Total marks – 120
Attempt Question 1-8
All questions are of equal value
Answer each question in the appropriate writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

(a) Find

(i) $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **2**

(ii) $\displaystyle\int \frac{dx}{x^2+2x+2}$ **2**

(b) Use the substitution $t=\tan\dfrac{\theta}{2}$ to find $\displaystyle\int \frac{dx}{5+4\cos x+3\sin x}$ **3**

(c) Use the substitution $u=-x$ to evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{e^x+1}$ **3**

(d) Evaluate the following definite integrals:

(i) $\displaystyle\int_0^1 \cos^{-1} x\,dx$ **2**

(ii) $\displaystyle\int_1^2 x(\ln x)^2\,dx$ **3**

Question 2 **(**15 marks) Start a new booklet

(a) If $z = 3 - 2i$ mark clearly on an Argand diagram the points represented by,

(i) $2z$ **1**

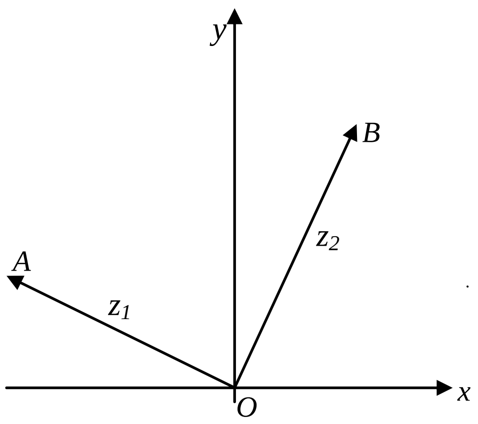
(ii) $-2iz$ **2**

(b) z is a complex number such that $\arg z = \frac{\pi}{3}$ and $|z| \le 2$.

(i) Sketch the locus of the point P representing z in the Argand diagram. **2**

(ii) Find the possible values of the principal argument of $z - 1$ for z on this locus. **2**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral **3**

(ii) Explain why $z_2 - z_1$ is equal to z_2 rotated by $\frac{\pi}{3}$ radians in a clockwise direction **2**

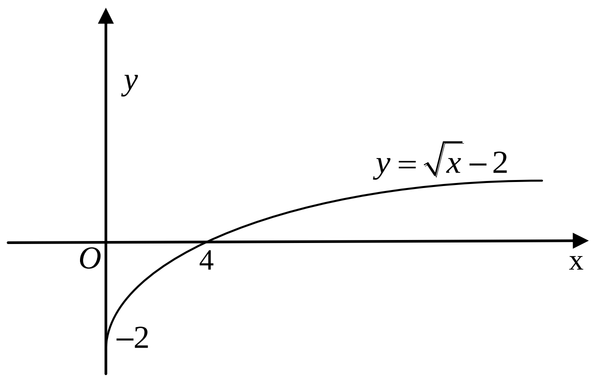
(iii) Express $z_2 - z_1$ in modulus-argument form. **3**

Question 3 **(**15 marks) Start a new booklet

(a) The polynomial $p(x) = x^4 - 2x^3 + 2x - 1$ has a root of multiplicity 3. Find this root and hence factorise $p(x)$ **3**

(b) Sketch the curve $y^2 = x^2\left(1 - x^2\right)$ clearly showing all relevant details. **6**

(c)



The diagram shows the graph of the function $f(x) = \sqrt{x} - 2$. On separate diagrams (each of half a page) sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = \left|f(x)\right|$ **1**

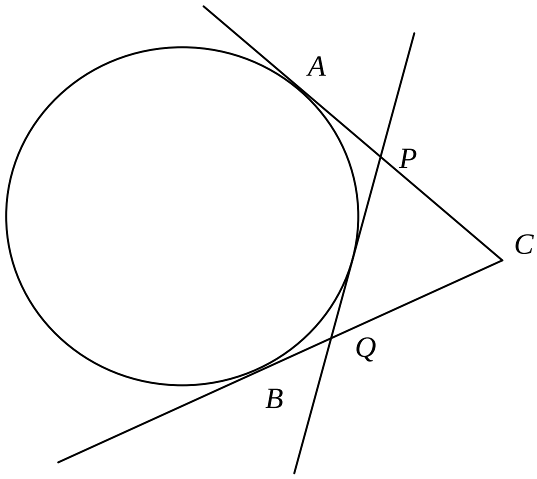
(ii) $y = \left[f(x)\right]^2$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \ln f(x)$ **2**

Question 4 **(**15 marks) Start a new booklet

(a) A and B are two points on a circle. Tangents at A and B meet at C. A third tangent cuts CA and CB in P and Q respectively, as shown in the diagram. Show that the perimeter of ΔCPQ is independent of PQ. **3**



(b) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **4**

(c) The polynomial $P(x) = x^4 + 3x^3 + 6x^2 + 12x + 8$ has one root $2i$. Find all the roots of $P(x)$. **4**

(d) If α, β are the roots of the equation $x^2 - px + q = 0$ and $S_n = \alpha^n + \beta^n$ where n is a positive integer, show that **4**

$$S_{n+2} - pS_{n+1} + qS_n = 0$$

Hence, or otherwise find S_3, S_4 in terms of p, q.

Question 5 **(**15 marks) Start a new booklet

(a) $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ are points on different branches of the hyperbola $xy = 9$.

(i) Find the equation of the tangent at *P*. **2**

(ii) Find the point of intersection, *T*, of the tangents at *P* and *Q*. **2**

(iii) If the chord *PQ* passes through the point $(0, 2)$, find the locus of *T*. **3**

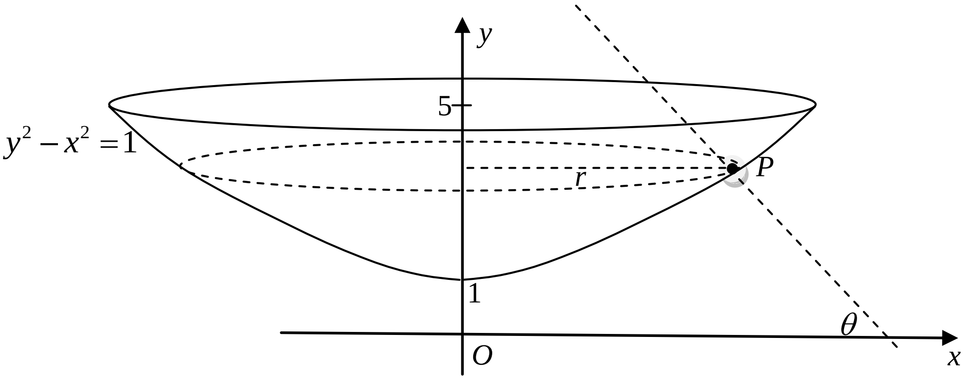
(b) The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved around the line $x = 3$. Derive the volume of the resulting solid as a definite integral. **Do not calculate the value of this integral**. **4**

(c) A solid has, as its base, the circular region in the *xy*-plane bounded by the graph of $x^2 + y^2 = a^2$, where $a > 0$. If every cross-section by a plane perpendicular to the *x*-axis is an equilateral triangle, with one side in the base, show that the volume of the solid is $\dfrac{4\sqrt{3}}{3}a^3$ units^3. **4**

Question 6 (15 marks) Start a new booklet

(a) A particle of mass m moves in a straight line away from a fixed point O in the line, such that at time t its displacement from O is x and its velocity is v. At time $t=0$, $x=1$ and $v=0$. Subsequently, the only force acting on the particle is one of magnitude $m\frac{k}{x^2}$, where k isa positive constant in a direction away from O. Show that v cannot exceed $\sqrt{(2k)}$. **4**

(b)



A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ about the y axis. A particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal as shown, where $\tan\theta = \frac{\sqrt{1+r^2}}{r}$. **4**

(ii) Draw a diagram showing the forces acting on P. **1**

(iii) By resolving these forces in the horizontal and vertical directions show that $r = \frac{\sqrt{g^2 - \omega^4}}{\omega^2}$ and the normal reaction $N = m\sqrt{2g^2 - \omega^4}$ **3**

(iv) Since P must be in contact with the surface of the bowl and the radius must be positive, prove $\sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$. **3**

Question 7 **(**15 marks) Start a new booklet

a) i) Expand $(2+i)(3+i)$ **1**

ii) By considering the arguments of each side, or otherwise show that **2**

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

iii) By first expanding $(p+q+i)(p^2+pq+1+iq)$ derive **4**

$$\tan^{-1}\left(\frac{1}{p+q}\right)+\tan^{-1}\left(\frac{q}{p^2+pq+1}\right)=\tan^{-1}\left(\frac{1}{p}\right)$$

b) i) Prove, by Mathematical Induction, that **3**

$\sin\theta=2^n\cos\frac{\theta}{2}\cos\frac{\theta}{4}....\cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$ for any positive integer n

ii) Hence show that $\dfrac{\sin\theta}{\theta}=\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}....\cos\left(\dfrac{\theta}{2^n}\right)\dfrac{\sin\left(\frac{\theta}{2^n}\right)}{\left(\frac{\theta}{2^n}\right)}$ **1**

iii) Hence find $\lim\limits_{n\to\infty}\left(\dfrac{\sin\theta}{\theta}\right)$ **1**

iv) Using $\prod$ as the product of terms, show that when $\theta=\dfrac{\pi}{2}$ **1**

$$\prod_{k=2}^{\infty}\cos\left(\frac{\pi}{2^k}\right)=\frac{2}{\pi}$$

v) Hence show, by applying the half angle formula for $\cos\left(\dfrac{\theta}{2}\right)$ **2**

$$\frac{2}{\pi}=\frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\times....$$

Question 8 **(**15 marks) Start a new booklet

(a) The ellipse $\mathcal{E}$: $\left(\frac{x}{5}\right)^2+\left(\frac{y}{3}\right)^2=1$ has foci $S(4,0)$ and $S'(-4,0)$.

(i) Sketch the ellipse $\mathcal{E}$ indicating its foci *S, S'* and its directrices. **1**

(ii) Find the tangent at $P(x_1, y_1)$ on the ellipse $\mathcal{E}$. **1**

(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through *S*. Show that **2**

$4(y_2 - y_1) = x_1y_2 - x_2y_1$.

(iv) If it is also known that $Q(x_2, y_2)$ lies on $\mathcal{E}$, find the point of intersection of the tangents at *P* and *Q* **1**

(v) By using the result in (iii) show the tangents at *P* and *Q* intersect on the directrix corresponding to *S*. **2**

(b) $I_n=\int_1^e (1-\ln x)^n\,dx$, $n=1,2,3,\ldots$

(i) Show $I_n=-1+nI_{n-1}$, $n=1,2,3,\ldots$ **3**

(ii) Hence evaluate $\int_1^e (1-\ln x)^3\,dx$. **2**

(iii) Show that $\frac{I_n}{n!}=e-\sum_{r=0}^{n}\frac{1}{r!}$, $n=1,2,3,\ldots$ **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a \neq 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Solutions

Question 1

(a) (i) $u=\sin\theta,\ du=\cos\theta d\theta$

$$\int\frac{du}{u^5}=-\frac{1}{4}u^{-4}+c$$

$$=-\frac{1}{4}\sin^{-4}\theta+c$$

(ii) $$\int\frac{dx}{(x+1)^2+1}=\tan^{-1}(x+1)+c$$

(b) $t=\tan\frac{x}{2},\ \therefore\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}\left(1+t^2\right)$

$$\sin x=\frac{2t}{1+t^2},\cos x=\frac{1-t^2}{1+t^2}$$

$$\int\frac{\frac{2dt}{1+t^2}}{5+\frac{4\left(1-t^2\right)}{1+t^2}+\frac{6t}{1+t^2}}$$

$$=\int\frac{2dt}{t^2+6t+9}=\int\frac{2dt}{(t+3)^2}$$

$$=2(t+3)^{-1}+c$$

$$=\frac{-2}{\tan\frac{x}{2}+3}+c$$

(c) $u=-x,-du=dx$

When $x=-1,u=1$ and when $x=1,u=-1$

$$\int_1^{-1}\frac{-du}{e^{-u}+1}=\int_{-1}^{1}\frac{du}{e^{-u}+1}=\int_{-1}^{1}\frac{du}{\frac{1}{e^u}+1}=\int_{-1}^{1}\frac{e^u du}{1+e^u}$$

$$=\left[\ln\left(e^u+1\right)\right]_{-1}^{1}=\ln\frac{e+1}{1/e+1}=\ln e=1$$

d)i) $$\int_0^1\cos^{-1}x\frac{d}{dx}(x)\,dx=x\cos^{-1}x\Big]_0^1-\int_0^1\frac{-x}{\sqrt{1-x^2}}dx$$

$$=0-\frac{1}{2}\int_0^1\frac{-2x}{\sqrt{1-x^2}}dx=\sqrt{1-x^2}\Big]_0^1=1$$

OR

Area= $\int_0^{\pi/2}\sin y\ dy=\left[-\cos y\right]_0^{\pi/2}=1$

(ii) $$\left[\frac{1}{2}x^2(\ln x)^2\right]_1^2-\int_1^2\frac{1}{2}x^2.2.\frac{1}{x}\ln x\ dx$$

$$=2(\ln 2)^2-\int_1^2 x\ln x\ dx$$

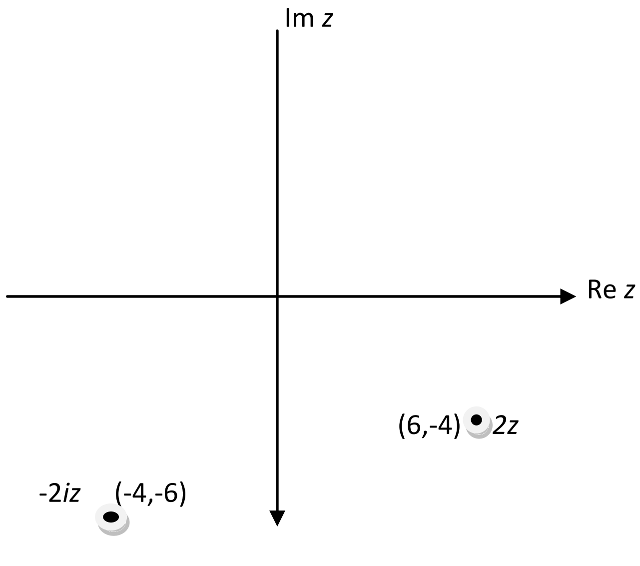
$$= 2(\ln 2)^2 - \left[\frac{1}{2}x^2.\ln x\right]_1^2 + \int_1^2 x^2.\frac{1}{x}\,dx$$

$$= 2(\ln 2)^2 - 2\ln 2 + \left[\frac{1}{4}x^2\right]_1^2$$

$$= 2(\ln 2)^2 - 2\ln 2 + \frac{3}{4}$$

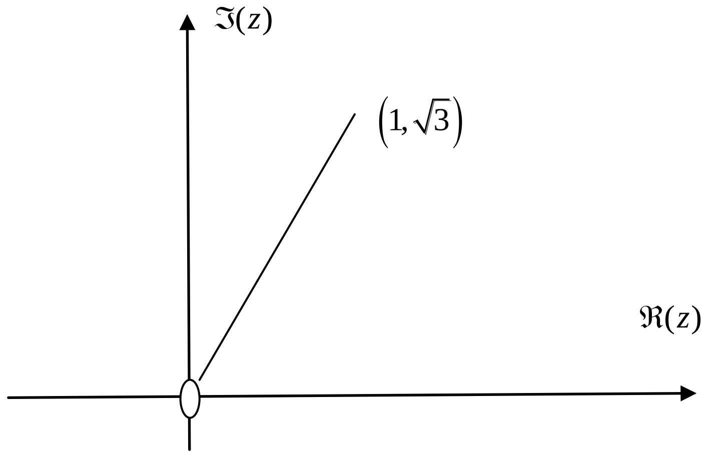
Question 2

(a)



(b)

i)



(ii) $\frac{\pi}{2} \le \arg(z-1) < \pi$

(i) $\therefore \arg \angle AOB = \arg z_1 - \arg z_2 = \frac{4\pi}{5} - \frac{7\pi}{15} = \frac{\pi}{3}$

$OA = OB = 2$

Since Triangle is Isosceles and the included angle is $\pi/3$

$\therefore \Delta OAB$ is equilateral.

(ii) The vector $\overrightarrow{AB}$ represents $z_2 - z_1$. Now, $\overrightarrow{AB}$ is a clockwise rotation of $\overrightarrow{OB}$ by $\frac{\pi}{3}$ since . $|AB| = |OB|$ and $\angle AOB = \frac{\pi}{3}$

iii)

$$\therefore z_2 - z_1 = z_2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$$

$$= 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$= 2\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)$$

<u>Question 3</u>

(a) $p(x) = x^4 - 2x^3 + 2x - 1$

$\therefore p'(x) = 4x^3 - 6x^2 + 2$

$\therefore p''(x) = 12x^2 - 12x = 0$

when $x = 0, 1$

$p'(1) = 4(1)^3 - 6(1)^2 + 2 = 0$

and $p(1) = 1 - 2 + 2 - 1 = 0$

$\therefore x = 1$ is the triple root

$\sum \alpha = +2 = 1 + 1 + 1 + \alpha$

$\therefore$ the fourth root $\alpha = -1$

$\therefore$ roots are 1, 1, 1, -1.

$b)$ $y^2 = x^2\left(1 - x^2\right)$

Since $y \ge 0$ $x^2 \le 1 \Rightarrow |x| \le 1$

when $y = 0$ $x = 0, \pm 1$

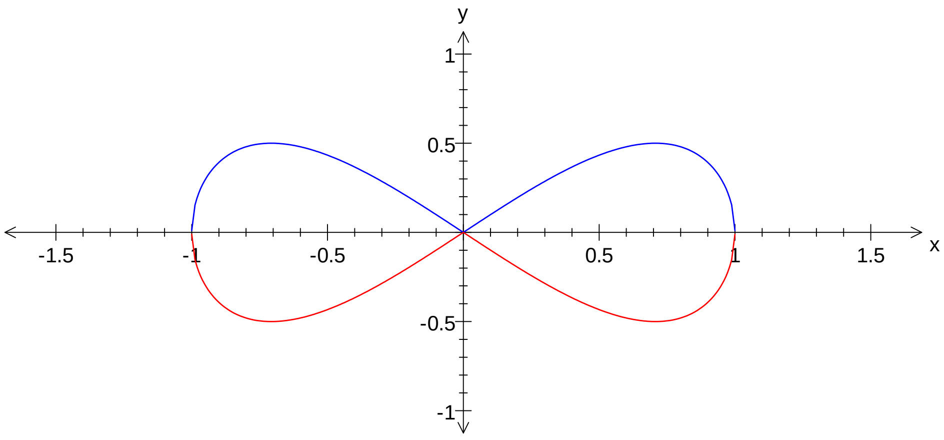
$$y\frac{dy}{dx} = 2x - 4x^3 = 2x\left(1 - 2x^2\right)$$

$$\therefore \frac{dy}{dx} = \frac{2x\left(1 - 2x^2\right)}{\pm\sqrt{x^2\left(1 - x^2\right)}} = \frac{2\left(1 - 2x^2\right)}{\pm\sqrt{\left(1 - x^2\right)}}$$

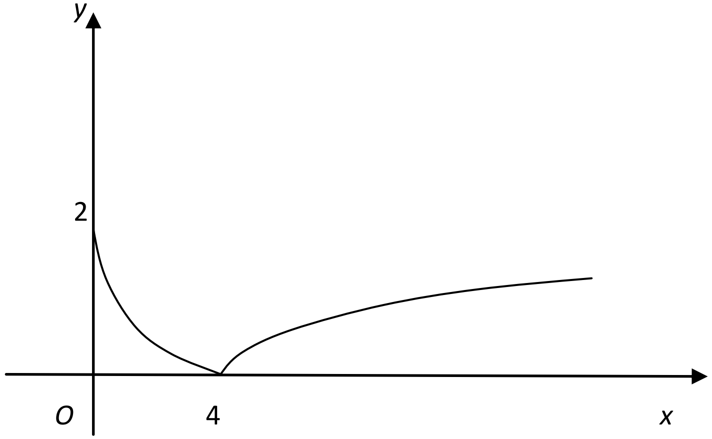
$= 0$ when $x = \pm\frac{1}{\sqrt{2}} \Rightarrow y = \pm\frac{1}{2}$

when $x \to \pm 1$ $\frac{dy}{dx} \to \infty \therefore$ vertical tangent

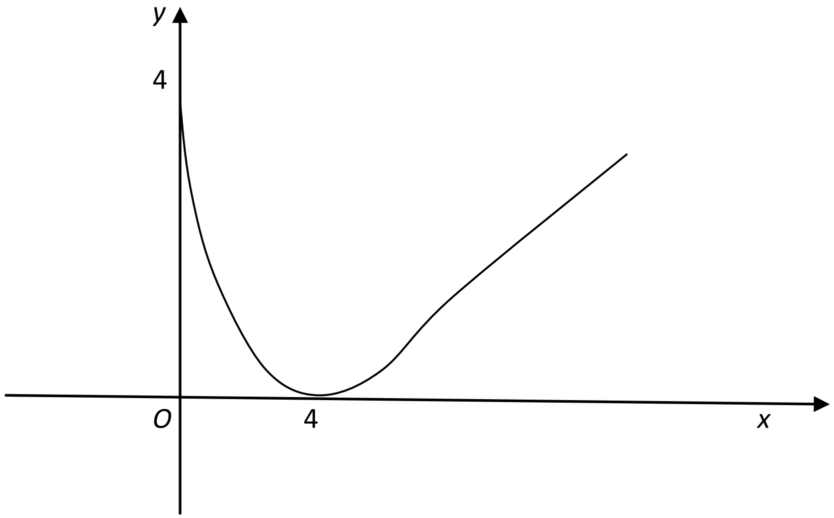
at $x = 0$ $\frac{dy}{dx} = \pm 1 \therefore$ tangents inclined at ± 45



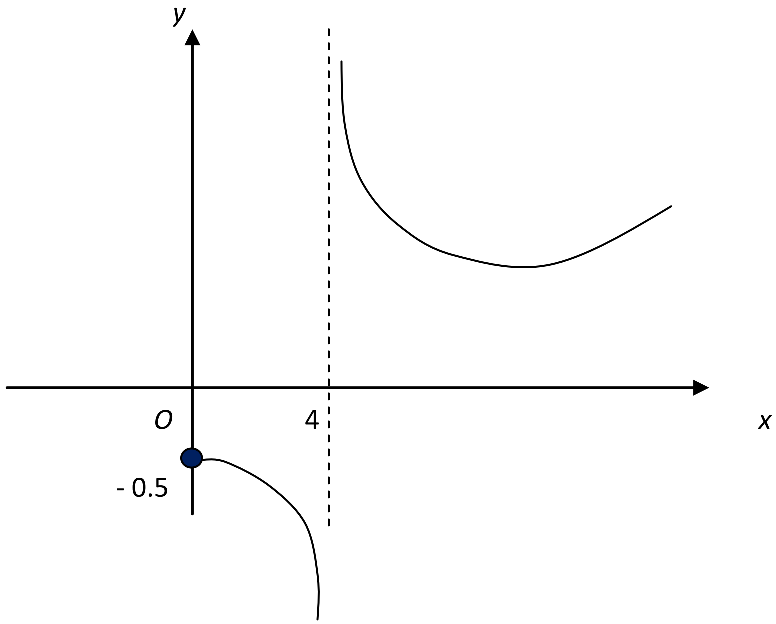
(c)(i) $y = \left| f\left(x\right) \right|$



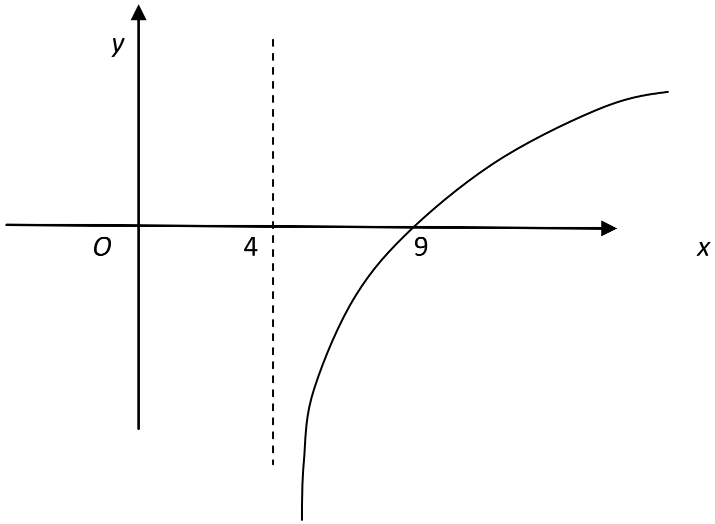
(ii) $y = \left[f\left(x\right) \right]^2$



(iii) $y = \dfrac{1}{f\left(x\right)}$



(iv) $y = \ln f\left(x\right)$



Question 4

(a) Perimeter of $\Delta CPQ = CP + CQ + PQ$

But $PQ = AP + BQ$ (tangents drawn from *P* are of equal length and
(tangents drawn from *Q* are of equal length)

Perimeter of $\Delta CPQ = CP + AP + CQ + BQ = CA + CB$

Which is independent of *PQ*.

b) $P(2) = 9,\ P(3) = 4$

$P(x) = Q(x).(x-2)(x-3) + R(x)$, where $R(x) = ax + b$

$P(2) = 0 + 2a + b = 9$

$P(3) = 0 + 3a + b = 4$

$a = -5,\ b = 19$

Remainder is $19 - 5x$

$c)$ Since all coefficients are real if $2i$ is a root, so is the conjugate $-2i$
Let other 2 roots be α, β $\therefore 2i + -2i + \alpha + \beta = \alpha + \beta = -3$
$\prod \alpha = 4\alpha\beta = 8$
solve simultaneously $4\alpha(-3-\alpha) = -12\alpha - 4\alpha^2 = 8$
$\Rightarrow \alpha^2 + 3\alpha + 2 = 0 \Rightarrow \alpha = -1, -2$
$\therefore P(x) = (x-2i)(x+2i)(x+1)(x+2)$

d)

$d)$ $x^2 - px + q = 0$, if roots are α, β
$\alpha^2 - p\alpha + q = 0$ and $\beta^2 - p\beta + q = 0 \Rightarrow \alpha^2 + \beta^2 = p(\alpha+\beta) - 2q = p^2 - 2q = S_2$
also $\alpha^n(\alpha^2 - p\alpha + q) = 0$ $\beta^n(\beta^2 - p\beta + q) = 0$
i.e. $\alpha^{n+2} - p\alpha^{n+1} + q\alpha^n = 0$ $\beta^{n+2} - p\beta^{n+1} + q\beta^n = 0$ add these results
$S_{n+2} - pS_{n+1} + qS_n = 0$
$S_3 - pS_2 + qS_1 = S_3 - p(p^2 - 2q) + q(p) = 0 \quad \Rightarrow S_3 = p^3 - 3pq$
$S_4 - pS_3 + qS_2 = S_4 - p(p^3 - 3pq) + q(p^2 - 2q) = 0$
$S_4 = p^4 - 4p^2q + 2q^2$

Question 5

(a) (i) $x\dfrac{dy}{dx} + y = 0,\ \therefore \dfrac{dy}{dx} = \dfrac{-y}{x}$

At *P*, $\dfrac{dy}{dx} = \dfrac{-3/p}{3p} = -\dfrac{1}{p^2}$

Required equation: $y - \dfrac{3}{p} = -\dfrac{1}{p}(x - 3p)$

Which gives $x + p^2 y = 6p$

(ii) tangent at *P* $x + p^2 y = 6p$

Tangent at Q $x + q^2 y = 6q$

When solved simultaneously, we get the coordinates of T:

$$\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$$

(iii) $m_{PQ} = \dfrac{\frac{3}{p} - \frac{3}{q}}{3p - 3q} = -\dfrac{1}{pq}$

Equation *PQ*: $y - \dfrac{3}{p} = -\dfrac{1}{pq}(x - 3p)$

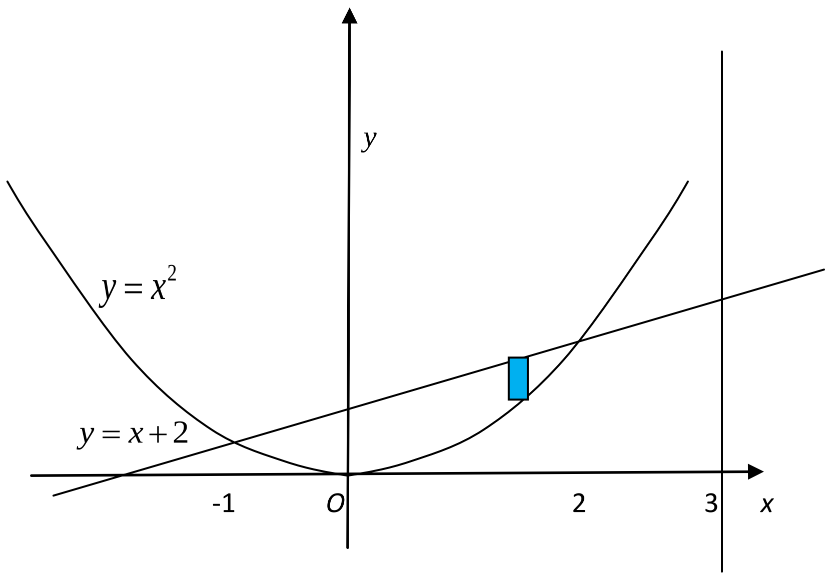
Now, when $x = 0, y = 2$

$\therefore \dfrac{p+q}{pq} = \dfrac{2}{3}$ or $p + q = \dfrac{2pq}{3}$

At *T*, $x = \dfrac{6pq}{p+q} = \dfrac{6pq}{\frac{2pq}{3}} = 9$

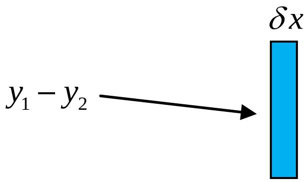
Therefore the locus is *x* = 9

b)



To find points of intersection: $x^2 = x + 2, \ \therefore x = -1, 2$

Consider a typical strip



Rotate the strip to form a shell. The volume of the shell is given by

$\delta V = 2\pi rh.\delta x$, where $r = 3 - x$ and $h = y_1 - y_2 = x + 2 - x^2$

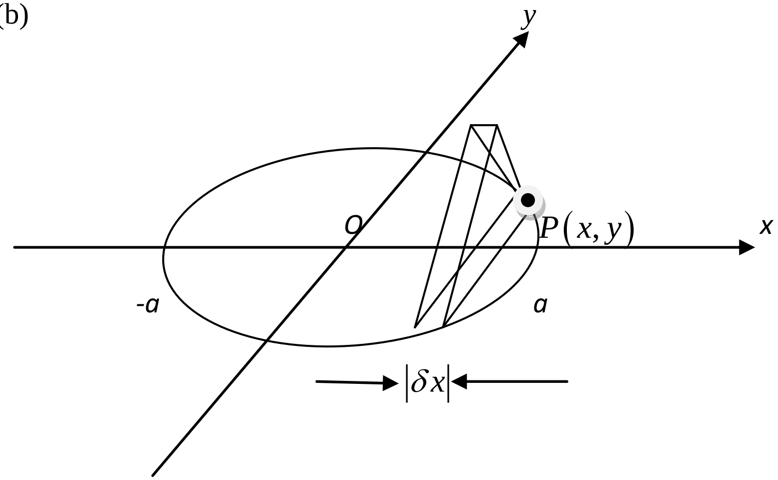
$$V \approx \sum_{x=-1}^{x=2} 2\pi(3-x)\left(x+2-x^2\right)\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=-1}^{x=2} 2\pi(3-x)\left(x+2-x^2\right)\delta x$$

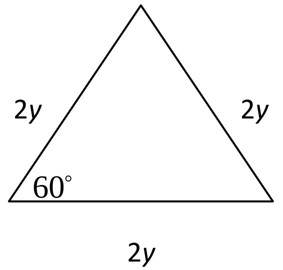
$$= \int_{-1}^{2} 2\pi(3-x)\left(x+2-x^2\right)dx$$

$$OR \ = \int_{-1}^{2} \pi\left(y^2 - 11y + 6\sqrt{y} + 16\right)dy$$

(b)



Consider a typical slice of width δx.



$$\delta V = \frac{1}{2}2y.2y.\sin 60^\circ.\delta x = y^2\sqrt{3}.\delta x$$

$$V \approx \sum_{x=-a}^{x=a} y^2\sqrt{3}.\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} y^2\sqrt{3}.\delta x = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} \left(a^2 - x^2\right)\sqrt{3}.\delta x$$

$$= \sqrt{3}\int_{-a}^{a} \left(a^2 - x^2\right)dx$$

$$= 2\sqrt{3}\int_{0}^{a} \left(a^2 - x^2\right)dx$$

$$= 2\sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{4\sqrt{3}}{3}a^3$$

Question 6

(a) Choose the initial direction as positive

$$\ddot{x} = \frac{k}{x^2},\ k > 0$$

$$v\frac{dv}{dx} = \frac{k}{x^2} \Rightarrow v\,dv = \frac{k}{x^2}dx$$

$$\frac{1}{2}v^2 = -\frac{k}{x} + c, \text{ where } c \text{ is constant}$$

Now, when $x = 1, v = 0 \Rightarrow c = k$

$$\therefore v^2 = 2k\left(1 - \frac{1}{x}\right)$$

Now, $x \geq 1 \qquad \therefore 0 \leq 1 - \frac{1}{x} < 1$

$$\therefore 0 \leq v^2 < 2k$$

Hence, v cannot exceed $\sqrt{2k}$

(b) (i) $y^2 - x^2 = 1 \Rightarrow 2y\frac{dy}{dx} - 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

At P, $\frac{dy}{dx} = \frac{r}{\sqrt{1+r^2}}$

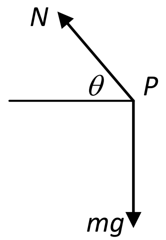
Hence, the gradient of the normal at P is $\frac{-\sqrt{1+r^2}}{r}$

Now, the gradient of the normal is the tangent of the angle made with the 'positive' x axis.

$$\therefore \tan\left(180^\circ - \theta\right) = \frac{-\sqrt{1+r^2}}{r}$$

$$\therefore \tan\theta = \frac{\sqrt{1+r^2}}{r}$$

(ii)



(iii) Resolve forces

Horizontally

$$mr\omega^2 = N\cos\theta$$

Vertically

$$mg = N\sin\theta$$

$\therefore \tan\theta = \frac{g}{r\omega^2} = \frac{\sqrt{1+r^2}}{r}$ from part (i)

$$1 + r^2 = \frac{g^2}{\omega^4} \Rightarrow r = \frac{\sqrt{g^2 - \omega^4}}{\omega^2}$$

Now, $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{mr\omega^2}{N}\right)^2 + \left(\frac{mg}{N}\right)^2 = 1$$

$$N^2 = m^2r^2\omega^4 + m^2g^2 = m^2\frac{g^2-\omega^4}{\omega^4}\omega^4 + m^2g^2$$

$$= m^2\left(2g^2-\omega^4\right)$$

$$\therefore N = m\sqrt{\left(2g^2-\omega^4\right)}$$

(iv) $r = \dfrac{\sqrt{g^2-\omega^4}}{\omega^2}$ and $r > 0$

$\therefore g^2 > \omega^4$

But $N > 0$ $\quad \therefore 2g^2 > \omega^4$

Both these conditions exist if $g > \omega^2$

Note: $y \le 5 \Rightarrow y^2 \le 25 \Rightarrow 1 + r^2 \le 25$

$$\therefore \frac{g^2}{\omega^4} > 25 \Rightarrow \omega \ge \sqrt{\frac{g}{5}}$$

$$\therefore \sqrt{\frac{g}{5}} \le \omega \le \sqrt{g}$$

Question 7

$a)\ i)\ (2+i)(3+i) = 6 - 1 + 2i + 3i = 5 + 5i$

$ii)\ \therefore \arg(2+i)(3+i) = \arg(5+5i)$

$\therefore \arg(2+i) + \arg(3+i) = \tan^{-1}1$

$\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{4}$

$a)\ iii)\ (p+q+i)\left(p^2+pq+1+qi\right)$

$= p^3 + p^2q + p + pqi + p^2q + pq^2 + q + q^2i + p^2i + pqi + i - q$

$= p\left(p^2+2pq+q^2+1\right) + i\left(p^2+2pq+q^2+1\right)$

$\therefore \arg(p+q+i)\left(p^2+pq+1+qi\right) = \arg\left(p\left(p^2+2pq+q^2+1\right) + i\left(p^2+2pq+q^2+1\right)\right)$

$\therefore \tan^{-1}\dfrac{1}{p+q} + \tan^{-1}\dfrac{q}{p^2+pq+1} = \tan^{-1}\dfrac{1}{p}$ as required

$b)i)$ let $n=1$ $LHS = \sin\theta$ $\quad RHS = 2\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2} = \sin\theta$ $\therefore$ true when $n = 1$.

assume true when $n = k$ i.e. $\sin\theta = 2^k\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}...\cos\dfrac{\theta}{2^k}\sin\dfrac{\theta}{2^k}$

now $\sin\dfrac{\theta}{2^k} = 2\cos\dfrac{\theta}{2^{k+1}}\sin\dfrac{\theta}{2^{k+1}}$ substitute in assumption

thus $\sin\theta = 2^k\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}...\cos\dfrac{\theta}{2^k}\times 2\cos\dfrac{\theta}{2^{k+1}}\sin\dfrac{\theta}{2^{k+1}}$

$\Rightarrow \sin\theta = 2^{k+1}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}...\cos\dfrac{\theta}{2^k}\cos\dfrac{\theta}{2^{k+1}}\sin\dfrac{\theta}{2^{k+1}}$

$\therefore$ if the result is true for $n = k$ it is also true for $n = k+1$

but it is true for $n = 1$ $\therefore$ it is true for $n = 1+1 = 2$ and so on for all positive integer n

$b)ii)$ Divide both sides by θ

thus $\dfrac{\sin\theta}{\theta} = \cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\ldots\cos\dfrac{\theta}{2^n}\sin\left(\dfrac{\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right)$

$iii)$ as $n \to \infty$ $\dfrac{\theta}{2^n} \to 0$ $\therefore \lim\limits_{n\to\infty}\sin\left(\dfrac{\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right) = 1$

$\therefore \lim\limits_{n\to\infty}\dfrac{\sin\theta}{\theta} = \cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\ldots\cos\dfrac{\theta}{2^n}\times 1$

$b)iii)$ Let $\theta = \dfrac{\pi}{2}$

$\therefore \lim\limits_{n\to\infty}\dfrac{\sin\theta}{\theta} = \dfrac{1}{\pi/2} = \cos\dfrac{\pi/2}{2}\cos\dfrac{\pi/2}{4}\ldots\cos\dfrac{\pi/2}{2^n} = \prod\limits_{k=2}^{\infty}\cos\dfrac{\pi}{2^k} = \dfrac{2}{\pi}$

$b)iv)$ $\therefore \dfrac{2}{\pi} = \cos\dfrac{\pi}{4}\cos\dfrac{\pi}{8}\cos\dfrac{\pi}{16}\ldots$

Now $\cos^2\theta = \dfrac{1}{2}(1+\cos 2\theta) \Rightarrow \cos\theta = \dfrac{1}{\sqrt{2}}\sqrt{1+\cos 2\theta}$

Now $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

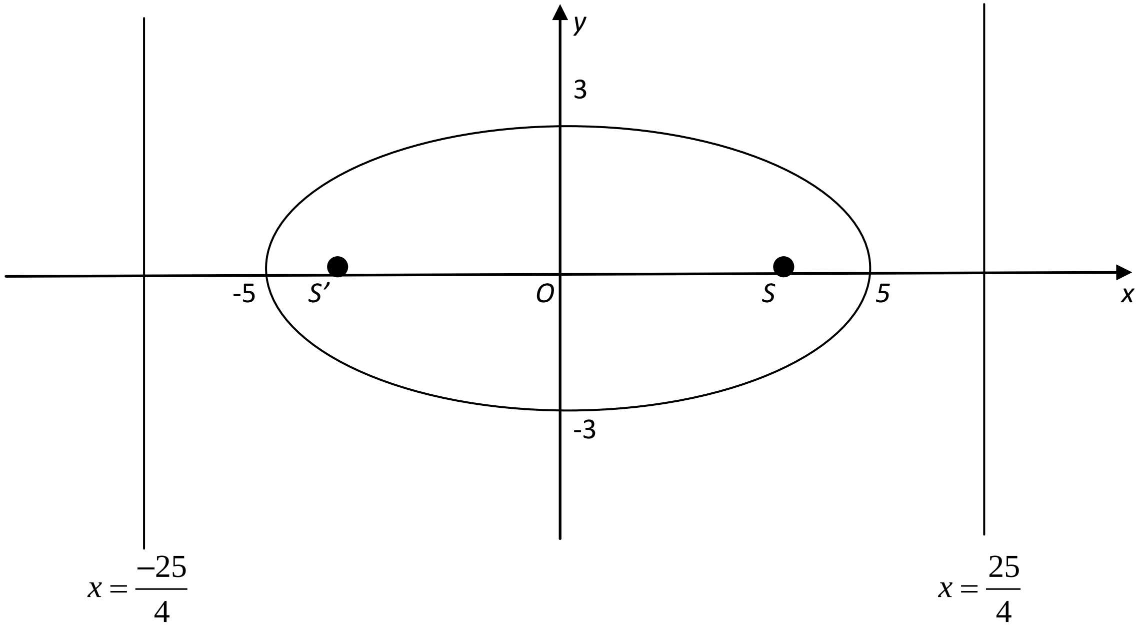
$\therefore \cos\dfrac{\pi}{8} = \dfrac{1}{\sqrt{2}}\sqrt{1+\cos\dfrac{\pi}{4}} = \dfrac{1}{\sqrt{2}}\sqrt{1+\dfrac{\sqrt{2}}{2}} = \dfrac{1}{\sqrt{2}}\sqrt{\dfrac{2+\sqrt{2}}{2}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

also $\cos\dfrac{\pi}{16} = \dfrac{1}{\sqrt{2}}\sqrt{1+\cos\dfrac{\pi}{8}} = \dfrac{1}{\sqrt{2}}\sqrt{1+\dfrac{\sqrt{2+\sqrt{2}}}{2}} = \dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

$\therefore \dfrac{2}{\pi} = \dfrac{\sqrt{2}}{2}\times\dfrac{\sqrt{2+\sqrt{2}}}{2}\times\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\times\ldots$

Question 8

(a) (i)



$ae = \pm 4,\ a = 5,\ \therefore e = \dfrac{4}{5}$

Hence, the directrices are $x = \dfrac{\pm 25}{4}$

(ii) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

$\dfrac{2x}{25} + \dfrac{2y\,{}^{dy}\!/\!_{dx}}{9} = 1$

At $P(x_1, y_1)$, $\frac{dy}{dx} = \frac{-9x_1}{25y_1}$

Required equation:

$$y - y_1 = \frac{-9x_1}{25y_1}(x - x_1)$$

$$9xx_1 + 25yy_1 = 9x_1^2 + 25y_1^2 = 225$$

Note: $P(x_1, y_1)$ lies on the curve $9x_1^2 + 25y_1^2 = 225$

(iii) $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

At $x = 4, y = 0$

$$-y_1(x_2 - x_1) = (y_2 - y_1)(4 - x_1)$$

$$4(y_2 - y_1) = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1y_2 - x_2y_1$$

(iv) Two equations are: $9xx_1 + 25yy_1 = 225$ and $9xx_2 + 25yy_2 = 225$

$$25y = \frac{9xx_1 - 225}{y_1} = \frac{9xx_2 - 225}{y_2} \quad \therefore 9x(x_1y_2 - x_2y_1) = 225(y_2 - y_1)$$

$$\therefore 9x.4(y_2 - y_1) = 225(y_2 - y_1) \qquad x = \frac{225}{36} = \frac{25}{4}$$

(i) $I_n = \int_1^e (1 - \ln x)^n \, dx$

$$= \left[x(1 - \ln x)^n \right]_1^e - \int_1^e nx(1 - \ln x)^{n-1}\left(-\frac{1}{x}\right) dx$$

$$= -1 + nI_{n-1}$$

(ii) $I_3 = -1 + 3I_2 = -1 + 3(-1 + 2I_1) = -4 + 6(-1 + I_0)$

$$= -10 + 6\int_1^e dx = -10 + 6(e - 1) = -16 + 6e$$

(iii) $I_r = -1 + rI_{r-1}$

$$\frac{I_r}{r!} = \frac{-1}{r!} + \frac{rI_{r-1}}{r!}$$

$$\frac{I_r}{r!} = \frac{-1}{r!} + \frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^{n} \frac{I_r}{r!} = \sum_{r=1}^{n} \frac{-1}{r!} + \sum_{r=1}^{n} \frac{I_{r-1}}{(r-1)!}$$

$$\frac{I_n}{n!} + \sum_{r=1}^{n-1} \frac{I_r}{r!} = \sum_{r=1}^{n} \frac{-1}{r!} + \sum_{r=0}^{n-1} \frac{I_r}{r!}$$

$$\frac{I_n}{n!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{I_0}{0!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{\int_1^e dx}{0!} = \sum_{r=1}^{n} \frac{-1}{r!} + \frac{e}{0!} + \frac{-1}{0!} = e - \sum_{r=0}^{n} \frac{1}{r!}$$