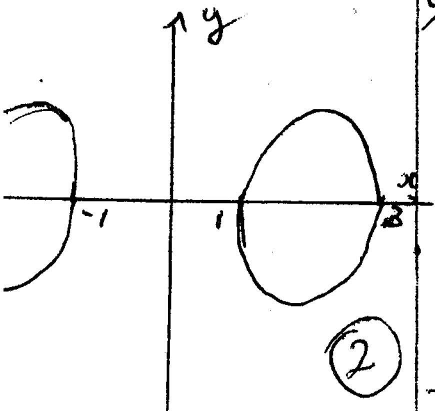
2

$|y| = g(|x|)$

If (x, y) lies on graph then so does $(-x, y)$, $(x, -y)$, $(-x, -y)$

Let $x > 0$, $y > 0$

$y = g(x)$



2

$\frac{x^2}{16} - \frac{y^2}{9} = 1$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$ 1

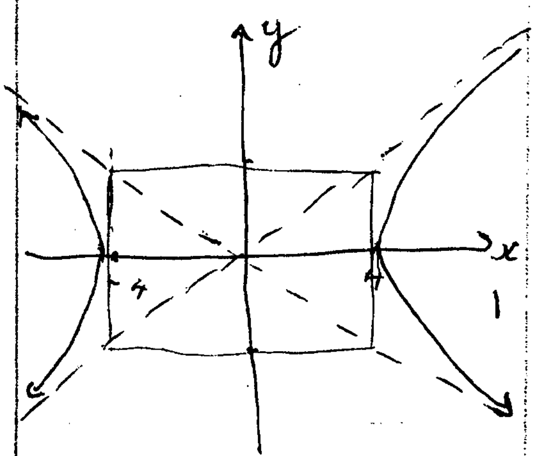
$(\pm ae, 0) = (\pm 5, 0)$

directrices

$x = \pm \frac{a}{e} = \pm \frac{16}{5} = \pm 3\frac{1}{5}$ 1

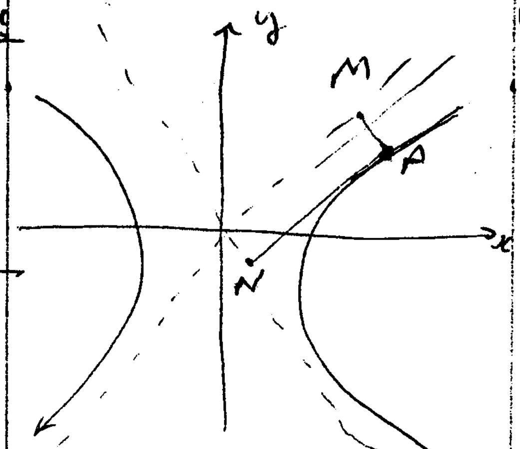
asymptotes

$y = \pm \frac{b}{a} x = \pm \frac{3}{4} x$ 1



1

(ii) $P(4\sec\theta, 3\tan\theta)$



asymptotes

$3x + 4y = 0$

$3x - 4y = 0$

$PM = \frac{|12\sec\theta - 12\tan\theta|}{\sqrt{3^2 + 4^2}}$

$= \frac{12|\sec\theta - \tan\theta|}{5}$

$PN = \frac{12|\sec\theta + \tan\theta|}{5}$

$PM \cdot PN$

$= \frac{144|\sec^2\theta - \tan^2\theta|}{25}$

$= \frac{144}{25}$

this is a constant hence independent of θ. 4

(b)

(i) Let (x_0, y_0) be a point of intersection of $C_1 = 0$ and $C_2 = 0$

$\therefore C_1(x_0, y_0) = 0$

and $C_2(x_0, y_0) = 0$

$\therefore C_1(x_0, y_0) + \lambda C_2(x_0, y_0) =$

$\therefore (x_0, y_0)$ lies on $C_1 + \lambda C_2 = 0$ 2

(ii)

$x^2 + 3y^2 - 9 + 2x^2 + y^2 - 3 =$

$3x^2 + 4y^2 = 12$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Ellipse

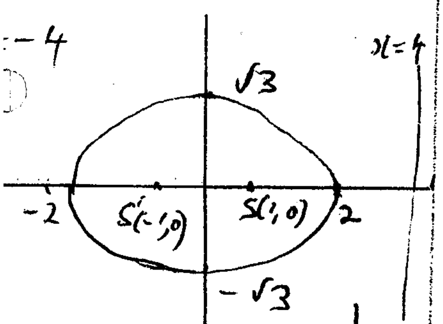
$\frac{3}{4} = 1-e^2$

$e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$ 1

$(\pm ae, 0) = (\pm 1, 0)$

directrices $x = \pm \frac{a}{e}$

$= \pm 4$ 1



1

)

curve is

$x^2 + 3y^2 - 9 + \lambda(2x^2 + y^2 - 3) = 0$

$(1+2\lambda) x^2 + (3+\lambda) y^2 = 9 + 3\lambda$

for circle

$1 + 2\lambda = 3 + \lambda$

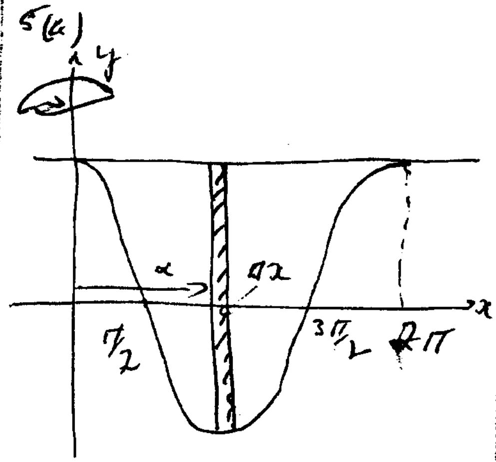
$\underline{\underline{\lambda = 2}}$

equation is

$5x^2 + 5y^2 = 15$

$x^2 + y^2 = 3$

2



$\Delta V = 2\pi x (2 - 2\cos x)\,\Delta x$ + error 1

$V = \lim_{n\to\infty} \sum_n \Delta V$ 1

n = no. of shells

$= \int_0^{2\pi} 2\pi x (2 - 2\cos x)\,dx$ 1

$= 4\pi \int_0^{2\pi} x(1-\cos x)\,dx$

$= 4\pi \int_0^{2\pi} x\,dx - 4\pi \int_0^{2\pi} x\cos x\,dx$ 1

$= 4\pi \left[\frac{x^2}{2}\right]_0^{2\pi} - 4\pi \times$

$= \underline{\underline{8\pi^3 \text{ unit}^3}}$ 1

Now

$\int_0^{2\pi} x \cos x \,dx$

$= \int_0^{2\pi} x \frac{d}{dx}(\sin x)\,dx$

$= \left[x \sin x\right]_0^{2\pi} - \int_0^{2\pi} \sin x\,dx$

$= 0 + \left[\cos x\right]_0^{2\pi}$

$= 0 + 0 = 0$

(b)

(i)

ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

∴ for solid S

$x = \sqrt{a^2\left(1 - \frac{y^2}{b^2}\right)}$

$= \frac{a}{b}\sqrt{(b^2 - y^2)}$

$QR = x \tan\alpha°$ 1

Area of ΔPQR

$= \frac{1}{2} PR \cdot QR$

$= \frac{1}{2} \frac{a}{b}\sqrt{b^2 - y^2}$

$\times \frac{a}{b}\sqrt{b^2 - y^2} \tan\alpha$ 1

$= \frac{a^2}{2b^2}(b^2 - y^2) \times \tan\alpha°$

)
taking slice at position $y \Rightarrow$

$\Delta V = \frac{a^2}{2b^2}(b^2-y^2)\tan\alpha^\circ$
$\times \Delta y$ + errors 1

$= \lim_{n\to\infty} \sum_n \Delta V$

n = no. of slices

$= \int_{-b}^{b} \frac{a^2}{2b^2}(b^2-y^2)\tan\alpha^\circ \times dy$ 1

$\frac{a^2}{2b^2}\tan\alpha^\circ \int_{-b}^{b}(b^2-y^2)dy$

[illegible]

$\frac{a^2}{b^2}\tan\alpha^\circ \int_{0}^{b}(b^2-y^2)dy$

as (b^2-y^2) is even

$\frac{a^2}{b^2}\tan\alpha^\circ\left[b^2y-\frac{y^3}{3}\right]_0^b$ 1

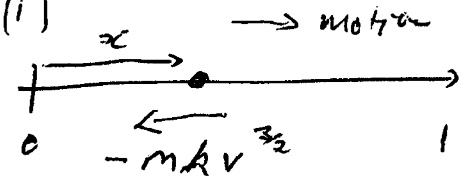
$\frac{a^2}{b^2}\tan\alpha^\circ\left[b^3-\frac{b^3}{3}\right]$

$\frac{a^2}{b^2}\tan\alpha^\circ\times\frac{2b^3}{3}$

$= \frac{2a^2b\tan\alpha^\circ}{3}$ unit^3 1

6 (a)

(i)



(ii) $\sum F = -mkv^{3/2}$

By Newton's second law

$\sum F = \frac{d}{dt}(mv)$

$-mkv^{3/2} = m\frac{dv}{dt}$

$\therefore \ddot{x} = -kv^{3/2}$ 1

(iii)

$\ddot{x} = v\frac{dv}{dx}$

assuming $v = v(x)$

$v\frac{dv}{dx} = -kv^{3/2}$

$\frac{dv}{dx} = -kv^{1/2}$

$\int \frac{dv}{v^{1/2}} = -k\int dx$

$2v^{1/2} = -kx + c$

at $x=0$ $v=u$

$2\sqrt{u} = c$

$2\sqrt{v} = -kx + 2\sqrt{u}$

$\sqrt{v} = \sqrt{u} - \frac{k}{2}x$

$v = \left(\sqrt{u} - \frac{k}{2}x\right)^2$ 3

(iv)

$\frac{dv}{dt} = -kv^{3/2}$

$\int\frac{dv}{v^{3/2}} = -k\int dt$

$-2v^{-1/2} = -kt + c$

$2v^{-1/2} = kt + c$

$t=0$, $v=u$

$2u^{-1/2} = c$

$2v^{-1/2} = kt + 2u^{-1/2}$

$\frac{2}{\sqrt{v}} = kt + \frac{2}{\sqrt{u}}$

$\frac{\sqrt{v}}{2} = \frac{1}{kt + \frac{2}{\sqrt{u}}}$

$= \frac{\sqrt{u}}{2 + k\sqrt{u}\,t}$

$' = \dfrac{2\sqrt{u}}{2 + k\sqrt{u}\, t}$

$= \dfrac{4u}{(2 + k\sqrt{u}\, t)^2}$ 3

) $v = \left(\sqrt{u} - \frac{k}{2}x\right)^2$

When $V = 0$

$\sqrt{u} - \frac{k}{2}x = 0$

$\Rightarrow x = \dfrac{2\sqrt{u}}{k}$

But

$v = \dfrac{4u}{(2 + k\sqrt{u}\, t)^2}$

> 0

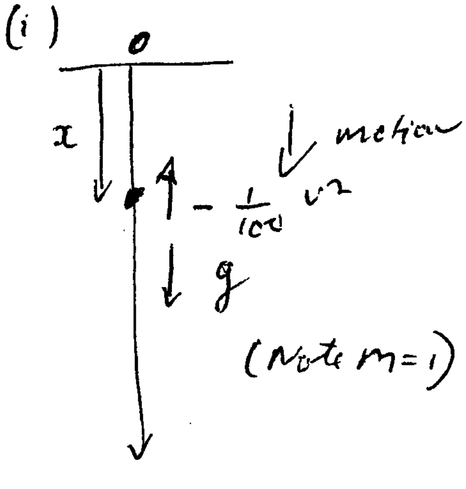
for all t

but as $t \to \infty$, $V \to 0$

So particle never comes to rest but is bounded by $x = \dfrac{2\sqrt{u}}{k}$ 2

(b)

(i)



By Newton's 2nd Law

$\Sigma F = \frac{d}{dt}(mv)$ $\quad m=1$

$g - \frac{1}{100}v^2 = \frac{dv}{dt}$ 1

$\ddot{x} = g - \frac{1}{100}v^2$

(ii) at terminal ~~velocity~~ speed

$\Sigma F = 0$

$\therefore g - \frac{1}{100}V_T^2 = 0$

$V_T^2 = 100g$

$= \sqrt{100g}$

$= 10\sqrt{g}$ 1

(iii)

$V\frac{dV}{dx} = g - \frac{1}{100}V^2$

$\displaystyle\int \frac{V\,dV}{g - \frac{1}{100}V^2} = \int dx$

$= x + c$

LHS $= \displaystyle\int \frac{100V}{100g - V^2}$

$= \displaystyle\int \frac{100V}{V_T^2 - V^2}$

$= -\frac{100}{2}\ln(V_T^2 - V^2)$

$\therefore -50\ln(V_T^2 - V^2) = x + c$

~~when $x=0$, $V=0$~~

$\cancel{c = -50\ln V_T^2}$

$\cancel{-50\ln(V_T^2 - x^2)}$

$V_T^2 - V^2 = Ae^{-x/50}$

$V^2 = V_T^2 - Ae^{-x/5}$

when $x = 0$, $V = 0$

$0 = V_T^2 - A$

$\Rightarrow A = V_T^2$

$v^2 = V_t^2\left(1 - e^{-x/50}\right)$

3

LHS $= -1 + \frac{1}{1-x^2}$

$= \frac{-1+x^2+1}{1-x^2}$

$= \frac{x^2}{1-x^2} =$ LHS (1)

$I_n = \int_0^1 (1-x^2)^n$

$= \int_0^1 (1-x^2)^n \frac{d}{dx}(x)\,dx$

$= \left[x(1-x^2)^n\right]_0^1$

$- \int_0^1 x \times n(1-x^2)^{n-1} \times (-2x)\,dx$

$2n\int_0^1 x^2(1-x^2)^{n-1}$

$2n \int_0^1 \frac{x^2(1-x^2)^n}{(1-x^2)}\,dx$

$2n \int_0^1 \left[-1 + \frac{1}{1-x^2}\right](1-x^2)^n\,dx$

$-2n\int_0^1 (1-x^2)^n\,dx$

$\int_0^1 (1-x^2)^{n-1}$

$\therefore I_n = -2nI_n + 2nI_{n-1}$

$(2n+1)I_n = 2nI_{n-1}$

$I_n = \frac{2n}{2n+1} I_{n-1}$ (3)

(β)

$I_4 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} I_0$

$I_0 = \int_0^1 1\,dx$

$= [x]_0^1 = 1$

$I_4 = \frac{128}{315} = 0.$ (2)

or $0.4\dot{0}6349\dot{2}$

(b)

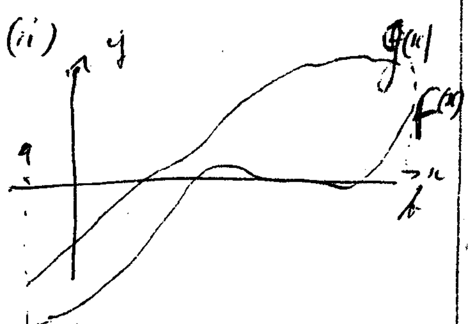
$\frac{d}{dx}(x\ln x - x)$

$= 1 \times \ln x + x \times \frac{1}{x} - 1$

$= \ln x$

provided $x > 0$ so that $\ln x$ is defined. (1)

(ii)



$\int_a^b (g(x) - f(x))\,dx$

= Area enclosed between curves > 0

$\therefore \int_a^b (g(x) - f(x))\,dx > 0$

$\int_a^b g(x)\,dx - \int_a^b f(x)\,dx > 0$

$\therefore \int_a^b f(x)\,dx < \int_a^b g(x)\,dx$ (1)

(iii)

Area of the rectangles

$= 1\times\ln 2 + 1\times\ln 3 + 1\times\ln 4 + \dots + 1\times\ln k$

$= \ln(2\times3\times4\times\dots k)$

$= \ln(1\times2\times3\times\dots\times k)$

$= \ln(k!)$ (2)

(iv)

3) $\int_2^{k+1} \ln(x-1)\,dx = \int_1^{k} \ln u\, du = \left[u \ln u - u\right]_1^{k}$

(put $u = x-1$) $= k\ln k - k + 1$

$\int_2^{k+1} \ln x\,dx = \left[x\ln x - x\right]_2^{k+1} = \{(k+1)\ln(k+1) - k - 1\}$
$- \{2\ln 2 - 2\}$ ①
$= (k+1)\ln(k+1) - 2\ln 2 - k + 1$

4) Now from (b)(ii) $\ln(x-1) \leq$ rectangle step fn $\leq \ln x$

Now integrating (and using this concept)

$\int_2^{k+1} \ln(x-1)\,dx < \ln(k!) < \int_2^{k+1} \ln x\,dx.$

$k\ln k - k + 1 < \ln(k!) < (k+1)\ln(k+1) - 2\ln 2 - k + 1$

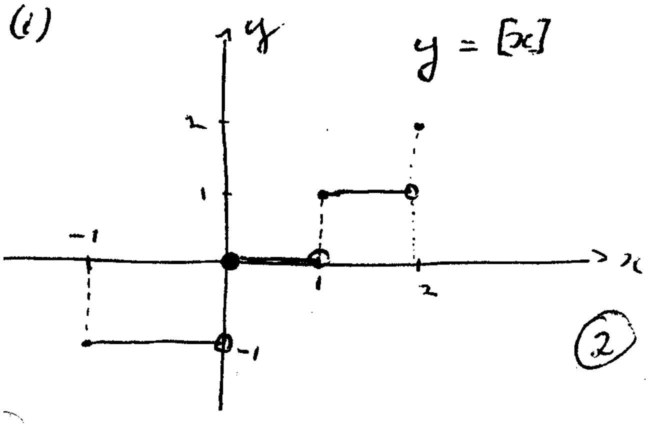
$\ln k^k < \ln(k!) + k - 1 < \ln\left[\frac{1}{4}(k+1)^{k+1}\right]$

$\ln k^k < \ln(k!\,e^{k-1}) < \ln\left[\frac{1}{4}(k+1)^{k+1}\right]$

$\Rightarrow$ as $\ln x$ is an invertible function

then $k^k < k!\,e^{k-1} < \frac{1}{4}(k+1)^{k+1}$ ④

(i)



(2)

(ii) $\int_{-1}^{2} [x]\,dx = -1 + 0 + 1 = 0$ (2)

(i) $0 \le |x| \le 1$ as $\sqrt{y} > 0$

$\Rightarrow -1 \le x \le 1$ (1)

(ii) If $x > 0$ then

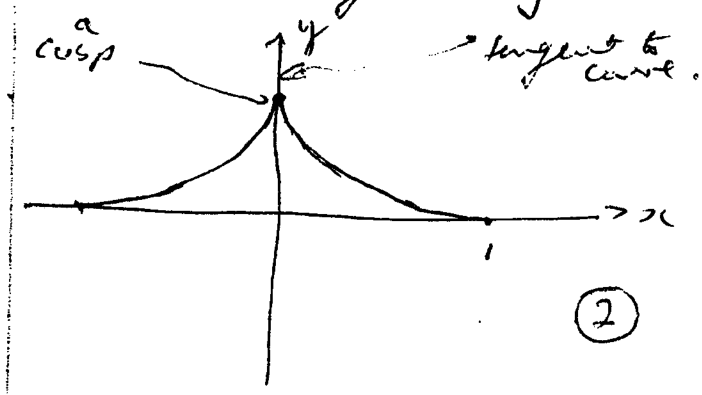
$\sqrt{x} + \sqrt{y} = 1$

$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} < 0$

as $\sqrt{y}, \sqrt{x}$ are both > 0 (2)

(iii) If (x, y) is on curve then so is $(-x, y) \Rightarrow$ y-axis is an axis of symmetry.



(2)

(c)

(i) Let $a = x$ and $c = \frac{1}{x}$

$\therefore x + \frac{1}{x} \ge 2\sqrt{x \cdot \frac{1}{x}} = 2$

$x + \frac{1}{x} \ge 2$ (1)

(ii)

(α) $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} = \frac{c+a}{ac}$

$\frac{b}{2} = \frac{ac}{a+c}$

$b = \frac{2ac}{a+c}$ (2)

(β) $\frac{\sqrt{ac}}{b} = \sqrt{ac} \times \frac{a+c}{2ac}$

$= \frac{1}{2} \frac{a+c}{\sqrt{ac}}$

$= \frac{1}{2}\left[\frac{a}{\sqrt{ac}} + \frac{c}{\sqrt{ac}}\right] = \frac{1}{2}\left[\frac{\sqrt{a}}{\sqrt{c}} + \frac{\sqrt{c}}{\sqrt{a}}\right]$

Let

letting $x = \frac{\sqrt{a}}{\sqrt{c}}$ in c(i) we get

$$\frac{\sqrt{a}}{\sqrt{c}} + \frac{\sqrt{c}}{\sqrt{a}} \geq 2$$

$$\therefore \frac{\sqrt{ac}}{b} \geq \frac{1}{2} \times 2$$

$$\frac{\sqrt{ac}}{b} \geq 1$$

2