-Total marks – 120

Attempt All Questions

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra booklets are available.

Question 1 *(15 Marks)* **Marks**

(a) Find $\int x^2 \sin(x^3)\, dx$. **2**

(b) Use integration by parts to evaluate $\int_0^1 \tan^{-1} x\, dx$. **3**

(c) (i) Find the real numbers a and b such that $\frac{x}{(x-1)(x+4)} \equiv \frac{a}{x-1} + \frac{b}{x+4}$. **2**

(ii) Find. $\int \frac{x}{(x-1)(x+4)}\, dx$. **2**

(d) Find $\int \frac{x+4}{x^2 - 4x + 13}\, dx$. **3**

(e) Use the substitution $t = \tan\frac{x}{2}$ to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x}$$ **3**

Question 2 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) Let $w = 1 + i$ and $z = 1 - i\sqrt{3}$, simplify the following

(i) $w\overline{z}$ **1**

(ii) $\dfrac{1}{w}$ **1**

(iii) $\dfrac{i(\text{Re}(z) - z)}{\text{Im}(z)}$. **2**

(b) Sketch the region on the Argand diagram where the inequalities $|z| \le 2$ and $\pi \ge \arg z \ge -\dfrac{\pi}{4}$ hold simultaneously. **3**

(c) Solve the equation $x^2 - 4x + (1 - 4i) = 0$. Answer should be expressed in the form $a + ib$ **4**

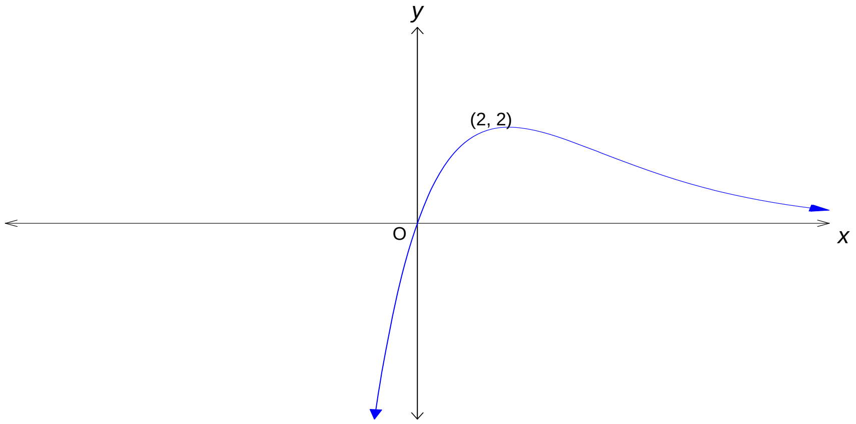
(d) The complex number $z = x + iy$, where x and y are real, satisfies the parametric equation $z = 1 + 2i + t(3 - 4i)$ where t is a real parameter.

(i) Show that the Cartesian equation of the locus of the point P which represents z in an Argand diagram is given by $4x + 3y = 10$. **2**

(ii) Hence find the minimum value of $|\ z\ |$. **2**

Question 3 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) The curve shown in the diagram is the equation $y = f(x)$. There is a maximum turning point at (2,2) and the curve crosses the x axis at (0,0). The graph has a horizontal asymptote at $y = 0$.



Sketch the following curves on separate diagrams, showing all of the essential features.

(i) $y = f(x+2)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \left(f(x)\right)^2$ **2**

(iv) $y = -x \times f(x)$. **2**

(b) (i) Show that $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3. **2**

(ii) Verify that $x = i$ is also a root of $P(x)$. **1**

(iii) Hence solve the equation $P(x) = 0$. **2**

(c) Let $\alpha,\ \beta,\ \gamma$ be the roots of the equation $x^3 - 2x^2 - 5x - 1 = 0$. Form an equation whose roots are $\dfrac{1}{\sqrt{\alpha}}, \dfrac{1}{\sqrt{\beta}}$ and $\dfrac{1}{\sqrt{\gamma}}$. **3**

Question 4 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) For what values of k does the equation $\frac{x^2}{5-k}+\frac{y^2}{k-3}=1$ represent:

(i) a circle? **2**

(ii) a hyperbola? **2**

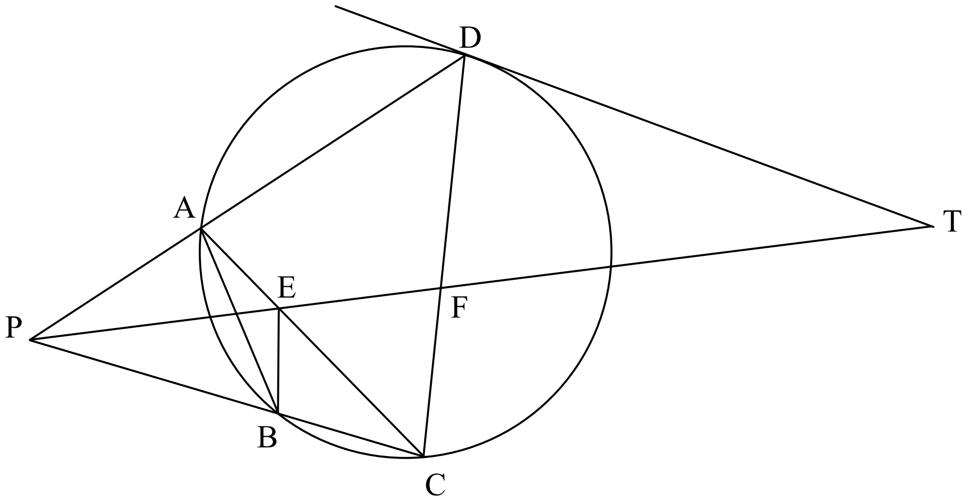
(b) The points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ are points on the rectangular hyperbola $xy = c^2$.
Tangents to the rectangular hyperbola at P and Q intersect at the point $R\left(X,Y\right)$.

(i) Show that the tangent to the rectangular hyperbola at any point $T\left(ct,\frac{c}{t}\right)$ has equation $x + t^2y - 2ct = 0$. **1**

(ii) Find the coordinates R. **2**

(iii) If P and Q are variable points on the rectangular hyperbola which move so that $p^2 + q^2 = 2$, show that the equation of the locus of R is given by $xy + y^2 = 2c^2$. **3**

(c) *ABCD* is a cyclic quadrilateral. *DA* is produced and *CB* produced meet at *P*. *T* is a point on the tangent at *D* to the circle through *A*, *B*, *C* and *D*. *PT* cuts *CA* and *CD* at *E* and *F* respectively. *TF* = *TD*.



Copy this diagram into your writing booklet.

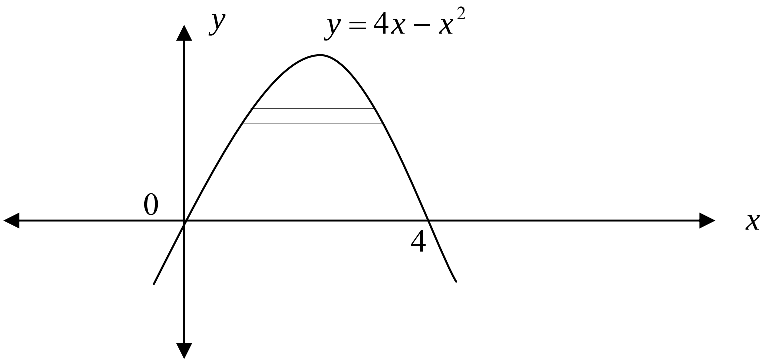
(i) Show that *AEFD* is a cyclic quadrilateral. **2**

(ii) Show that *PBEA* is a cyclic quadrilateral. **3**

Question 5 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) Find the general solution for the equation $\cos 3x = -\sin 2x$ **3**

(b)



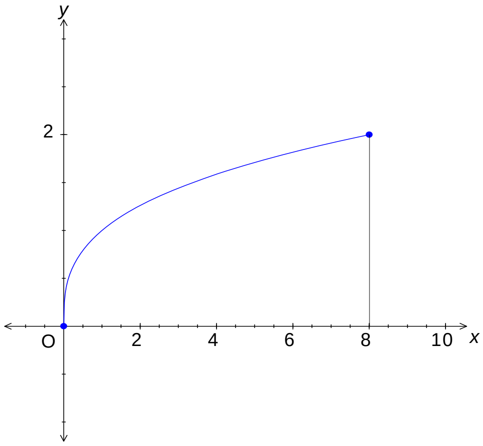
The area bounded by the curve $y = 4x - x^2$ and the x-axis is rotated about the y-axis.

(i) Use strips perpendicular to the axis of rotation and show the x-coordinates of the end-points of these strips are $2-\sqrt{4-y}$ and $2+\sqrt{4-y}$. **2**

(ii) Find the maximum value y. **1**

(iii) Hence find the volume of the solid of revolution, in terms of π. **5**

(c) The sketch below shows the region enclosed by the curve $y = x^{\frac{1}{3}}$, the x axis and the ordinate $x = 8$.



Find the volume generated when this region is rotated about the line $x = 8$, using the method of cylindrical shells. **4**

Question 6 ***(15 Marks)*** **Use a SEPARATE writing booklet.** **Marks**

(a) Given that $z^n - \frac{1}{z^n} = 2i\sin n\theta$ and $\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

(i) prove that: $\sin^5\theta = \frac{1}{16}\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$ **3**

(ii) find the general solutions of the equation $16\sin^5\theta = \sin 5\theta$. **4**

(b) A particle, of mass *m*, is projected vertically upwards in a resisted medium where the resistance is proportional to its velocity and *mk* is the constant of variation. The velocity of projection is given by $u\ ms^{-1}$.

(i) Show that after a time *t* seconds, the height above the ground is:

$$x_1 = \frac{g + ku}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}.$$ **5**

(ii) At the same time another particle is dropped from a height *h* metres vertically above the first particle. Given that at the time *t* seconds, its distance from the ground is:

$$x_2 = h + \frac{g}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k},$$

show that the two particles will meet at a time *T* where

$$T = \frac{1}{k}\log\left(\frac{u}{u - kh}\right).$$ **3**

Question 7 *(15 Marks)* **Use a SEPARATE writing booklet.** **Marks**

(a) i) Find the greatest and least values of e^{x^2-x} in the domain $0 \le x \le 2$. **2**

ii) Hence show that $2e^{-\frac{1}{4}} < \int_0^2 e^{x^2-x}\,dx < 2e^2$ **1**

(b) (i) Using the substitution $u = a - x$, prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence show that $\int_0^\pi x\cos^2 x\,dx = \frac{\pi^2}{4}$. **3**

(c) Given that $\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$ where n is a positive integer,

(i) prove that

$$\cos(n\theta) = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \text{.........} \text{ and}$$

$$\sin(n\theta) = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \text{..........}$$ **4**

(ii) hence deduce that

$$\tan(n\theta) = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \text{.........}}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \text{........}}$$ **3**

Question 8 *(15 Marks)* Use a SEPARATE writing booklet. Marks

(a) Use the compound angle formulae for $\cos(x+y)$ and $\cos(x-y)$ to prove the result
$\cos S - \cos T = -2\sin\left(\frac{S+T}{2}\right)\sin\left(\frac{S-T}{2}\right)$. **2**

(b) For $n = 0,1,2,3,.....$,define $I_n = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2nx}{\sin 2x}\,dx$.

(i) Evaluate I_1 **2**

(ii) Using the result proven in part (a), show that for $r \geq 1$:
$$I_{2r+1} - I_{2r-1} = \frac{1-(-1)^r}{2r}.$$ **3**

(iii) Hence evaluate I_9 **3**

(c) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ are points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose eccentricity is *e*.

(i) Find the equation of the chord *PQ*. **2**

(ii) If *PQ* is a focal chord of this ellipse show that $e = \frac{\sin(\phi-\theta)}{\sin\phi - \sin\theta}$. **3**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax\ \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

Solutions THGS Ext 2 Trial 2009

$Q1a \int x^2 \sin\left(x^3\right)dx = -\frac{1}{3}\cos\left(x^3\right) + C$

$Q1b \int_0^1 \tan^{-1} x \, dx = \int_0^1 \tan^{-1} x \frac{d}{dx}(x)\,dx = \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$

$= \frac{\pi}{4} - 0 - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$

$Q1c\ i)\quad \frac{x}{(x-1)(x+4)} \equiv \frac{a}{x-1} + \frac{b}{x+4}$

$\Rightarrow x \equiv a(x+4) + b(x-1)$

$let\ x = 1 \Rightarrow a = \frac{1}{5}$

$let\ x = -4 \Rightarrow b = \frac{4}{5}$

$Q1cii)\quad \int \frac{x}{(x-1)(x+4)}dx = \frac{1}{5}\int \frac{1}{x-1}dx + \frac{4}{5}\int \frac{1}{x+4}dx$

$= \frac{1}{5}\ln|x-1| + \frac{4}{5}\ln|x+4| + C$

$Q1d)\quad \int \frac{x+4}{x^2-4x+13}dx = \frac{1}{2}\int \frac{2x-4}{x^2-4x+13}dx + 6\int \frac{1}{(x-2)^2+9}dx$

$= \frac{1}{2}\ln|x^2-4x+13| + 2\tan^{-1}\left(\frac{x-2}{3}\right) + C$

$Q1e)\quad If\ t = \tan\frac{x}{2}\ then\ x = 2\tan^{-1} t\ since\ 0 \le x \le \frac{\pi}{2} \Rightarrow dx = \frac{2}{1+t^2}dt$

$\cos x = \frac{1-t^2}{1+t^2}\quad and\ x = 0 \Rightarrow t = 0,\ x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x} = \int_0^1 \frac{\frac{2}{1+t^2}dt}{1+\frac{1-t^2}{1+t^2}} = \int_0^1 \frac{2dt}{1+t^2+1-t^2} = \int_0^1 dt = [t]_0^1 = 1$

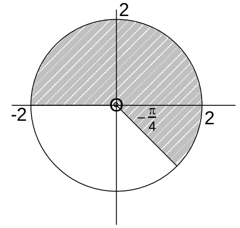
$Q2a)\quad w = 1+i\quad z = 1-i\sqrt{3}$

$i)\quad w\overline{z} = (1+i)\left(1+i\sqrt{3}\right) = 1-\sqrt{3} + i\left(1+\sqrt{3}\right)$

$ii)\quad \frac{1}{w} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{2}$

$iii)\quad \frac{i\left(\Re e(z) - z\right)}{\Im m(z)} = \frac{i\left(1-1+i\sqrt{3}\right)}{-\sqrt{3}} = 1$

$Q2b\quad |z| \le 2\quad \pi \ge \arg z \ge -\frac{\pi}{4}$



$Q2c)\quad x^2 - 4x + (1-4i) = 0$

$x = \frac{4 \pm \sqrt{16 - 4(1-4i)}}{2} = \frac{4 \pm \sqrt{12+16i}}{2} = 2 \pm \sqrt{3+4i}$

$let\ \sqrt{3+4i} = a + ib\ where\ a, b\ are\ real\ = ----(1)$

$\therefore 3+4i = a^2 - b^2 + 2abi \Rightarrow a^2 - b^2 = 3 --(2),\ \left(2ab = 4 \Rightarrow b = \frac{2}{a} --(4)\right)$

$taking\ modulus\ of\ both\ sides\ of\ (1) \Rightarrow 5 = a^2 + b^2\ ---(3)$

$Add\ (2)\ and\ (3)\quad 2a^2 = 8 \Rightarrow a = \pm 2 \Rightarrow b = \pm 1$

$\left(OR\ using\ (2)\ and\ (4) \right.$

$a^2 - \left(\frac{2}{a}\right)^2 = 3 \Rightarrow \left(a^2\right)^2 - 3\left(a^2\right) - 4 = 0$

$\left(a^2-4\right)\left(a^2+1\right) = 0 \Rightarrow a^2 = 4,\ reject\ a^2 = -1\ \sin ce\ a\ is\ real$

$\left. \therefore a = \pm 2 \Rightarrow b = \pm 1 \right)$

$\therefore x = 2 + (2+i)\ or\ 2 - (2+i) \Rightarrow 4+i,\ or\ -i$

$Q2di)\quad z = 1 + 2i + 3t - (4t)i = 1 + 3t + i(2-4t)$

$thus\ x = 1+3t\ ---(\mathbf{1})\quad y = 2-4t\ ---(\mathbf{2})$

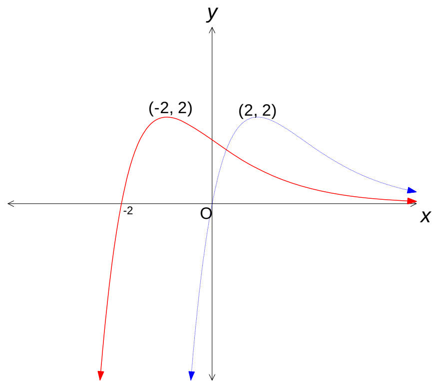
$4 \times (1) + 3 \times (2) \Rightarrow 4x + 3y = 10$

$Q2dii)\quad |z| =$ distance from origin

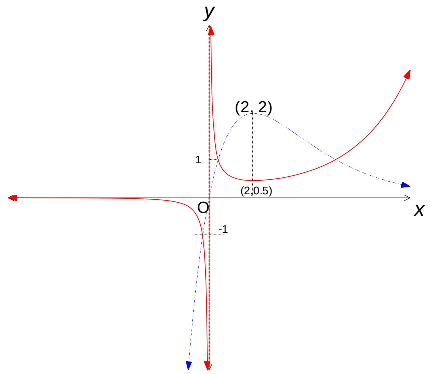
$\therefore \min |z| =$ perpendicular distance from origin

$= \frac{|4(0) + 3(0) - 10|}{\sqrt{4^2+3^2}} = 2$

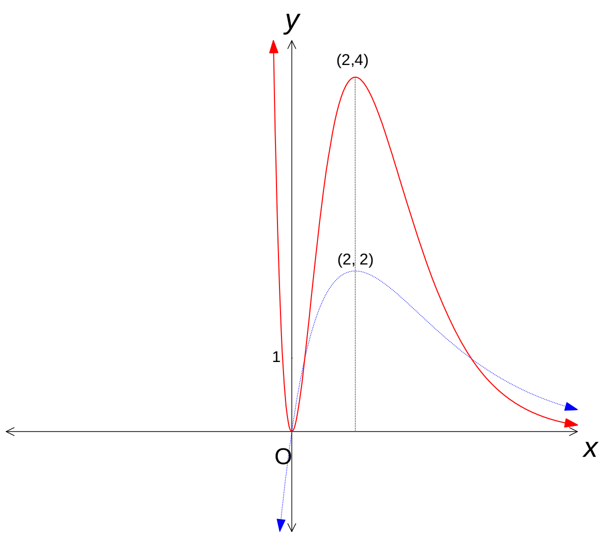
Q3ai)



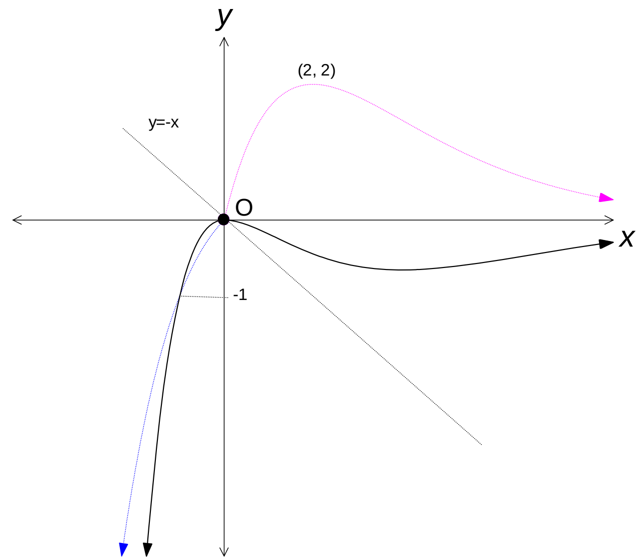
Q3aii)



Q3aiii



Q3aiv)



$Q3bi)\quad P(x) = x^5 - 3x^4 + 4x^3 - 4x \ + 3x - 1$

$P(1) = 1 - 3 + 4 - 4 + 3 - 1 = 0$

$P'(x) = 5x^4 - 12x^3 + 12x^2 - 8x + 3$

$P'(1) = 5 - 12 + 12 - 8 + 3 = 0$

$P''(x) = 20x^3 - 36x^2 + 24x - 8$

$P''(1) = 20 - 36 + 24 - 8 = 0$

$\therefore P(1) = P'(1) = P''(1) = 0 \qquad \therefore x = 1 \text{ is a triple root}$

$Q3bii)\quad P(i) = i^5 - 3i^4 + 4i^3 - 4i^2 + 3i - 1$

$= i - 3 - 4i + 4 + 3i - 1 = 0 \therefore x = i \text{ is a root}$

$Q3biii)\quad \text{If } \ x = i \text{ is a root so is } x = -i$

$\therefore \text{roots are } 1, 1, 1, i, -i$

$Q3c)$ let $y=\frac{1}{\sqrt{x}} \Rightarrow x=\frac{1}{y^2}$

Substitute in $x^3-2x^2-5x-1=0$

$$\left(\frac{1}{y^2}\right)^3-2\left(\frac{1}{y^2}\right)^2-5\left(\frac{1}{y^2}\right)-1=0$$

$$\frac{1}{y^6}-\frac{2}{y^4}-\frac{5}{y^2}-1=0$$

$$\Rightarrow 1-2y^2-5y^4-y^6=0$$

$\therefore y^6+5y^4+2y^2-1=0$ has roots $\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\gamma}}$

$Q4a)$ $\quad \frac{x^2}{5-k}+\frac{y^2}{k-3}=1$

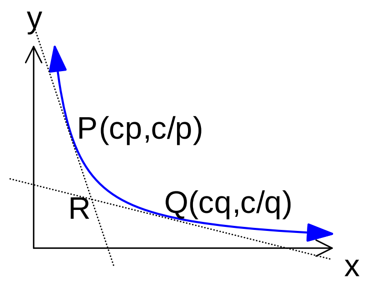
i) Circle if $5-k=k-3 \Rightarrow k=4$

ii) Hyperbola if $5-k<0$ and $k-3>0$

OR $5-k<0$ and $k-3>0$

ie $(5-k)(k-3)<0 \Rightarrow k>5$ or $k<3$

$Q4bi)$



$xy=c^2 \Rightarrow y=\frac{c^2}{x} \quad \therefore y'=-\frac{c^2}{x^2}=-\frac{1}{t^2}$ at $x=ct$

$\therefore$ tangent is $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct) \Rightarrow x+t^2y-2ct=0$

$Q4bii)$ Tangent at P $x+p^2y-2cp=0\ ---(1)$

Tangent at Q $x+q^2y-2cq=0\ ---(2)$

$(1)-(2)$ $\left(p^2-q^2\right)y=2c(p-q)$

$$y=\frac{2c}{p+q} \quad \text{as } p\neq q$$

substitute in (1) $x=2cp-p^2\left(\frac{2cp}{p+q}\right)=\frac{2cp^2+2cpq-2cp^2}{p+q}$

$$\Rightarrow x=\frac{2cpq}{p+q}$$

$\therefore R$ is $\left(\frac{2cpq}{p+q}.\frac{2c}{p+q}\right)$

$Q4biii)$ $\therefore x=\frac{2cpq}{p+q}\ ---(1)$

$$y=\frac{2c}{p+q}\ ---(2)$$

$(1)\div(2)$ $\frac{x}{y}=pq$ and from (2) $p+q=\frac{2c}{y}$

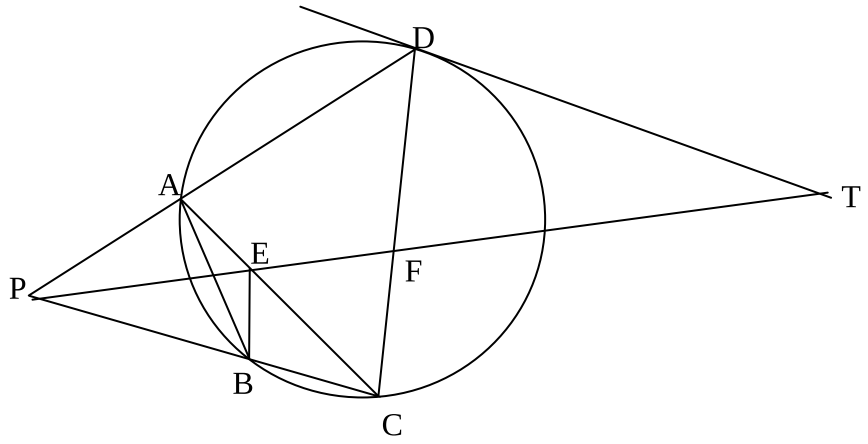
Now $p^2+q^2=2$ given $\therefore (p+q)^2-2pq=2$

$$\left(\frac{2c}{y}\right)^2-2\left(\frac{x}{y}\right)=2 \Rightarrow 4c^2-2xy=2y^2$$

$$\therefore xy+y^2=2c^2$$

OR substitute $R\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$ in $xy+y^2=2c^2$ to show true.

$Q4c)$



i) $TD = TF$ *given*

$\therefore \angle TFD = \angle TDF$ *base angles of isosceles triangle are equal*

$\angle TDF = \angle CAD$ *angle between tan gent and chord at point of contact equals angle in the alternate seqment*

$\therefore \angle TFD = \angle CAD$

$\therefore$ *AEFD is cyclic quad since exterior angle equals interior opposite angle*

ii) $\angle PEA = \angle ADF$ *exterior angle of cyclic quad AEFD equals interior opposite angle*

$\angle PBA = \angle ADF$ *exterior angle of cyclic quad ABCD equals interior opposite angle*

$\therefore \angle PEA = \angle PBA$

$\therefore$ *PBEA is cyclic since these two angles stand on the interval AP and are on the same side of the interval*

Q5a) $\cos 3x = -\sin 2x = \sin(-2x)$ *since odd function*

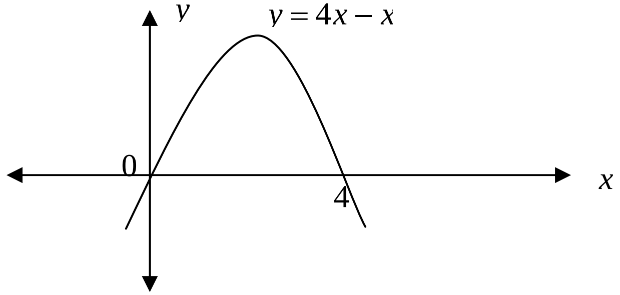
$$\therefore \cos 3x = \cos\left(\frac{\pi}{2} + 2x\right)$$

$$\Rightarrow 3x = 2n\pi \pm \left(\frac{\pi}{2} + 2x\right)$$

$$\therefore x = \frac{\pi}{2}(4n+1) \quad or \quad 5x = \frac{\pi}{2}(4n-1)$$

$$\Rightarrow x = \frac{\pi(4n+1)}{2} \quad or \quad x = \frac{\pi(4n-1)}{10}$$

Q5b

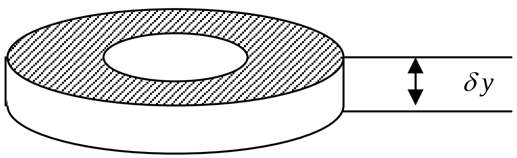


i) $x^2 - 4x + 4 = 4 - y$

$$x - 2 = \pm\sqrt{4-y} \Rightarrow x = 2 + \sqrt{4-y},\ 2 - \sqrt{4-y}$$

ii) $y_{\max} = f(2) = 8 - 4 = 4$

iii) Rotating the slice indicated creates a disc



$$Volume\ of\ disc = \delta V = \pi\left(x_2^2 - x_1^2\right)\delta y$$

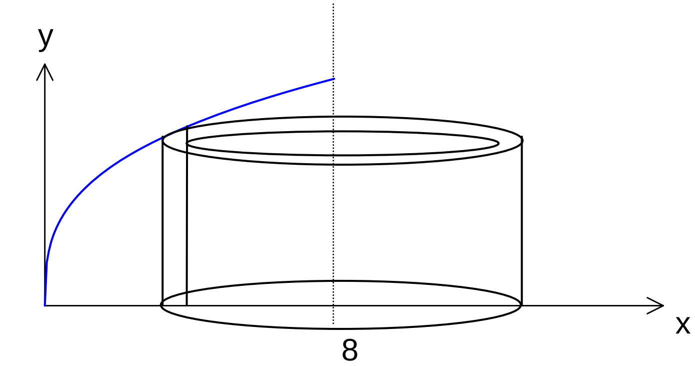
$$V = \lim_{\delta y \to 0}\sum_{y=0}^{y=4} \pi\left(x_2^2 - x_1^2\right)\delta y$$

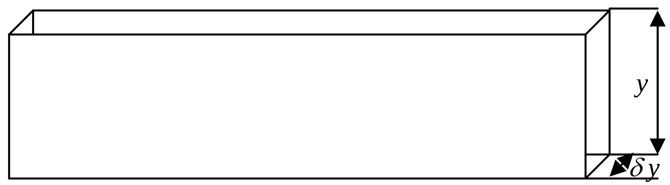
$$= \pi\int_0^4 \left(x_2^2 - x_1^2\right)dy = \pi\int_0^4 \left(x_2 - x_1\right)\left(x_2 + x_1\right)dy$$

$$= \pi\int_0^4 2\sqrt{4-y} \times 4\,dy = 8\pi\int_0^4 (4-y)^{\frac{1}{2}}\,dy$$

$$= 8\pi \times \frac{-2}{3}\left[(4-y)^{\frac{3}{2}}\right]_0^4 == 8\pi \times \frac{-2}{3}[0-8] = \frac{128\pi}{3} unit^3$$

Q5c





$y = x^{\frac{10}{3}}$ $\qquad 2\pi(8-x)$

$$Volume\ of\ shell = \delta V \approx 2\pi(8-x)\,y\,\delta x = 2\pi(8-x)\,x^{\frac{1}{3}}\delta x$$

$$V = \lim_{\delta x \to 0}\sum_{x=0}^{x=8} \delta V = 2\pi\int_0^8 \left(8x^{\frac{1}{3}} - x^{\frac{4}{3}}\right)dx = 2\pi\left[8 \times \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{7}x^{\frac{7}{3}}\right]_0^8$$

$$= 2\pi\left(6 \times 16 - \frac{3}{7} \times 128\right) = \frac{576\pi}{7} unit^3$$

$Q6ai)$ Given $\left(z-\frac{1}{z}\right)^5 = z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}$

$$= z^5-\frac{1}{z^5}-5\left(z^3-\frac{1}{z^3}\right)+10\left(z+\frac{1}{z}\right)$$

$\therefore (2i\sin\theta)^5 = 2i(\sin5\theta-5\sin3\theta+10\sin\theta)$

$\therefore \sin^5\theta = \frac{1}{16}(\sin5\theta-5\sin3\theta+10\sin\theta)$

$Q6aii)$ from $i)$ $16\sin^5\theta = \sin5\theta-5\sin\theta+10\sin\theta$

if $16\sin^5\theta=\sin5\theta$ then $-5\sin3\theta+10\sin\theta=0$

$\therefore 2\sin\theta=\sin3\theta=\sin(2\theta+\theta)=\sin2\theta\cos\theta+\cos2\theta\sin\theta$

$$=2\sin\theta\cos^2\theta+(1-2\sin^2\theta)\sin\theta=3\sin\theta-4\sin^3\theta$$

$\therefore 4\sin^3\theta-\sin\theta=0 \Rightarrow \sin\theta(4\sin^2\theta-1)=0$

$\sin\theta=0,\ \pm\frac{1}{2} \Rightarrow \theta=n\pi,\ n\pi\pm\frac{\pi}{6}$ for integer n

$Q6b)$ $R=-mg-mkv\ \therefore \ddot{x}=-g-kv$ $\uparrow +ve\quad \downarrow mg \downarrow mkv$

$$\frac{dt}{dv}=\frac{-1}{g+kv}$$

$\therefore t=\int_u^V \frac{-1}{g+kv}dv \Rightarrow t=\left[\frac{-1}{k}\ln|g+kv|\right]_u^V$

$$t=\frac{-1}{k}\ln\left|\frac{g+kV}{g+ku}\right| \Rightarrow -kt=\ln\left|\frac{g+kV}{g+ku}\right|$$

$\frac{g+kV}{g+ku}=e^{-kt} \Rightarrow g+kV=(g+ku)e^{-kt}\ \therefore\ V=\left(\frac{g+ku}{k}\right)e^{-kt}-\frac{g}{k}$

$\therefore x=\int_0^t\left(\left(\frac{g+ku}{k}\right)e^{-kt}-\frac{g}{k}\right)dt=\left[-\left(\frac{g+ku}{k^2}\right)e^{-kt}-\frac{gt}{k}\right]_0^t$

$\therefore x=\left(\frac{g+ku}{k^2}\right)(1-e^{-kt})-\frac{gt}{k}$

$Q6bii)$ $t=T$ and $x_1=x_2$

$\therefore \frac{g+ku}{k^2}(1-e^{-kT})-\frac{gT}{k}=h+\frac{g}{k^2}(1-e^{-kT})-\frac{gT}{k}$

$(1-e^{-kT})\left[\frac{g+ku}{k^2}-\frac{g}{k^2}\right]=h$

$$1-e^{-kT}=\frac{hk}{u} \Rightarrow e^{-kT}=\frac{u-hk}{u}$$

$$\therefore -kT=\ln\left(\frac{u-hk}{u}\right)$$

$$T=\frac{1}{k}\ln\left(\frac{u}{u-hk}\right)$$

$Q7ai)$ Consider $y=e^{x^2-x}$

$$y'=(2x-1)e^{x^2-x} \Rightarrow y'=0 \text{ when } x=\frac{1}{2}$$

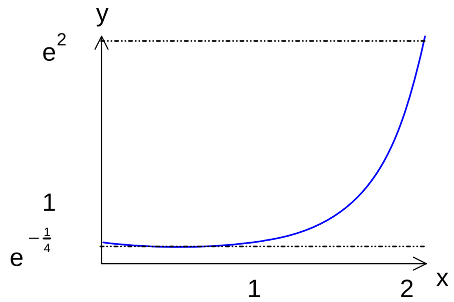
x	0	$\frac{1}{2}$	1
y'	$-ve$	0	$+ve$

$\therefore$ Minimum at $\left(\frac{1}{2},e^{-\frac{1}{4}}\right)$

also $f(0)=e^0=1$ $\quad f(2)=e^2$ $\quad\therefore$ Maximum at $(2,e^2)$

$areaOABC<\int_0^2 e^{x^2-x}dx<areaOADE$

$\therefore 2e^{-\frac{1}{4}}<\int_0^2 e^{x^2-x}dx<2e^2$



$Q7bi)\quad let\ u = a - x \Rightarrow du = -dx,\ x = 0 \Rightarrow u = a,\ x = a \Rightarrow u = 0$

$$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(u)\times -du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx$$

$ii)$
$$\int_0^{\pi} x\cos^2 x\,dx = \int_0^{\pi} (\pi - x)\cos^2(\pi - x)\,dx = \int_0^{\pi}(\pi - x)\cos^2 x\,dx$$

$$\therefore 2\int_0^{\pi} x\cos^2 x\,dx = \pi\int_0^{\pi}\cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi}(\cos 2x + 1)\,dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\pi} = \frac{\pi}{2}(\pi - 0)$$

$$\therefore \int_0^{\pi} x\cos^2 x\,dx \frac{\pi^2}{4}$$

$Q7c)\quad \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$$= \cos^n\theta + \binom{n}{1}\cos^{n-1}\theta(i\sin\theta) + \binom{n}{2}\cos^{n-2}\theta(i\sin\theta)^2 + \binom{n}{3}\cos^{n-3}\theta(i\sin\theta)^3 + \ldots$$

$$= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \ \ldots$$

$$+i\left(\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ \ldots\right)$$

using $i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ \ldots$

Equating real and imaginary parts

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \ \ldots \quad ---(1)$$

$$\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ \ldots \quad ----(2)$$

$ii)\ (2)\div(1)$
$$\frac{\sin n\theta}{\cos n\theta} = \tan n\theta = \frac{\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \ \ldots}{\cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \ \ldots}$$

divide top and bottom by $\cos^n\theta$

$$\tan n\theta = \frac{\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \ \ldots}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \ \ldots}$$

$Q8a)$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y \quad ---(1)$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y \quad ---(2)$$

$$(1)-(2) \Rightarrow \cos(x+y) - \cos(x-y) = -2\sin x\sin y$$

$$let\ x + y = S,\quad x - y = T \ \Rightarrow x = \frac{S+T}{2},\ \ y = \frac{S-T}{2}$$

$$\therefore \cos S - \cos T = -2\sin\frac{S+T}{2}\sin\frac{S-T}{2}$$

$Q8b$
$$I_n = \int_0^{\frac{\pi}{4}}\frac{1-\cos 2nx}{\sin 2x}\,dx$$

$i)$
$$I_1 = \int_0^{\frac{\pi}{4}}\frac{1-\cos 2x}{\sin 2x}\,dx = \int_0^{\frac{\pi}{4}}\frac{2\sin^2 x}{2\sin x\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}\,dx = -\left[\ln|\cos x|\right]_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \frac{1}{2}\ln 2$$

$Q8bii)$
$$I_{2r+1} - I_{2r-1} = \int_0^{\frac{\pi}{4}}\left(\frac{1-\cos(4xr+2x)}{\sin 2x} - \frac{1-\cos(4xr-2x)}{\sin 2x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}}\frac{-2\sin 4xr.\sin(-2x)}{\sin 2x}\,dx \qquad from(a)$$

$$= 2\int_0^{\frac{\pi}{4}}\sin 4xr\,dx = \frac{-1}{2r}\left[\cos 4xr\right]_0^{\frac{\pi}{4}} = \frac{-1}{2r}\left[\cos r\pi - 1\right]$$

$$= \frac{-1}{2r}\left[(-1)^r - 1\right] = \frac{1-(-1)^r}{2r}$$

$Q8biii)\quad I_9 = I_{2\times 4+1}\ \therefore r = 4 \Rightarrow I_9 - I_7 = 0 \quad ----(1)$

$$r = 3 \Rightarrow I_7 - I_5 = \frac{1+1}{6} = \frac{1}{3} \quad ---(2)$$

$$r = 2 \Rightarrow I_5 - I_3 = 0 \quad -----(3)$$

$$r = 1 \Rightarrow I_3 - I_1 = \frac{1+1}{2} = 1 \quad ----(4)$$

$$(1)+(2)+(3)+(4) \Rightarrow I_9 - I_1 = \frac{4}{3}$$

$I_1 = \frac{1}{2}\ln 2\ \ from\ )i)$

$$\therefore I_9 = \frac{4}{3} + \frac{1}{2}\ln 2$$

$Q8ci)\quad P\left(a\cos\theta, b\sin\theta\right)\quad Q\left(a\cos\phi, b\sin\phi\right)$

$$m_{PQ} = \frac{b\left(\sin\phi - \sin\theta\right)}{a\left(\cos\phi - \cos\theta\right)}$$

$\therefore$ *equation of chord PQ is* $y - b\sin\theta = \dfrac{b\left(\sin\phi - \sin\theta\right)}{a\left(\cos\phi - \cos\theta\right)}\left(x - a\cos\theta\right)$

$$\Rightarrow ay\left(\cos\phi - \cos\theta\right) - ab\sin\theta\left(\cos\phi - \cos\theta\right) = bx\left(\sin\phi - \sin\theta\right) - ab\cos\theta\left(\sin\phi - \sin\theta\right)$$

$ii)$ *Focal chord through* $\left(ae, 0\right)$

$$\therefore -ab\sin\theta\left(\cos\phi - \cos\theta\right) = bae\left(\sin\phi - \sin\theta\right) - ab\cos\theta\left(\sin\phi - \sin\theta\right)$$

$$e\left(\sin\phi - \sin\theta\right) = \cos\theta\sin\phi - \cos\theta\sin\theta - \sin\theta\cos\phi + \sin\theta\cos\theta$$

$$= \sin\left(\phi - \theta\right)$$

$$\Rightarrow e = \frac{\sin\left(\phi - \theta\right)}{\left(\sin\phi - \sin\theta\right)}$$