THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2004

MATHEMATICS

EXTENSION 2

Time Allowed: Three hours (plus 5 minutes reading time)

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Attempt all questions
- All questions are of equal value
- All necessary working should be shown
- Approved calculators may be used
- Start each question with a new booklet

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

E2
chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E3
uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

E4
uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

E6
combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

E7
uses the techniques of slicing and cylindrical shells to determine volumes

E8
applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

E9
communicates abstract ideas and relationships using appropriate notation and logical argument

Question One Marks

(a) Find:

(i) $\int x\sqrt{x^2-5}\,dx$ 2

(ii) $\int (1-x^2)^3\,dx$ 2

(b) Evaluate:

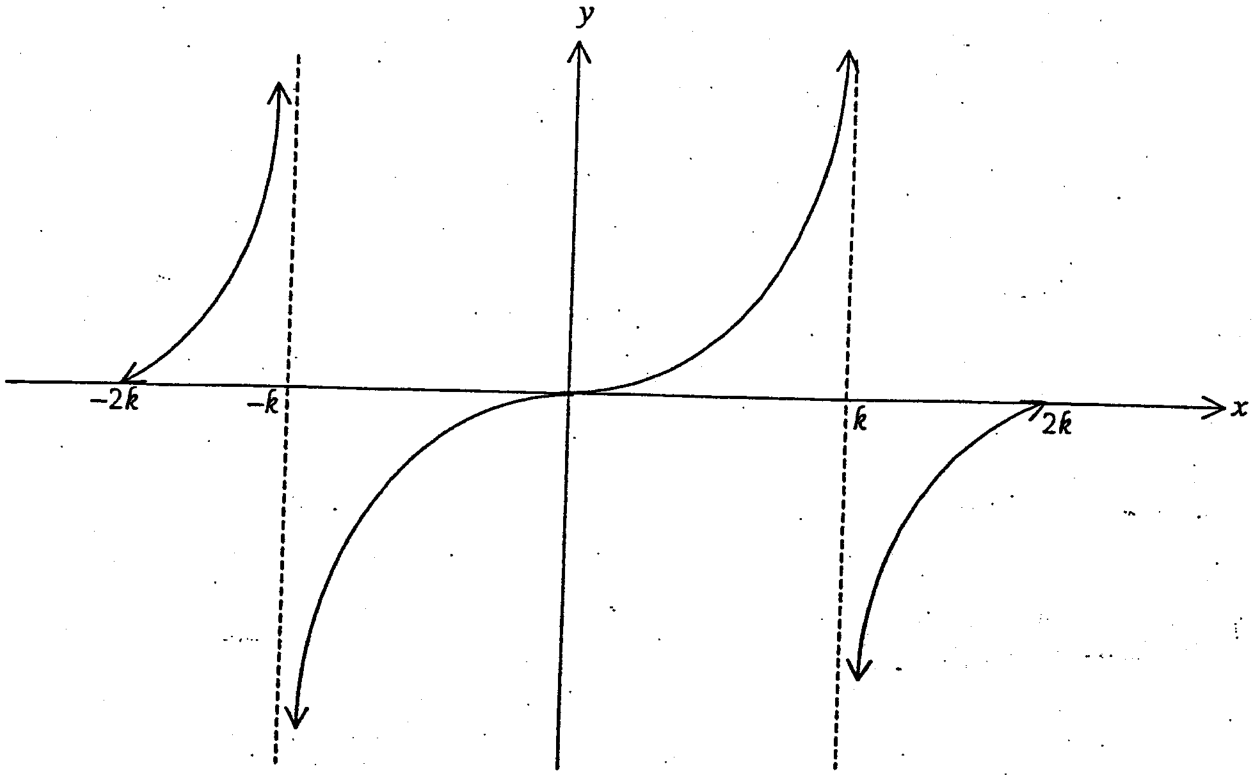
(i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$ 3

(ii) $\int_0^1 \frac{x^2-5x-2}{(2-x)(4+x^2)}\,dx$ 4

(iii) $\int_1^2 x^2.e^x\,dx$ 4

Question Two **(Start a new booklet)**

(a) The graph of $f(x)$ is shown below:



Draw neat sketches of the following: 8

(i) $y = f(x-k)$

(ii) $y = [f(x)]^2$

(iii) $y = \dfrac{1}{f(x)}$

(iv) $y = f'(x)$

(b) Consider the curve $y = 4x^2(2-x^2)$.

(i) Sketch the curve, clearly indicating the important features. 4

(ii) Hence sketch the curve $y^2 = 4x^2(2-x^2)$ 1

(iii) Sketch the curve $y = \log_e 4x^2(2-x^2)$ 2

Question Three (Start a new booklet)

(a) Express $w = 1 + i$ and $z = \sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$.

Hence find the modulus and argument of

(i) wz

(ii) $w^{-1}z$ 4

(b) An equilateral triangle has its vertices on the circle $|z| = 2$. One vertex is the point representing $\sqrt{3} + i$. Find the other two vertices and make a neat sketch. 4

(c) Solve for z:

$$\frac{z - 2i}{1 + iz} = \frac{4}{3}$$

expressing your answer in modulus-argument form. 4

(d) Shade the region of the Argand diagram consisting of those points z for which

(i) $R(z) \le 2$ and $I(z) > -1$ 1

(ii) $|z - 1 - i| \le 1,\quad 0 \le \arg z \le \frac{\pi}{4}$ 2

Question Four (Start a new booklet)

(a) Given that $P(x)=(x^4-1)(x^2-2)$, factorise $P(x)$ completely over:

(i) The real numbers R

(ii) The complex numbers C. 3

(b) (i) If $x=\alpha$ is a double root of the polynomial equation $Q(x)=0$, show that $x=\alpha$ is a root of the equation $Q'(x)=0$. 2

(ii) If the polynomial $P(x)=x^4+x^2+6x+4$ has a rational zero of multiplicity 2, find all the zeros of $P(x)$ over the complex field. 4

(c) Consider the polynomial $P(x)=x^4-4x^3+11x^2-14x+10$.

(i) If $P(x)$ has roots $a+bi$ and $a-2bi$ [where a and b are real], find the values of a and b. 3

(ii) Hence find the zeros of $P(x)$ over the complex field and express $P(x)$ as the product of two quadratic factors. 3

Question Five (Start a new booklet)

(a) A solid has its base in the shape of an ellipse with major axis 8 units and minor axis 6 units. If every section perpendicular to the major axis is an equilateral triangle, show that the volume of the solid formed is $48\sqrt{3}$ cubic units. 5

(b) Using the method of cylindrical shells find the volume of the solid of revolution obtained by rotating about the y-axis, the region bounded by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{2}$. 5

(c) Find the volume obtained by rotating the area enclosed by the x-axis, the curve $y = \tan^{-1} x$ and the line $x = 1$ about the line $x = 1$. 5

Question Six **(Start a new booklet)**

(a) Consider the curves $\frac{x^2}{16}+\frac{y^2}{7}=1$ and $x^2-\frac{y^2}{8}=1$.

(i) Show that both curves have the same foci. **4**

(ii) Find the equation of the circle through the points of intersection of the two curves. **4**

(b) (i) Show that the tangent to the ellipse $\frac{x^2}{12}+\frac{y^2}{4}=1$ at the point $P(3, 1)$ has equation $x+y=4$. **3**

(ii) If this tangent cuts the directrix in the fourth quadrant at the point T, and S is the corresponding focus, show that SP and ST are at right angles to each other. **4**

Question Seven (Start a new booklet)

(a) Consider the function $f(x) = \dfrac{3x}{(x-1)(4-x)}$.

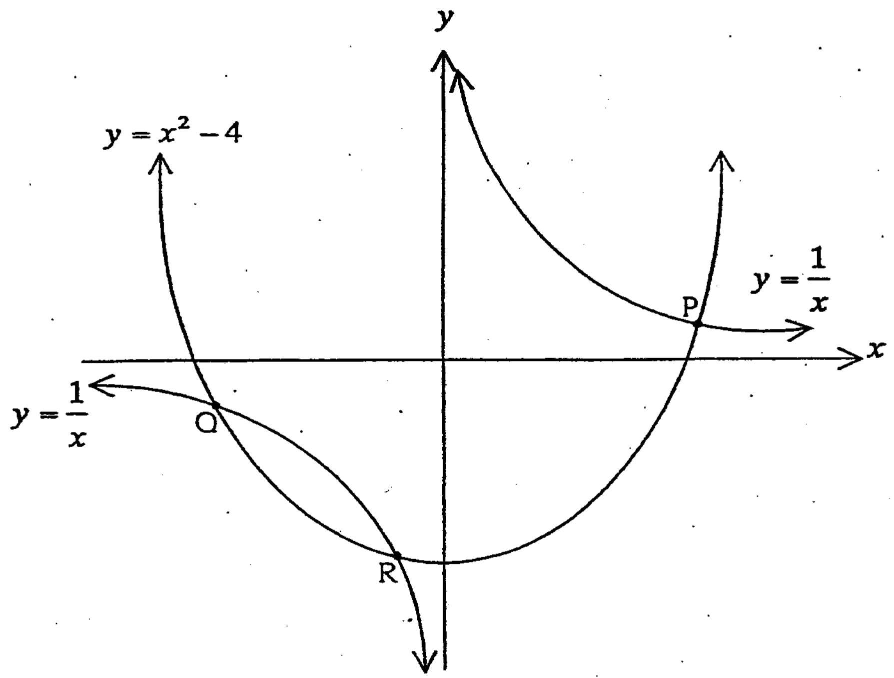
(i) Express $f(x)$ in partial fractions. 2

(ii) Find the co-ordinates and the nature of any turning points of the graph $y = f(x)$. 4

(iii) Sketch the graph of $y = f(x)$ showing clearly the co-ordinates of any turning points and the equation of asymptotes. 3

(iv) Find the area of the region bounded by the curve $y = f(x)$ and the x-axis between the lines $x = 2$ and $x = 3$. 2

(b)



The curves $y = x^2 - 4$ and $y = \dfrac{1}{x}$ intersect at the points P, Q and R whose x- coordinates are α, β and λ respectively.

(i) Show that α, β and λ are roots of the equation $x^3 - 4x - 1 = 0$. 1

(ii) Find a polynomial equation which has roots α^2, β^2 and λ^2. 3

Question Eight **(Start a new booklet)**

(a) Use the substitution $x = \frac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\, dx = \frac{\pi}{3}$.

Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$. 5

(b) (i) If $z = \cos\theta + i\sin\theta$ use de Moivres Theorem to show that

$z^n + \frac{1}{z^n} = 2\cos n\theta$.

(ii) By expanding $\left(z + \frac{1}{z}\right)^4$ show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. 5

(c) Show that $\frac{d}{dx}\left[\ln(x + \sqrt{x^2 + 4})\right] = \frac{1}{\sqrt{x^2 + 4}}$.

Hence or otherwise, prove that $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x + 4}} = 2\ln\left(\frac{\sqrt{5} + 1}{2}\right)$. 5

END OF PAPER

EXTENSION 2 TRIAL 2004 SOLUTIONS

QUESTION ONE

i) $\int x\sqrt{x^2-5}\,dx$

$= \int x(x^2-5)^{1/2}\,dx$ ✓

$= \frac{1}{3}(x^2-5)^{3/2}+c$ ✓ (2)

ii) $\int (1-x^2)^3\,dx$

$\int 1+3x^4-3x^2-x^6$ ✓

$\Rightarrow x+\frac{3x^5}{5}-x^3-\frac{x^7}{7}+c$ ✓ (2)

Evaluate

i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta$

let $t=\tan\frac{\theta}{2}$

$\frac{d\theta}{dt}=\frac{2}{1+t^2}$

$\theta=\frac{\pi}{2}\ \therefore t=1$

$\theta=0\ \therefore t=0$ ✓

$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\frac{d\theta}{dt}\,dt$

$\int_0^1 \frac{1}{1+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$ ✓

$\int_0^1 \frac{2}{1+t^2+2t}\,dt$

$\int_0^1 \frac{2}{(1+t)^2}\,dt$ (3)

$2\int_0^1 (1+t)^{-2}\,dt$

$\left[\frac{-2}{1+t}\right]_0^1=1$ ✓

ii) $\int \frac{x^2-5x-2}{(2-x)(4+x^2)}\,dx$

$\frac{x^2-5x-2}{(2-x)(4+x^2)}\equiv\frac{a}{2-x}+\frac{bx+c}{4+x^2}$

$x^2-5x-2\equiv a(4+x^2)+(bx+c)(2-x)$ ✓

$x^2-5x-2\equiv 4a+ax^2+2bx-bx^2+2c-cx$

$x^2-5x-2\equiv(a-b)x^2+(2b-c)x+4a+2c$

$\therefore a-b=1,\quad 2b-c=-5,\quad 4a+2c=-2$

Solve simultaneously

$a=-1\quad b=-2\quad c=1$

$\int_0^1 \frac{-1}{2-x}-\frac{2x-1}{4+x^2}\,dx$

$\int \frac{-1}{2-x}-\frac{2x}{4+x^2}+\frac{1}{4+x^2}\,dx$ ✓

$\Rightarrow\left[\ln(2-x)-\ln(4+x^2)+\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^1$

$\left[\ln 1-\ln 5+\frac{1}{2}\tan^{-1}\frac{1}{2}\right]-\left[\ln 2-\ln 4+\frac{1}{2}\tan^{-1}0\right]$

$-\ln 5+\frac{1}{2}\tan^{-1}\frac{1}{2}-\ln 2+\ln 4$ ✓

$\Rightarrow \ln\frac{4}{2\times 5}+\frac{1}{2}\tan^{-1}\frac{1}{2}$ (4)

$\Rightarrow \ln\frac{2}{5}+\frac{1}{2}\tan^{-1}\frac{1}{2}$ ✓

iii) $\int_1^2 x^2 e^x \, dx$ integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$u = x^2$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x$ $\quad v = e^x$

$\Rightarrow \left[e^x . x^2\right]_1^2 - \int_1^2 2x . e^x \, dx$

$\Rightarrow \left[e^x . x^2\right]_1^2 - 2\int x . e^x \, dx$

$\Rightarrow (4e^2 - e) - 2\left[x . e^x\right]_1^2 - \int_1^2 e^x \, dx$

$\Rightarrow 4e^2 - e - 2(2e^2 - e) - \left[e^x\right]_1^2$

$\Rightarrow 4e^2 - e - 2\left[2e^2 - e - (e^2 - e)\right]$

$\Rightarrow 4e^2 - e - 2e^2$

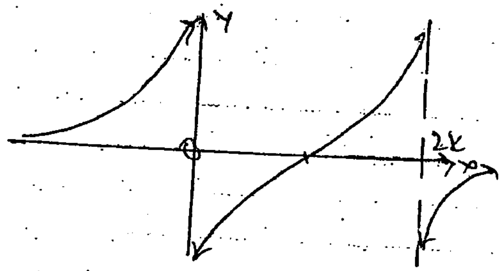
$\Rightarrow 2e^2 - e.$

4

15 marks

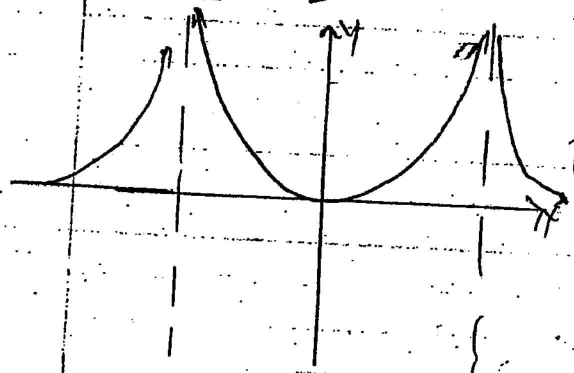
Question Two

i) $y = f(x-k)$



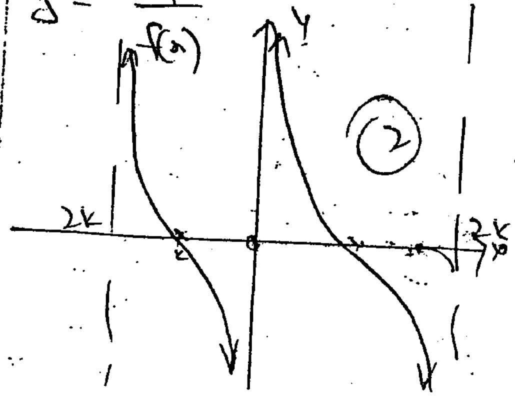
2

ii) $y = [f(x)]^2$



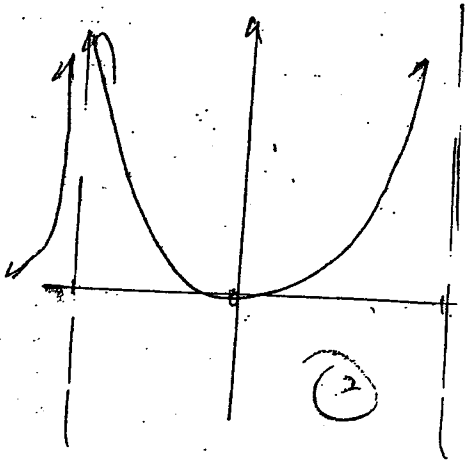
2

iii) $y = \frac{1}{f(x)}$



2

iv) $y = f'(x)$



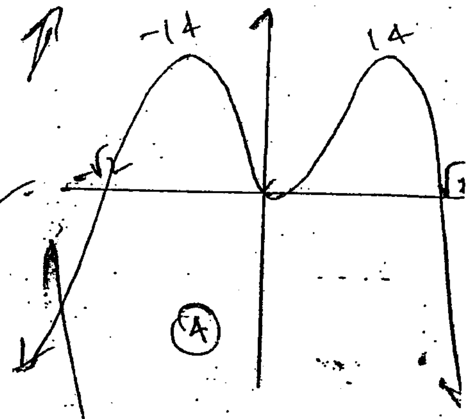
2

) $y = 4x^2(2-x^2)$

a) Sketch the curve.

i) $y = 4x^2(\sqrt{2}-x)(\sqrt{2}+x)$

when $y=0$ $x=0$ $x=\pm\sqrt{2}$



(4)

ii) S.V $dy/dx=0$

$y = 8x^2 - 4x^4$

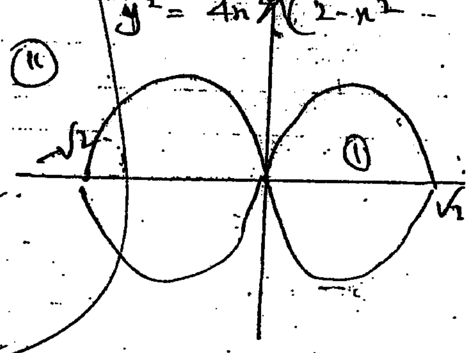
$y' = 16x - 16x^3$

$y'=0$

$16x(1-x^2)$

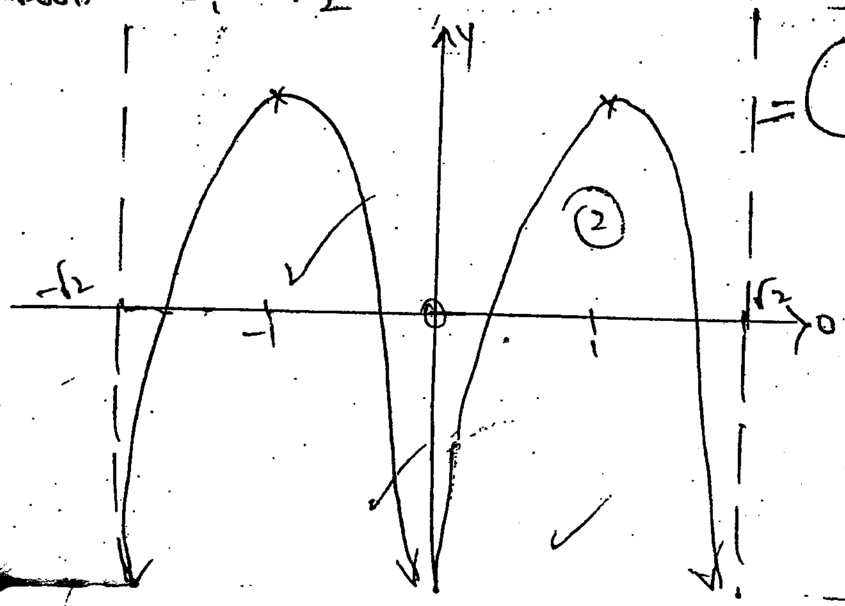
$y'=0$ at $x=0$ $x=\pm1$

SV $(0,0)$ $(1,4)$ $(-1,4)$



(ii) (1)

Even function since roots $0, \pm\sqrt{2}$

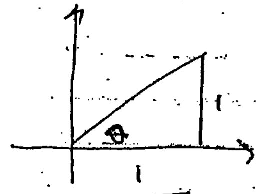


(2)

(7)

Question Three

$w = 1+i$ $z = \sqrt{3}-i$



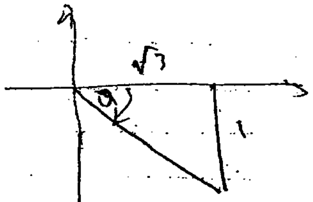
$|w| = \sqrt{1^2+1^2}$

$|w| = \sqrt{2}$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$w = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



$|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

$\operatorname{Arg} z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$z = 2\left(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}\right)$

i) $\therefore wz = \sqrt{2}\operatorname{cis}\frac{\pi}{4} \times 2\operatorname{cis}-\frac{\pi}{6}$

$wz = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$ Mod $|wz| = 2\sqrt{2}$

$wz = 2\sqrt{2}\operatorname{cis}\frac{\pi}{12}$ Arg $wz = \frac{\pi}{2}$

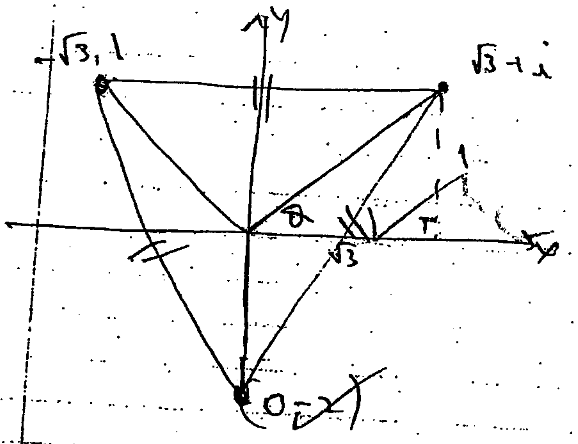
ii) $w^{-1}z = \frac{z}{w}$

$= \dfrac{2\operatorname{cis}-\frac{\pi}{6}}{\sqrt{2}\operatorname{cis}\frac{\pi}{4}}$

$= \sqrt{2}\operatorname{cis}\frac{-5\pi}{12}$

(4)

(b) Circle $(0,0)$



$|z| = 2$ given

$\tan\theta = \sqrt{3}$

$\theta = \tan^{-1}(\sqrt{3})$

$\theta = 60^\circ = \frac{\pi}{3}$

(4)

(c) $\frac{z-2i}{1+iz} = \frac{4}{3}$ Solve.

$3z + 6i = 4 + 4iz$

$3z - 4iz = 4 + 6i$

$(3-4i)z = 4 + 6i$

$z = \frac{4+6i}{3-4i} \times \frac{3+4i}{3+4i}$

$\Rightarrow \frac{12 + 16i + 18i - 24}{9+16}$

$\Rightarrow \frac{-12 + 34i}{25}$

$|z| \; \sqrt{\frac{12^2 + 34^2}{25^2}}$

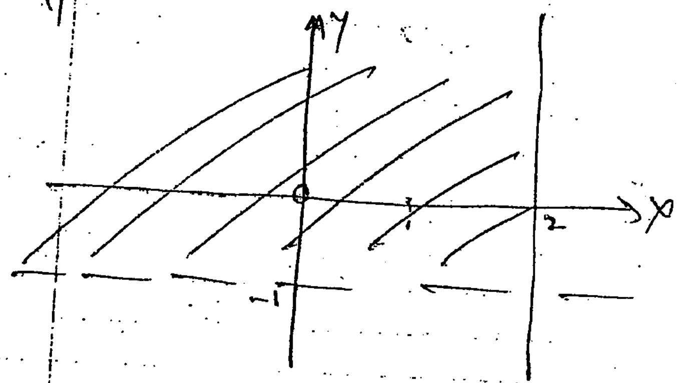
$\frac{\sqrt{1300}}{25} = \frac{10\sqrt{13}}{25}$

$\theta = \pi - \tan^{-1}\frac{34}{12} \doteq 1.91$

(4)

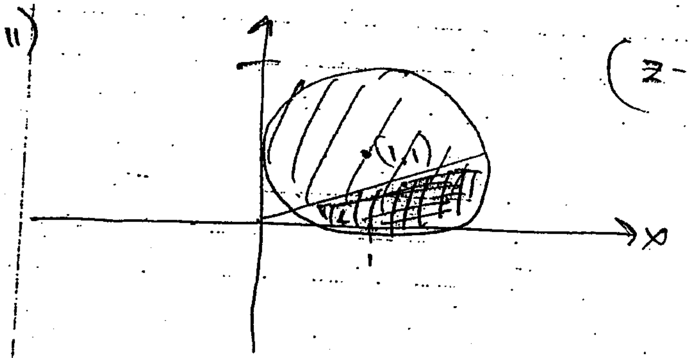
(d) $R(z) \leq 2$ $I(z) > -1$

i)



(1)

ii)



$(z - 1 - i) \leq 1$

centre $(1,1)$

(2)

Question Four

$P(x) = (x^4-1)(x^2-2)$

$= (x^2+1)(x^2-1)(x^2-2)$

$= (x+1)(x-1)(x^2-2)(x^2+1)$

Real $\Rightarrow (x+1)(x-1)(x^2+1)(x-\sqrt{2})(x+\sqrt{2})$ 3

Complex $(x+1)(x-1)(x-\sqrt{2})(x+\sqrt{2})(x+i)(x-i)$

If $x=\alpha$ and a double root Then $x-\alpha$ is a repeat factor.

$\therefore P(x) = (x-\alpha)(x-\alpha)\ Q(x)$

$P(x) = (x-\alpha)^2\ Q(x)$ $\boxed{\text{product Rule}}$

$P'(x) = 2\,Q(x)(x-\alpha) + (x-\alpha)^2\ Q'(x)$

$= (x-\alpha)\left[2\,Q(x) - (x-\alpha)\,Q'(x)\right]$ 2

$\therefore (x-\alpha)$ is a factor of $P'(x)=0$

$P(x) = x^4+x^2+6x+4$ has a rational zero of multiplicity 2

$P'(x) = 4x^3+2x+6$

$P(x) = (x+1)^2\ Q(x)$

$x^4+x^2+6x+4 = (x+1)^2\ Q\,x.$

$$\begin{array}{r l}
 & x^2 - 2x + 4 \\
x^2+2x+1 \,\big)\!\! & x^4 + 0x^3 + x^2 + 6x + 4 \\
 & x^4 + 2x^3 + x^2 \\ \hline
 & -2x^3 + 0x^2 + 6x \\
 & -2x^3 - 4x^2 - 2x \\ \hline
 & 4x^2 + 8x + 4 \\
 & 4x^2 + 8x + 4 \\ \hline
 & 0
\end{array}$$

$P(x) = (x+1)^2(x^2-2x+4)$ 4

Consider $x^2-2x+4=0$

$x^2-2x = -4$

$x^2-2x+(-1)^2 = -4+(-1)^2$

$(x-1)^2 = -3$

$(x-1) = \pm\sqrt{-3}$

$x = 1 \pm \sqrt{-3}$ $\rightarrow$ $x = 1 \pm i\sqrt{3}$

Roots are $x=-1,\ x=-1$ and $x = 1\pm\sqrt{3}$

(c) $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

has roots $a+bi$ and $a-2bi$

Sum Roots = $a+bi + a-bi + a-2bi + a-2bi = 4$

$\Rightarrow 4a = 4$

$a = 1$

Product Roots = $(a+bi)(a-bi)(a-2bi)(a+2bi) = 10$

$(a^2+b^2)(a^2+4b^2) = 10$

and $a = 1$

$\Rightarrow (1+b^2)(1+4b^2) = 10$

by inspection $b = \pm 1$

roots are $1+i,\ 1-i,\ 1+2i,\ 1-2i$ (3)

c ii) hence find zero values over complex field.
express as two quadratics

$P(x) = [x-(1+i)][x-(1-i)][x-(1+2i)][x-(1-2i)]$

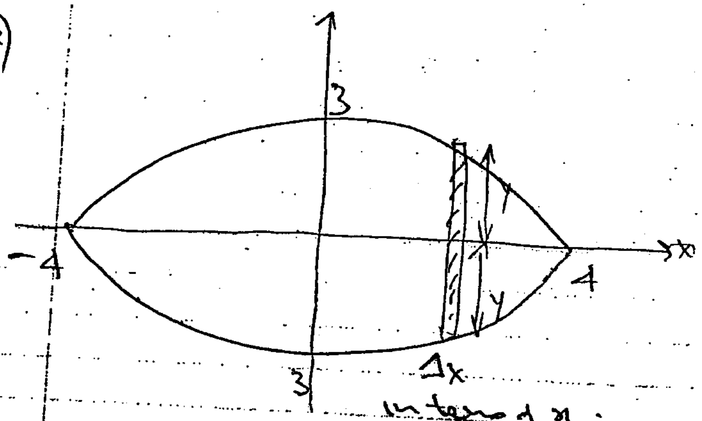
$= (x-1-i)(x-1+i)(x-1-2i)(x-1+2i)$

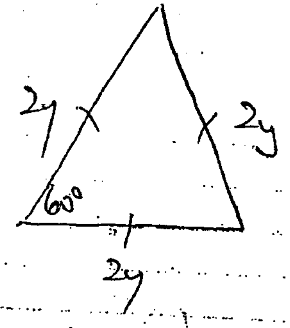
$\Rightarrow [(x-1)^2+1][(x-1)^2+4]$

$\Rightarrow (x^2-2x+2)(x^2-2x+4)$ (3)

Question Five

a)





ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$V = \text{Area} \times \Delta x$

$= \sqrt{3}y^2 \times \Delta x$

$V = \int_{-4}^{4} \sqrt{3}\, y^2\, dy$

$V = \int_{-4}^{4} \frac{9}{\sqrt{3}}\left(1-\frac{x^2}{16}\right) dy$

$= 18\sqrt{3}\left[x - \frac{x^3}{48}\right]_0^4$

$= 18\sqrt{3}\left(4 - \frac{64}{48}\right)$

$= 18\sqrt{3}\ (8)$

$(\times \text{Area}) = \frac{1}{2}ab\sin C$

$A = \frac{1}{2} \times 2y \times 2y \times \sin 60$

$= 2y^2 \sin 60°$

$= 2y^2 \times \frac{\sqrt{3}}{2}$

Area $= \sqrt{3}y^2$

$48\sqrt{3}$

(5)

Question Six

a) $\frac{x^2}{16}+\frac{y^2}{7}=1$ — (1) $x^2-\frac{y^2}{8}=1$ — (2)

i) $a=4$ $b=\sqrt{7}$ $a=1$ $b=\sqrt{8}$

$b^2=a^2(1-e^2)$
$7=16(1-e^2)$
$\frac{7}{16}=1-e^2$
$e^2=\frac{9}{16}$
$e=3/4$

$b^2=a^2(1-e^2)$
$8=1(e^2-1)$
$e^2=9$
$e=3$

foci $S(\pm ae,0)$
$S(\pm 4\times\frac{3}{4},0)$
$S(\pm 3,0)$

$S(\pm ae,0)$
$(\pm 1\times 3,0)$
$S(\pm 3,0)$

4

ii) Eqn of circle through intersection of the curves.

$x^2=16-\frac{16y^2}{7}$

Sub into (2)

$x^2-\frac{y^2}{8}=1$

$16-\frac{16y^2}{7}-\frac{y^2}{8}=1$ $(\times 56)$

$\Rightarrow 896-128y^2-7y^2=56$

$\Rightarrow -135y^2=-840$

$y^2=\frac{56}{9}$

$x^2=16-\frac{16y^2}{7}$

$\therefore x^2=16-\frac{16}{7}\times\frac{56}{9}$

$x^2=\frac{16}{9}$

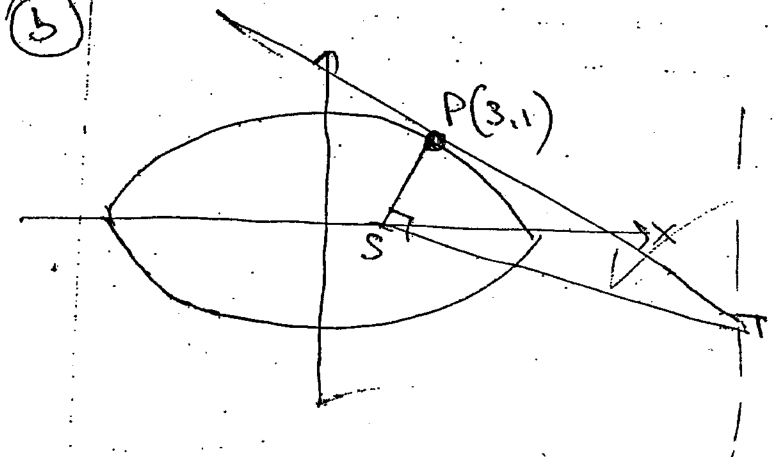
$x^2+y^2=r^2$

$\frac{16}{9}+\frac{56}{9}=r^2$ $9r^2=72$ $r^2=8$

$\boxed{x^2+y^2=8}$

4

b)



$\frac{x^2}{12}+\frac{y^2}{4}=1$

$a^2=12$ $b^2=4$

$b^2=a^2(1-e^2)$
$4=12(1-e^2)$
$4=12-12e^2$

) $y = \sin x$

Volume Shell $= 2\pi x y \; \Delta x$

$$V = 2\pi \int_0^{\frac{\pi}{2}} x \sin x \; dx$$

Integration by parts $\quad u = x \quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1 \quad v = -\cos x$

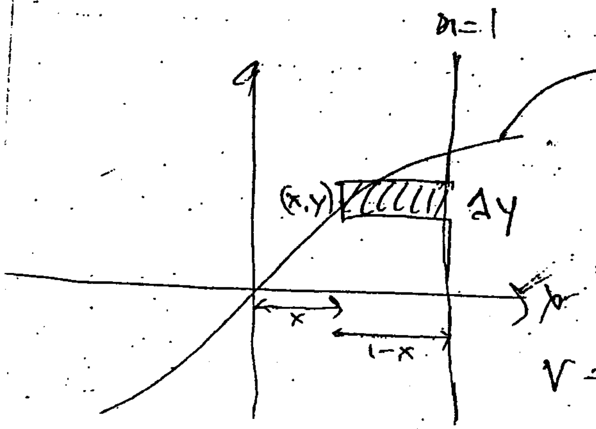
$$V = 2\pi \left[-x \cos x \right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= 2\pi \left[[0] + [\sin x]_0^{\frac{\pi}{2}} \right]$$

$$= 2\pi (1)$$

5

$= 2\pi$ cubic units



V (disc) $= \pi (1-x)^2$

$$V = \pi \int (1-x)^2 \, dy$$

$$V = \pi \int (1 - 2x + x^2) \, dy$$

$x = \tan y$

$x = 1 \quad y = \frac{\pi}{4}$

$x = 0 \quad y = 0$

$$V = \pi \int_0^{\frac{\pi}{4}} (1 - 2\tan y + \tan^2 y) \, dy$$

$$= \pi \int (1 + \tan^2 y - 2\tan y) \, dy$$

$$= \pi \int_0^{\frac{\pi}{4}} (\sec^2 y - 2\tan y) \, dy$$

$$= \pi \left[\tan y + 2\ln \cos y \right]_0^{\frac{\pi}{4}}$$

Note $\frac{2 \sin y}{\cos y} \quad \frac{f'(x)}{f(x)}$

$= 2\ln \cos y$

$$V = \pi \left[\left(\tan\frac{\pi}{4} + 2\ln \cos\frac{\pi}{4} \right) - (0 + 0) \right]$$

$$V = \pi \left(1 + 2\ln \frac{1}{\sqrt{2}} \right) \text{ cubic units.}$$

Foci

$(ae, 0)$

$\left(2\sqrt{3} \cdot \frac{\sqrt{2}}{\sqrt{3}}, 0\right)$

$S(\ldots, 0)$

$\boxed{\text{Directrix}}$

$x = \frac{a}{e} = 2\sqrt{3} \times \frac{\sqrt{3}}{2}$

$x = \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$x = 3\sqrt{2}$

$T(3\sqrt{2}, 4 - 3\sqrt{2})$

$3\sqrt{2} + y = 4$

$y = 4 - 3\sqrt{2}$

$M(PS) \times M(ST)$

$\frac{1-0}{3-2\sqrt{2}} \times \frac{4-3\sqrt{2}-0}{3\sqrt{2}-2\sqrt{2}}$

$\frac{1}{3-2\sqrt{2}} \times \frac{4-3\sqrt{2}}{\sqrt{2}}$

$= \frac{4-3\sqrt{2}}{3\sqrt{2}-4}$ ∴ $PS \perp ST$

$= -1$

4

$\ldots\frac{y^2}{4} = 1$ at $(3,1)$

$\ldots\frac{2y}{4}\, dy/dx = 0$

$\frac{2y}{4}\, dy/dx = \frac{-2x}{12}$

$\frac{dy}{dx} = \frac{-2x}{12} \times \frac{4}{2y}$

$dy/dx = \frac{-8x}{24y}$

$dy/dx = \frac{-x}{3y}$

$\frac{dy}{dx} = \frac{-x}{3y}$ at $(3,1)$

∴ $m = -1$ $P(3,1)$

$y - 1 = -1(x-3)$

$y - 1 = -x + 3$

$\boxed{x + y - 4 = 0}$

3

Question Series

a) $f(x) = \frac{3x}{(x-1)(4-x)}$

i) $\frac{3x}{(x-1)(4-x)} = \frac{A}{x-1} + \frac{B}{4-x}$

$3x = A(4-x) + B(x-1)$

let $x = 4$ $B = 4$

let $x = 1$ $A = 1$

$f(x) = \frac{1}{x-1} + \frac{4}{4-x}$

2

ii) $f(x) = (x-1)^{-1} + 4(4-x)^{-1}$

$f'(x) = -(x-1)^{-2} \times 1 - 4(4-x)^{-2} \times -1$

$= \dfrac{-1}{(x-1)^2} + \dfrac{4}{(4-x)^2}.$

S.V when $f'(x) = 0$

$\Rightarrow -(4-x)^2 + 4(x-1)^2 = 0$

$-(16 - 8x + x^2) + 4(x^2 - 2x + 1) = 0$

$-16 + 8x - x^2 + 4x^2 - 8x + 4 = 0$

$3x^2 - 12 = 0 \quad \rightarrow \quad x = \pm 2$

$3x^2 = 12$

$x^2 = 4$

SV $(2,3)$ $(-2, \frac{1}{3})$

$f''(x) = \dfrac{2}{(1-x)^3} + \dfrac{8}{(4-x)^3}$

$f''(2) > 0$ Min TP

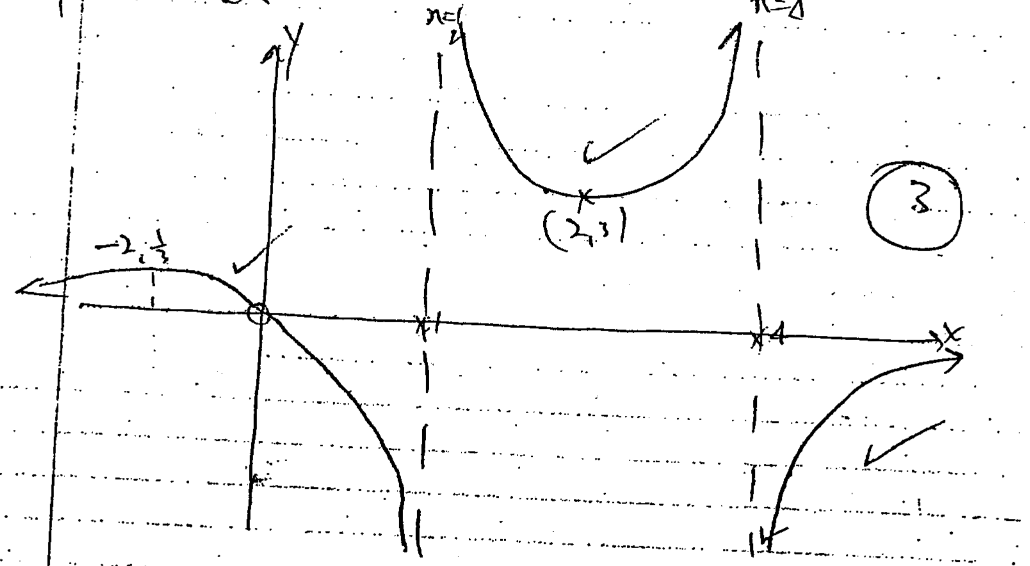
$f''(-2) < 0$ Max TP.

4

asymptotes $x-1=0$, $4-x=0$

$\therefore x=1$ and $x=4$

iii) Sketch



3

iv) Area $= \displaystyle\int_2^3 \frac{3x}{(x-1)(4-x)}\,dx$

$\Rightarrow \displaystyle\int_2^3 \frac{1}{x-1} + \frac{4}{4-x}\,dx$

$\Rightarrow \Big[\ln(x-1) - 4\ln(4-x)\Big]_2^3$

$\Rightarrow (\ln 2 - 4\ln 1) - (\ln 1 - 4\ln 2)$

$\ln 2 - 4\ln 2$

$5\ln 2$ unit^2

2

(i) $y = x^2 - 4$, $y = \frac{1}{x}$

$x^2 - 4 = \frac{1}{x}$

$x^3 - 4x = 1$

$x^3 - 4x - 1 = 0$

1

$\alpha^2 = x$ $\quad x^3 - 4x - 1 = 0$

$\therefore x = \sqrt{x}$ $\quad (\sqrt{x})^3 - 4(\sqrt{x}) - 1 = 0$

$\Rightarrow x^{3/2} - 4x^{1/2} = 1$

$\Rightarrow x^{\frac{1}{2}}(x - 4) = 1$

$\Rightarrow x(x-4)^2 = 1^2$

3

$x(x^2 - 8x + 16) = 1$

$x^3 - 8x^2 + 16x - 1 = 0$

Question Eight

$x = \frac{2}{3}\sin\theta$

prove $\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\,dx = \frac{\pi}{3}$

let $x = \frac{2}{3}\sin\theta$

$\frac{dx}{d\theta} = \frac{2}{3}\cos\theta\,d\theta$

when $x = 2$ $\theta = \frac{\pi}{2}$

when $x = 0$ $\theta = 0$

$\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\,dx = \frac{2}{3}\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta$

$\Rightarrow \frac{2}{3}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta.$

$\Rightarrow \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$

5

$= \frac{\pi}{3}$

area enclosed by ellipse

$9x^2 + y^2 = 4$

$A = 4\int_0^{\frac{2}{3}} \sqrt{4 - 9x^2}\,dx$

$= 4 \times \frac{\pi}{3}$

$= \frac{4\pi}{3}$ units^2

b) i) $z = \cos\theta + i\sin\theta$ show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + \frac{1}{(\cos\theta + i\sin\theta)^n}$$

3

$$\Rightarrow \cos n\theta + i\sin n\theta + \frac{1}{\cos\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

$$\Rightarrow \cos n\theta + i\sin n\theta + \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$$

$$\Rightarrow \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$\Rightarrow 2\cos n\theta$$

ii) $\left(z + \frac{1}{z}\right)^4$

$$\Rightarrow z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4}$$

$$\Rightarrow z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$\therefore (2\cos\theta)^4 = 2\cos 4\theta + 4[2\cos 2\theta] + 6$$

$$\Rightarrow 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

2

$$\cos^4\theta = \frac{1}{8}\left[\cos 4\theta + 4\cos 2\theta + 3\right]$$

(c) $\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}$

here $f(x) = x + (x^2+4)^{1/2}$

$$f'(x) = 1 \quad x(x^2+4)^{-\frac{1}{2}} = 1 + \frac{x}{\sqrt{x^2+4}}$$

$$\text{so } \frac{d}{dx}\left[\ln\left(x + \sqrt{x^2+4}\right)\right] = \frac{1 + \frac{x}{\sqrt{x^2+4}}}{x + \sqrt{x^2+4}}$$

3

$$\Rightarrow \frac{\sqrt{x^2+4} + x}{\sqrt{x^2+4}} \Big/ \frac{x\sqrt{x^2+4} + x^2 + 4}{\sqrt{x^2+4}}$$

$$\Rightarrow \frac{\sqrt{x^2+4} + x}{x\sqrt{x^2+4} + x^2 + 4} = \frac{1}{\sqrt{x^2+4}}$$

Irena

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x \, du}{\sqrt{\tan^2 x + 4}} \qquad u = \tan x$$

$$du = \sec^2 x \, dx$$

$$x = \frac{\pi}{4} \quad u = 1$$

$$x = 0 \quad u = 0$$

$$2\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{\sqrt{\tan^2 x + 4}}$$

$$= 2\int_0^1 \frac{du}{\sqrt{u^2+4}} = 2\left[\ln\left(u + \sqrt{u^2+4}\right)\right]_0^1$$

$$= 2\left[\ln(1+\sqrt{5}) - \ln(2)\right]$$

$$= 2\ln\left(\frac{1+\sqrt{5}}{2}\right)$$

End Solution