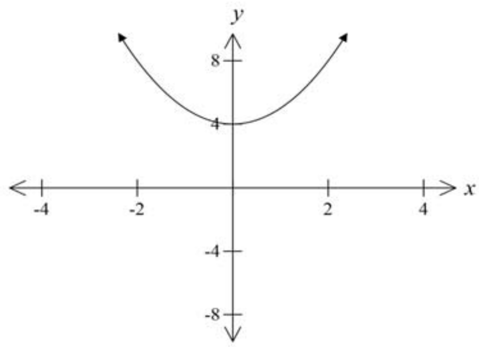
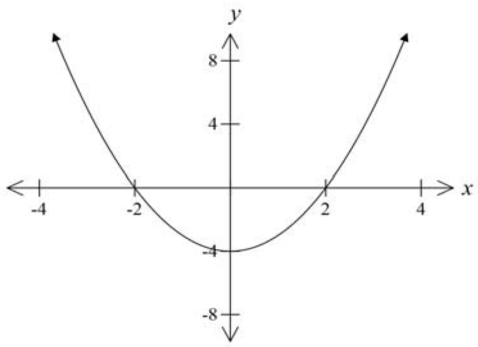
Year 12 Mathematics Ext. 2
Trial Examination 2012
Section 1 Objective Response Questions

1 Given that $f(x) = 4 - x^2$, which one of the following graphs best fits the graph of $y = |f(x)|$?

(A)



(B)



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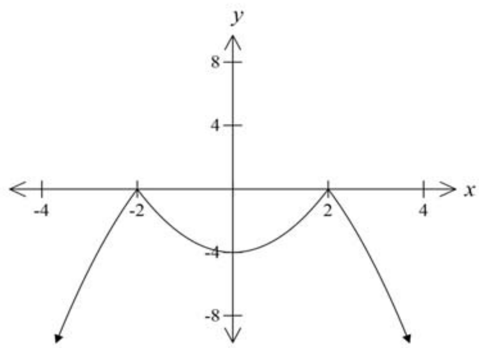
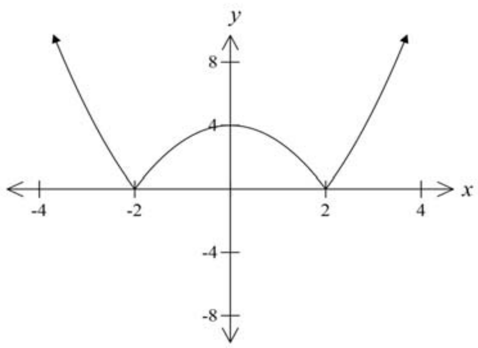


(D)



2 It is given that $3+i$ is a root of $P(z) = z^3 + az^2 + bz + 10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

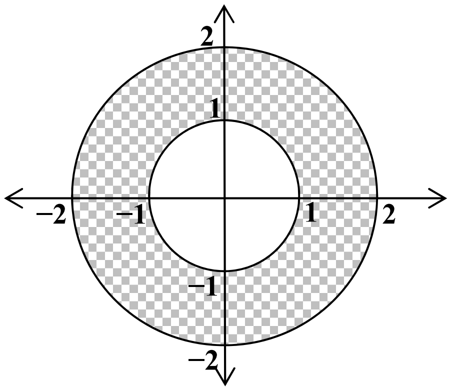
(A) $(z-1)(z^2 + 6z - 10)$

(B) $(z-1)(z^2 - 6z - 10)$

(C) $(z+1)(z^2 + 6z + 10)$

(D) $(z+1)(z^2 - 6z + 10)$

3 Consider the Argand diagram below.



Which inequality best defines the shaded area?

(A) $0 \le |z| \le 2$

(B) $1 \le |z| \le 2$

(C) $0 \le |z-1| \le 2$

(D) $1 \le |z-1| \le 2$

4 The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct equation?

(A) $\tan\theta\tan\phi = -\frac{b^2}{a^2}$

(B) $\tan\theta\tan\phi = -\frac{a^2}{b^2}$

(C) $\tan\theta\tan\phi = \frac{b^2}{a^2}$

(D) $\tan\theta\tan\phi = \frac{a^2}{b^2}$

5 Consider the hyperbola with the equation $\frac{x^2}{16}-\frac{y^2}{9}=1$.

What are the coordinates of the foci of this hyperbola?

(A) $(\pm 4, 0)$

(B) $(0, \pm 4)$

(C) $(0, \pm 5)$

(D) $(\pm 5, 0)$

6 Which one of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}\,dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

7 The area bounded by $y=x^3$, the line $y=8$, and the y axis, is rotated about the y axis to form a solid. The volume of this solid is:

(A) $\frac{2\pi}{5}$ cubic units

(B) $\frac{3\pi}{5}$ cubic units

(C) $\frac{93\pi}{5}$ cubic units

(D) $\frac{96\pi}{5}$ cubic units

8 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant. Which of the following is the correct expression for square of the velocity where x is the distance of the particle fallen from rest?

(A) $v^2=\frac{g}{k}\left(1-e^{-2kx}\right)$

(B) $v^2=\frac{g}{k}\left(1+e^{-2kx}\right)$

(C) $v^2=\frac{g}{k}\left(1-e^{2kx}\right)$

(D) $v^2=\frac{g}{k}\left(1+e^{2kx}\right)$

9 Let α, β and γ be the roots of the equation $x^3 + 3x^2 + 4 = 0$. Which one of the following polynomial equations have roots α^2, β^2 and γ^2?

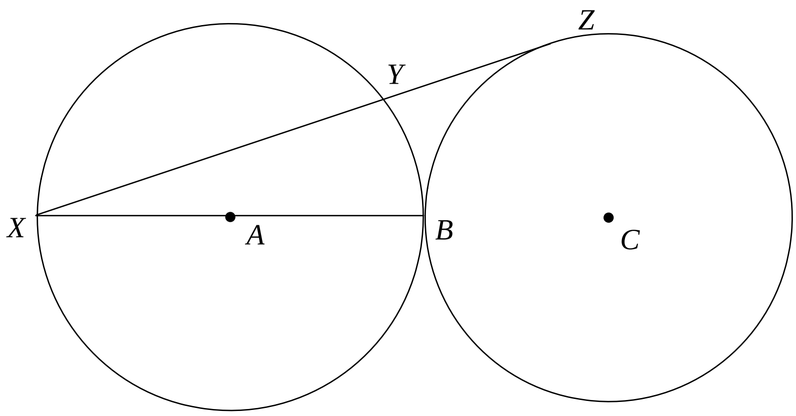
(A) $x^3 - 9x^2 - 24x - 4 = 0$

(B) $x^3 - 3x^2 - 12x - 4 = 0$

(C) $x^3 - 9x^2 - 24x - 16 = 0$

(D) $x^3 - 3x^2 - 12x - 16 = 0$

10



The diagram above shows two circles of equal radii with centres A and C respectively. The two circles touch externally at B and the line XB is a diameter. The line XZ is the tangent to the circle centre C, at Z, cutting the circle, centre A, in Y. Which is the correct expression that relates the length of XZ to the length of XY?

(A) $3XZ = 4XY$

(B) $XZ = 2XY$

(C) $2XZ = 3XY$

(D) $2XZ = 5XY$

SECTION II Extended Response Questions

Answers for this section should be recorded in the booklets provided.
Show all necessary working.
Start a new booklet for each question

Question 1 (15 marks) **Marks**

a) Find $\int \frac{x}{\sqrt{2-x^2}}\,dx$ using the substitution $x = \sqrt{2}\sin\theta$. **3**

b) i) Find the real numbers a and b such that

$$\frac{1}{x(2x+1)} = \frac{a}{x} + \frac{b}{2x+1}.$$

1

ii) Hence, evaluate $\int_{\frac{1}{2}}^{1} \frac{dx}{x(2x+1)}$. **2**

c) Use integration by parts to show that $\int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$. **3**

d) Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$, where $n \geq 0$ is an integer.

i) Prove that $I_n = \frac{(n-1)}{n} I_{n-2}$, for $n \geq 2$. **4**

ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x\,dx$. **2**

Question 2 Begin in a new booklet (15 marks) **Marks**

a) The complex number v has modulus 1 and argument $\frac{\pi}{6}$, and the complex number w has modulus 2 and argument $\frac{-2\pi}{3}$.

i) Express wv and iv in the modulus-argument form where each argument is between $-\pi$ and π. **2**

ii) Show that v is a solution of the equation $Z^4 = i\,Z$. Hence, or otherwise, state the other two non-zero roots of this equation. You may leave your answers in modulus-argument form. **3**

iii) Mark, on an Argand diagram, the points P, Q, R and S representing v, w, wv and iv respectively. **1**

iv) Hence, or otherwise, show that PS is parallel to RQ. **2**

v) Hence, or otherwise, find a real number u such that $iv - v = u\,(w - wv)$. **2**

b) Let $f(x) = \dfrac{11-x}{x^2-x-2}$,

i) Draw a one-third page sketch of the graph of $y = f(x)$ showing clearly all asymptotes. (Do not calculate co-ordinates of any turning points) **3**

ii) Hence or otherwise, draw a one- third page sketch of the graph of $y = \dfrac{1}{f(x)}$ showing all asymptotes. (Do not calculate the co-ordinates of any turning points) **2**

Question 3 Begin in a new booklet (15 marks) **Marks**

a)

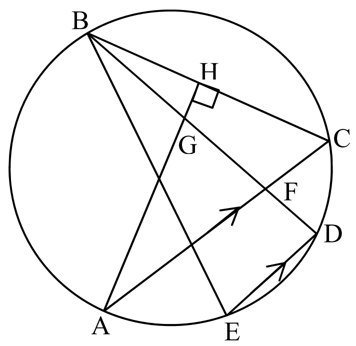


Diagram not to scale

The diagram shows the points A, B, C, D and E on a circle, such that BE is a diameter and AC is parallel to ED. Also, AH is perpendicular to BC, and BD intersects AH and AC at G and F respectively.

i) Copy the diagram into your answer booklet.

ii) Prove angle BFC is 90°. **2**

iii) Prove $CFGH$ is a cyclic quadrilateral. **2**

iv) Hence, or otherwise, show that $AB \times BG = BE \times BH$. **3**

b) The hyperbola, H, has the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

i) Find the eccentricity of H. **1**

ii) Find the co-ordinates of the foci of H. **1**

iii) Draw a neat, one-third page sketch of H. **2**

iv) The line $x = 6$ cuts H at A and B. Find the co-ordinates of A and B, if A is in the first quadrant. **2**

v) Derive the equation of the tangent to H at A. **2**

Question 4 Begin in a new booklet (15 marks) **Marks**

a) If p, q and r are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, find the polynomial equation whose roots are $\frac{1}{p}$, $\frac{1}{q}$ and $\frac{1}{r}$. **2**

b) i) Let k be a zero of the polynomial $F(x)$ and also of its derivative $F'(x)$. Prove that k is a zero of $F(x)$ of multiplicity at least 2. **3**

ii) Show that $y = 1$ is a root of multiplicity at least 2, of the equation $y^{2t} - ty^{t+1} = 1 - ty^{t-1}$, where $t \geq 2$ is a positive integer. **2**

c) The polynomial $P(x)$ gives remainders 1 and – 2 when divided by $2x - 1$ and $x - 2$ respectively. What is the remainder when $P(x)$ is divided by $2x^2 - 5x + 2$? **3**

d)

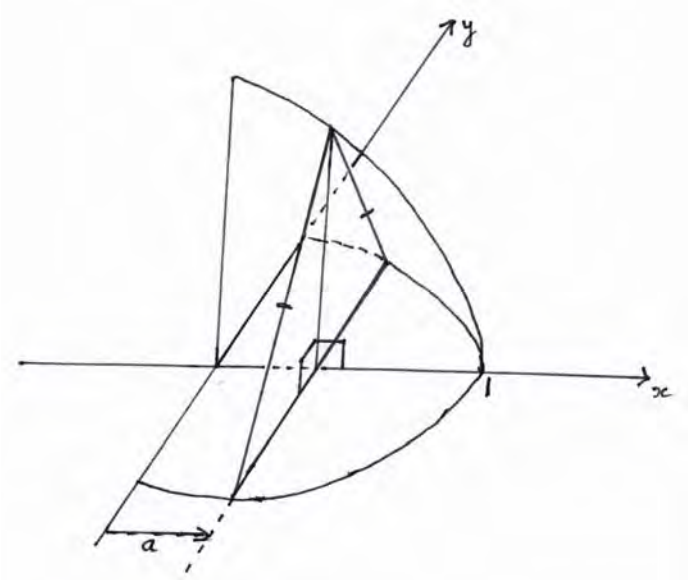


Diagram not to scale

The base of a solid is formed by the area bounded by $x^2 + y^2 = 1$ for $0 \leq x \leq 1$ as shown in the diagram above. Vertical cross sections of the solid taken parallel to the y – axis are in the shape of isosceles triangles with the two equal sides being of length three- quarters the length of the third side which is in the base of the solid.

i) Show that the area of the triangular cross-section at $x = a$, is $\frac{\sqrt{5}}{2}\left(1 - a^2\right)$. **3**

ii) Hence or otherwise find the volume of the solid. **2**

Question 5 Begin in a new booklet (15 marks) **Marks**

a) Prove by mathematical induction that, for integers $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}\ .$$ **4**

b) i) If $\theta = \tan^{-1} A + \tan^{-1} B$, show that $\tan\theta = \dfrac{A+B}{1-AB}$. **1**

ii) Hence solve the equation $\tan^{-1} 3x + \tan^{-1} 2x = \dfrac{\pi}{4}$. **3**

c) i) Sketch the graph of the function $y = \sin^{-1}\left(\dfrac{x}{2}\right)$. **1**

ii) Show that the equation of the tangent, l, to the curve $y = \sin^{-1}\left(\dfrac{x}{2}\right)$ at the point where $x = \sqrt{3}$ is $y = x + \dfrac{\pi}{3} - \sqrt{3}$. **2**

iii) The region, where $x \geq 0$, bounded by $y = \sin^{-1}\left(\dfrac{x}{2}\right)$, the y - axis and the line l is rotated about the y – axis to form a solid of volume V.

1) Show $V = \pi\sqrt{3} - 4\pi\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 y\, dy$. **2**

2) Hence, or otherwise, find V. **2**

Question 6 Begin in a new booklet (15 marks) **Marks**

a) A body is projected vertically upwards, under gravity, from the ground in a medium that produces a resistance force per unit mass of kv^2, where v is the velocity and k is a positive constant.
The acceleration due to gravity is g.

i) If the initial velocity of the body is v_0, prove that the maximum height, H, of the body above the ground is given by

$$H = \frac{1}{2k}\log_e\left(1 + \frac{kv_0^2}{g}\right).$$ **4**

ii) In a second projection vertically upwards of the body, it is noticed that the maximum height reached is $2H$. Show that the initial velocity was $\left(e^{2kH} + 1\right)^{\frac{1}{2}} v_0$. **3**

b) A body is moving in a horizontal straight line. At time t seconds, its displacement is x metres from a fixed point O on the line, and its acceleration is $\frac{-1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$ where $v \geq 0$ is its velocity.
The body is initially at O with velocity $V > 0$.

i) Show that $t = 20\log_e\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$. **3**

ii) Hence, or otherwise, prove that the body comes to rest. **1**

iii) Find the distance travelled before the body comes to rest. **4**

END OF ASSESSMENT

Solution to Year 12 Ext 2 Trial Examination

SECTION 1 MCQ

1. D
2. D
3. B
4. B
5. D
6. C
7. D
8. A
9. C
10. C

SECTION 11 EXTENDED RESPONSES

Question 1

$$a)\int \frac{x}{\sqrt{2-x^2}}dx \qquad x=\sqrt{2}\sin\theta$$

$$\therefore \quad dx=\sqrt{2}\cos\theta$$

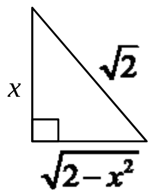
$$=\int \frac{\sqrt{2}\sin\theta.\sqrt{2}\cos\theta}{\sqrt{2-\left(\sqrt{2}\sin\theta\right)^2}}$$

$$=\int \frac{\sqrt{2}\sin\theta\cos\theta}{\cos\theta}\ d\theta$$

$$=\int \sqrt{2}\sin\theta\ d\theta$$

$$=-\sqrt{2}\cos\theta+c$$

$$=-\sqrt{2}.\frac{\sqrt{2-x^2}}{\sqrt{2}}+c$$



$$=-\sqrt{2-x^2}+c$$

$$c)\int_0^1 \tan^{-1}x\ dx$$

$$=\int_0^1 \frac{d(x)}{dx}.\tan^{-1}x\ dx$$

$$=\left[x\tan^{-1}x\right]_0^1-\int_0^1 x.\frac{1}{1+x^2}\ dx$$

$$=\left(\frac{\pi}{4}-0\right)-\frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\ dx$$

$$=\frac{\pi}{4}-\frac{1}{2}\left[\ln\left(1+x^2\right)\right]_0^1$$

$$=\frac{\pi}{4}-\frac{1}{2}\left[\ln 2-\ln 1\right]$$

$$=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

Question 1 conti

1 $bi)$ $\dfrac{1}{x(2x+1)} = \dfrac{a}{x} + \dfrac{b}{2x+1}$

$a(2x+1) + bx \equiv 1$

Put $x = 0 \Rightarrow a = 1$

Put $x = \dfrac{-1}{2} \Rightarrow b = -2$

$\therefore \dfrac{1}{x(2x+1)} = \dfrac{1}{x} - \dfrac{2}{2x+1}$

$bii)\int_{\frac{1}{2}}^{1} \dfrac{dx}{x(2x+1)}$

$= \int_{\frac{1}{2}}^{1} \dfrac{dx}{x} - \int_{\frac{1}{2}}^{1} \dfrac{2dx}{(2x+1)}$

$= \left[\ln|x| - \ln|2x+1|\right]_{\frac{1}{2}}^{1}$

$= \left[\ln\left|\dfrac{x}{2x+1}\right|\right]_{\frac{1}{2}}^{1}$

$= \ln\dfrac{1}{3} - \ln\left|\dfrac{\frac{1}{2}}{2}\right|$

$= \ln\dfrac{4}{3}$

Question 1 continued

$$di) I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x \, dx$$

$$= \left[\sin x \cos^{n-1} x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (n-1) \cos^{n-2} x \sin x . \sin x \, dx$$

$$= (0-0) + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$nI_n = (n-1) I_{n-2}$$

$$\therefore I_n = \frac{(n-1)}{n} I_{n-2}$$

$$dii) \ I_6 = \int_0^{\frac{\pi}{2}} \cos^6 x \, dx$$

$$= \frac{5}{6} I_4$$

$$= \frac{5}{6} . \frac{3}{4} . \frac{1}{2} I_2$$

$$= \frac{15}{48} \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{15}{48} \left[x^0 \right]_0^{\frac{\pi}{2}}$$

$$= \frac{15\pi}{96} = \frac{5\pi}{32}$$

Question 2

$$2ai)\quad wv = 2\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$iv = \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)$$

$$aii)\quad v = \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = z$$

$$z^4 = \cos\frac{\pi}{6}\times 4+i\sin\frac{\pi}{6}\times 4$$

$$= \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right) = iv$$

$$= iz$$

$\therefore v$ is a solution to $z^4 = iz$

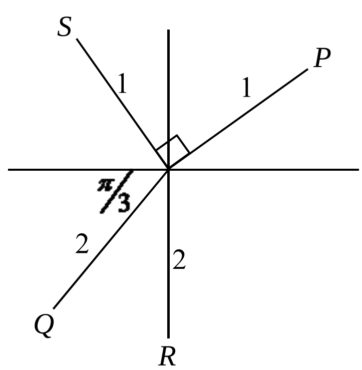
if $z \neq 0$, then $z^3 = i$

non zero roots are $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = \frac{1}{2}\left(\sqrt{3}+i\right)$

$$\cos\frac{5\pi}{6}+i\sin\frac{\pi}{6} = \frac{1}{2}\left(-\sqrt{3}+i\right) = v\text{cis}\left(\frac{2\pi}{3}\right)$$

and $\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right) = -i = \frac{wv}{2}$

a iii)



Question 2 conti

$aiv)$ Method 1

In $\Delta OQR \quad x = \frac{1}{2}\left(180^\circ - 30^\circ\right) = 75^\circ$

In $\Delta OSP \quad x = \left[180^\circ - 45^\circ - \left(90^\circ - 30^\circ\right)\right] = 75^\circ$

$\therefore \overrightarrow{PS} \,//\, \overrightarrow{RQ}$

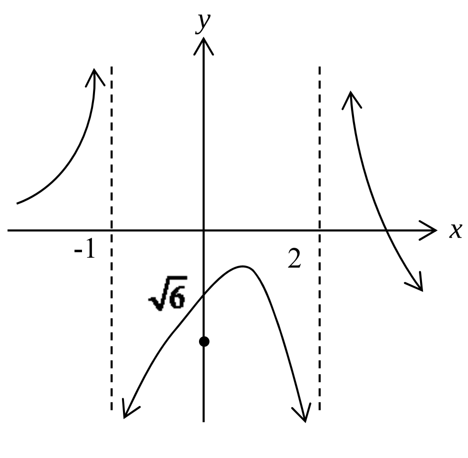
Method 2

Show $\overrightarrow{PS} = \lambda \ \overrightarrow{RQ}$ where $\lambda \in$ reals

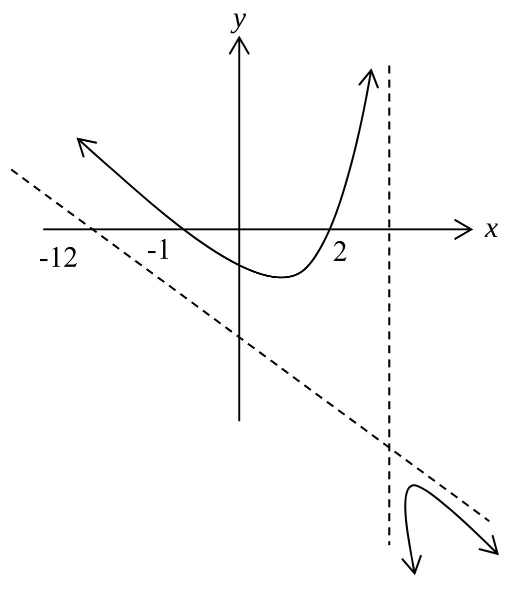
$$\begin{aligned}\overrightarrow{RQ} &= w - wv \\ &= 2cis\left(-\frac{2\pi}{3}\right) - \left(2i\right) \\ &= \frac{2}{2}\left[-1 - i\sqrt{3}\right] + 2i \\ &= -1 + i\left[2 - \sqrt{3}\right]\end{aligned}$$

$$\begin{aligned}\overrightarrow{PS} &= iv - v \\ &= cis\left(\frac{2\pi}{3}\right) - cis\left(\frac{\pi}{6}\right) \\ &= \frac{1}{2}\left[-1 + i\sqrt{3}\right] - \frac{1}{2}\left[\sqrt{3} + i\right] \\ &= 2\frac{\left(\sqrt{3}+1\right)}{2} \times \overrightarrow{RQ} \quad \text{by inspection note} \left(\sqrt{3}+1\right) \times \left(2 - \sqrt{3}\right) = \sqrt{3-1} \\ &\Rightarrow \overrightarrow{PS} \,//\, \overrightarrow{RQ}\end{aligned}$$

bi)



bii)



Question 3

$3aii)$ $\angle EDB = 90°$ $\quad$ (angle in a semi circle)

$\angle BFC = 90°$ $\quad$ (alt angles, $AC // ED$, suppl angles)

$iii)$ $\angle GHC + \angle GFC = 90 + 90 = 180^{\circ}$

$\therefore$ $CFGH$ is a cyclic quad (opps angles are suppl)

$iv)$ In ΔBGH, ΔBEA

$\angle BGH = \angle BAE = 90$ $\quad$ (diameter subtends right angle at cfce)

$\angle BCA = \angle BEA$ $\quad$ (angles subtended by a common chord BA)

but $\angle BCA = \angle BGH$ $\quad$ (exterior angle of cyclic quad equal to int opp anlges)

$\therefore \angle BGH = \angle BEA$

$\therefore$ $\Delta BGH \equiv \Delta BEA$ $\quad$ (equi-angular)

$\therefore \dfrac{BG}{BE} = \dfrac{BH}{AB}$ $\quad$ (corresp sides of similarΔ)

$\therefore AB.BG = BE.BH$

$3bi)\ \ a = 5, b = 3$

$$b^2 = a^2\left(e^2 - 1\right)$$

$$9 = 25\left(e^2 - 1\right)$$

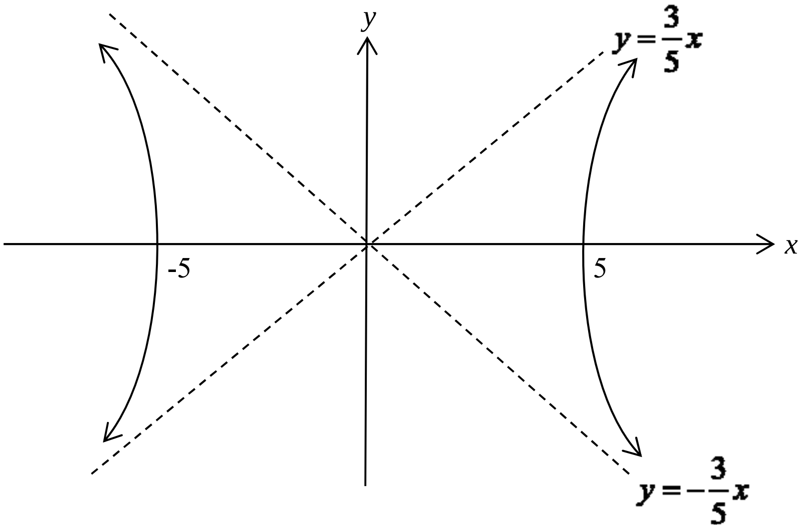
$$e^2 = \frac{34}{25}$$

$$e = \frac{\sqrt{34}}{5}$$

$ii)$ Foci $S\left(ae, 0\right) \ \ = \left(\sqrt{34}, 0\right)$

$S'\left(-ae, 0\right) = \left(-\sqrt{34}, 0\right)$

$iii)$



Question 3 conti

$3biv)\ \dfrac{36}{25}-\dfrac{y^2}{9}=1$

$$y^2=\frac{9\times 11}{25}$$

$$y=\frac{\pm 3\sqrt{11}}{5}$$

$$A\left(6,\frac{3\sqrt{11}}{5}\right)\quad B\left(-6,\frac{-3\sqrt{11}}{5}\right)$$

$3bv)\ \dfrac{x^2}{25}-\dfrac{y^2}{9}=1$

$$\frac{2x}{25}-\frac{2y}{9}.\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}=\frac{2x}{25}.\frac{9}{2y}$$

$$\therefore \frac{dy}{dx}=\frac{9x}{25y}$$

$$\text{at } \left(6.\frac{99}{5}\right),\frac{dy}{dx}=\frac{54}{25.\sqrt{99}}$$

$$\therefore \text{ equation is } y-\frac{3\sqrt{11}}{5}=\frac{18}{5\sqrt{11}}(x-6)$$

$$y=\frac{18}{5\sqrt{11}}x-6+\frac{3\sqrt{11}}{5}$$

Question 4

$4a)$ $x^3+4x^2-3x+1=0$ has no root $x=0.$

let $y=\frac{1}{x}$ and $x=\frac{1}{y}$

the equation below has roots

$\frac{1}{p},\frac{1}{q},\frac{1}{r}$

$\frac{1}{y^3}+\frac{4}{y^2}-\frac{3}{y}+1=0$

$y^3-3y^2+4y+1=0$

$bi)$ k is zero of $F(x)$

$\Rightarrow F(x)=(x-4)q(x)$

$F'(x)=(x-k)q'(x)+q(x)$

k is zero of $F'(x)$

$F'(k)=0$

$=(k-k)q'(k)+q(k)=0$

$\Rightarrow q(k)=0$

$\Rightarrow q(k)=(x-4)r(x)$

hence $F(x)=(x-4)^2\,r(x)$

$\Rightarrow k$ is at least a root of multiplicity 2

$bii)$ $y^{2t}-ty^{t+1}=1-ty^{t-1}$

$t=1,2,3.....\Rightarrow y^{2t}-ty^{t+1}-1+ty^{t-1}=0$

let $P(y)=$ LHS of above $\therefore P(1)=1-t\times1+t\times1-1=0$

$\therefore y=1$ is a root of $P(y)=0$

$P'(y)=2ty^{2t-1}-t(t+1)y^t+t(t-1)y^{t-2}$

for $t=2,3,4,...\Rightarrow P'(1)=2t-t^2-t+t^2-t=0$

$\therefore$ need to consider the case where $t=1$ to complete the proof

$P(y)=y^2-y^2+1\times y^0-1$

So $t=1$ $\Rightarrow P(y)=y^2-y^2+1-1\equiv0$ $[$omit this case $t\geq2]$

Question 4 conti

$4c)\ P(x) = (2x^2 - 5x + 2)Q(x) + R(x)$

Since degree $D(x) > \deg R(x)$

$\deg R(x) < 2$

let $R(x) = ax + b$

$\therefore\ P(x) = (2x-1)(x-2)Q(x) + ax + b$

$P\left(\frac{1}{2}\right) = \frac{a}{2} + b = 1$

$P(2) = 2a + b = -2$

$-3b = -6$

$b = 2, a = -2$

$\therefore\quad R(x) = -2x + 2$

$4d)$ when $x = a, y = 2\sqrt{1-a^2}$

$\therefore$ length of base $= 2\sqrt{1-a^2}$

$$h^2 = \left(\frac{3}{2}\sqrt{1-a^2}\right)^2 - \left(\sqrt{1-a^2}\right)^2$$

$$= \frac{a}{4}(1-a^2) - (1-a^2)$$

$$= \frac{5}{4}(1-a^2)$$

$$h = \frac{\sqrt{5(1-a^2)}}{2}, h > 0$$

Question 5

5 (a) Prove true for $n = 2$

1.
$$LHS = 1-\frac{1}{2}2 \qquad RHS = \frac{2+1}{2\times 2}$$
$$=\frac{3}{4} \qquad\qquad =\frac{3}{4}$$

$\therefore$ True form = 2

2. Assume true form $n = k$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{k^2}\right)=\left(\frac{k+1}{2k}\right)$$

Step 3. Prove true for $n = k+1$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+1+1}{2(k+1)}$$

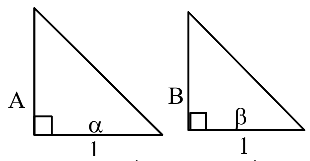
$LHS\ \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right)$ from ②

$$=\frac{k+1}{2k},\frac{(k+1)^2-1}{(k+1)^2}$$
$$=\frac{k+1}{2k},\frac{k^2+2k+1-1}{(k+1)^2}$$
$$=\frac{(k+1)}{2k},\frac{k(k+2)}{k+1^2}$$
$$=\frac{k+1+1}{2(k+1)}$$
$$=RHS$$

$\therefore$ true for $n = k+1$

$\therefore$ true by MI

5 b(i)



Let $\tan^{-1} A = \alpha$

$\tan^{-1} B = \beta$

$$\theta = \tan^{-1}\beta + \tan^{-1} A$$
$$= \alpha + \beta$$
$$\tan\theta = \tan(\alpha+\beta)$$
$$= \frac{\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta}$$
$$\therefore \tan\theta = \frac{A+B}{1-AB}$$

Question 5 conti

(ii) from **(i)**

$$\tan\frac{\pi}{4}=\frac{3x+2x}{1-3x_1 2x}\Rightarrow 1=\frac{5x}{1-6x^2}$$

$$\therefore 6x^2+5x-1=0$$

$$(6x-1)(x+1)=0$$

$$x=\frac{1}{6},-1$$

$$x=\frac{1}{6}$$

Test: $\tan\frac{\pi}{4}=\frac{3\times\frac{1}{6}+2\times\frac{1}{6}}{1-3\times\frac{1}{6}\times 2\times\frac{1}{6}}$

$=1 \quad \therefore$ True

Test $x=-1$

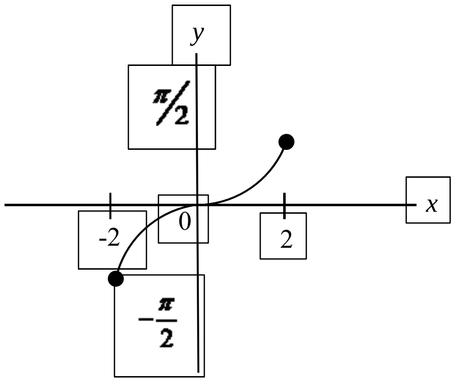
$$\tan\frac{\pi}{4}=\frac{3\times-1+2\times-1}{1-3\times-1\times 2\times-1}$$

$=-1 \quad \therefore$ not true

$\therefore$ Solution $\boxed{x=\frac{1}{6}}$

Question5 cont

c(i) $y = \sin^{-1}\frac{x}{2}$



(ii) $y' = \frac{1}{2} \times \frac{1}{\sqrt{\frac{4-x^2}{4}}}$

When $x = \sqrt{3}$

$y' = 1$

$x = \sqrt{3}, y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

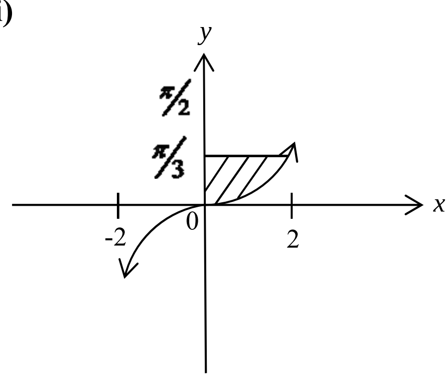
Equation tan:

$y - \frac{\pi}{3} = 1\left(x - \sqrt{3}\right)$

$y = x - \sqrt{3} + \frac{\pi}{3}$

$\therefore y \text{ int } \left(\frac{\pi}{3}, -\sqrt{3}\right)$

(iii)



$$\Delta V = \left(\frac{-\pi}{3} + \sqrt{3}\right) - \pi x^2 \Delta y \qquad \sin y = \frac{x}{2} \therefore x = 2\sin y$$

$$\Delta V = \left(\left(\frac{-\pi}{3} + \sqrt{3}\right) - \pi\left(2\sin y\right)^2\right)\Delta y$$

$$V = \lim_{\Delta V \to 0} \sum_{y=0}^{y=\frac{\pi}{3}} \left(\left(\frac{-\pi}{3} + \sqrt{3}\right) - \pi\left(2\sin y\right)^2\right)\Delta y$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{-\pi}{3} + \sqrt{3}\right) - \pi\left(2\sin y\right)^2 \, dy$$

$$=$$

5c (iii) 1)

$$y = \sin^{-1}\left(\frac{x}{2}\right)$$

$$\frac{x}{2} = \sin y$$

$$x^2 = 4\sin^2 y$$

$$V = \int_0^{\frac{\pi}{3}} 4\sin^2 y \, dy$$

So Volume = Vol of cone - vol of shaded area about *y*- axis

$$\therefore V = \frac{1}{3}\pi\left(\sqrt{3}\right)^2\left[\frac{\pi}{3} - \left(\frac{\pi}{3} - \sqrt{3}\right)\right] - \int_0^{\frac{\pi}{3}} 4\sin^2 y \, dy$$

$$V = \pi\sqrt{3} - 4\pi\int_0^{\frac{\pi}{3}} \sin^2 y \, dy$$

5 c(iii) 2)

$$V = \pi\sqrt{3} - 4\pi\int_0^{\frac{\pi}{3}}\left(1-\cos 2y\right)dy$$

$$= \pi\sqrt{3} - 2\pi\left[y - \frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{3}}$$

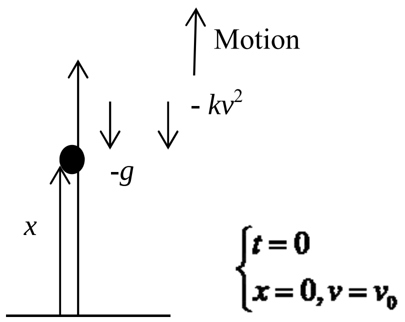
$$= \pi\sqrt{3} - 2\pi\left[\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$= \pi\sqrt{3} - \frac{2\pi^2}{3} + \frac{\pi\sqrt{3}}{2}$$

$$= \frac{\pi 3\sqrt{3}}{2} - \frac{2\pi^2}{3}u^3$$

Question 6

(a)



(i) $\sum F = -\left(kv^2 + g\right)$

$$\frac{dv}{dt} = -\left(kv^2 + g\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\left(kv^2 + g\right)$$

$$v\frac{dv}{dx} = -\left(kv^2 + g\right)$$

$$\int \frac{v\ dv}{kv^2 + g} = -\int dx$$

$$\frac{1}{2k}\ln\left(kv^2 + g\right) = -x + c$$

$$\ln\left(kv^2 + g\right) = -2kx + c$$

$x = 0 \quad v = v_0$

So $c = \ln\left[kv_0^{\ 2} + g\right]$

At *x=H, v=0*

$\Rightarrow$

$$\ln\left(g\right) = -2kH + \ln\left(kv_0^{\ 2} + g\right)$$

$$= \ln\left(1 + \frac{kv_0^{\ 2}}{g}\right)$$

$$\therefore \text{H} = \frac{1}{2k}\log_e\left(1 + \frac{kv_0^{\ 2}}{g}\right)$$

Alternate solution to **6ai)**

$$F = -mg - mkv^2$$
$$ma = --mg - mkv^2$$
$$a = -g - kv^2$$
$$\frac{vdv}{dx} = -g - kv^2$$
$$\int_{V_o}^{0} \frac{vdv}{g + kv^2} = \int_0^H dx$$

$$\frac{1}{2k}\left[\ln\left|g + kv^2\right|\right]_{V_o}^{0} = \left[x\right]_o^H$$

$$\frac{1}{2k}\left[\ln|g| - \ln\left|g + kv^2\right|\right] = -H$$
$$\therefore H = \frac{1}{2k}\left[\ln\frac{g + kv^2}{g}\right], \therefore H = \frac{1}{2k}\ln\left|1 + \frac{kv_0^2}{g}\right|$$

(6aii)

$$H = \frac{1}{2k}\ln\left[1 + \frac{kv_0^{\ 2}}{g}\right]$$
$$\Rightarrow \frac{kv_0^{\ 2}}{g} = \quad e^{2kH} - 1$$
Let V_0 be the initial velocity in the second projection.
$$\therefore \frac{kv_0^{\ 2}}{g} = e^{4kh} - 1$$

$$\frac{V_0^{\ 2}}{v_0^{\ 2}} = \frac{e^{kh}-1}{e^{2kh}-1}$$

$$= \frac{\left(e^{2kh}+1\right)\left(e^{2kh}-1\right)}{\left(e^{2kh}-1\right)}$$

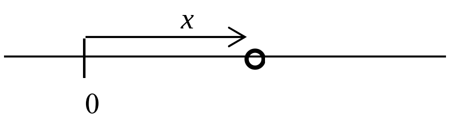
$$= e^{2kh}+1$$

$$\therefore \left(\frac{V_0}{v_0}\right) = \left(e^{2kh}+1\right)$$

by $V_0 > 0, v_0 > 0$

$$V_0 = \left(e^{2kh}+1\right)^{\frac{1}{2}} v_0$$

Question 6 (b)



t = 0 x = 0, v = V > 0

(i) $\dfrac{dv}{dt} = -\dfrac{\sqrt{v}}{10}\left(1+\sqrt{v}\right)$

$$\int \frac{2dv}{2\sqrt{v}\left(1+\sqrt{v}\right)} = -\int \frac{dt}{10}$$

$$2\ln\left(1+\sqrt{v}\right) = -\frac{t}{10}+c$$

$20\ln\left(1+\sqrt{v}\right) = -t+c$ at *t = 0, v = V > 0*

$$c = 20\ln\left(1+\sqrt{v}\right)$$

$$\therefore t = 20 \geq \left[\ln\left(1+\sqrt{v}\right)\right.$$
$$\left. -\ln\left(1+\sqrt{v}\right)\right]$$

$$= 20\ln\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$$

6 (bii) $t = 20\ln\left(\dfrac{1+\sqrt{V}}{1+\sqrt{v}}\right)$

When $v = 0$ $\quad t = 20\ln\left(1+\sqrt{v}\right)$ so particle comes to rest (as $t > 0$)

6 (biii) $\dfrac{dv}{dt} = v \quad \dfrac{dv}{dx} = -\dfrac{\sqrt{v}}{10}\left(1+\sqrt{v}\right)$

$$\int \frac{v\ \ dv}{\sqrt{v}\left(1+\sqrt{v}\right)} = -\frac{1}{10}\int dx$$

$$\int \frac{\sqrt{v}\ \ dv}{1+\sqrt{v}} = -\frac{x}{10}+c$$

$$LHS = \int \frac{\sqrt{v}}{1+\sqrt{v}}dv$$

Let $y = \sqrt{v} > 0$

$$dy = \frac{1}{2\sqrt{v}}dv$$

$$\therefore \quad dv \ = 2\sqrt{v}\ dy$$
$$= 2y\ dy$$

$$\therefore LHS = \int \frac{y}{1+y}\times 2y \ \ dy$$
$$= 2\int \frac{y^2-1+1}{(y+1)}\ dy$$
$$= 2\int \left(y-1+\frac{1}{y+1}\right) dy$$
$$= y^2 - 2y + 2\ln(y+1)$$
$$= (Vv)^2 - 2\sqrt{v} + 2\ln\left(\sqrt{2}+1\right)$$
$$= v - 2\sqrt{2} + 2\ln\left(\sqrt{v}+1\right)$$
$$v - 2\sqrt{v} + 2\ln\left(\sqrt{v}+1\right)$$
$$= -\frac{x}{10}+c$$

Where $x = 0, v = V > 0$

$$\therefore C = V - 2\sqrt{v} + 2\ln\left(\sqrt{v}+1\right)$$

Hence $v = 0$ when x = D

D = distance travelled before body comes to rest.

$$\frac{D}{10} = C$$
$$\therefore D = 10\left[V - 2\sqrt{v} + 2\ln\left(\sqrt{v}+1\right)\right]$$

Remark

D(0) = *0*

$\frac{d}{dv}D(V)=\frac{\sqrt{v}}{1+\sqrt{v}}>0$ for V>0

$\therefore D(V)$ is an increasing function $\Rightarrow$

$D(V)\geq 0$ for $V\geq 0$

End of solutions