Section I

10 Marks
Attempt Questions 1–10
Allow about 20 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

1 Let α, β and γ be roots of the equation $x^3 + 3x^2 + 4 = 0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

A. $x^3 - 9x^2 - 24x - 4 = 0$

B. $x^3 - 9x^2 - 12x - 4 = 0$

C. $x^3 - 9x^2 - 24x - 16 = 0$

D. $x^3 - 9x^2 - 12x - 16 = 0$

2. If $w = \sqrt{3}cis(\pi)$, then w^4 is equal to

A. $9cis(0)$

B. $9cis\ (\pi)$

C. $81cis\ (4\pi)$

D. $4\sqrt{3}cis\ (\pi^4)$

3. $\displaystyle\int \frac{3x^2}{\sqrt{x^3 - 2}}\,dx$ is equal to

A. $2\sqrt{x^3 - 2} + c$

B. $\dfrac{\sqrt{x^3 - 2}}{2} + c$

C. $\dfrac{2}{3(x^3 - 2)^{\frac{3}{2}}} + c$

D. $\dfrac{3}{2(x^3 - 2)^{\frac{3}{2}}} + c$

4. If $y = \tan 2x$, then $\dfrac{d^2y}{dx^2}$ is equal to

A. $\dfrac{8\sin 2x}{\cos^3 2x}$

B. $2\sec^2 2x$

C. $\dfrac{8\sin 2x}{\cos 2x}$

D. $\dfrac{-8\sin 2x}{\cos 2x}$

5. A light elastic spring of natural length 2 metres hangs vertically from a ceiling. A mass of 0.5 kg is attached to the free end of the spring and hangs in equilibrium. Given that the stiffness of the spring is 0.25g (where g is gravity in m/s^2), then the length of the spring in metres would be

A. $\dfrac{2}{g}$

B. 2

C. $2 + \dfrac{2}{g}$

D. 4

6. The displacement *x*, in metres, of a particle from a fixed point at time *t*, in seconds, $t \geq 0$, is given by $x = 2\cos(3t)$.
The number of oscillations made by the particle per second is

A. $\dfrac{1}{\pi}$

B. $\dfrac{3}{2\pi}$

C. 2

D. 3

7. If $z = x + yi$, then the set of points in the complex plane given by $(2-i)z + (2+i)\overline{z} = 1$ is

A. a horizontal line through $(0, 2)$
B. a vertical line through $(-2, 0)$
C. a straight line with gradient 2 and y-intercept -1
D. a straight line with gradient -2 and y-intercept $\frac{1}{2}$

8. A solution to the differential equation $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 1$ is

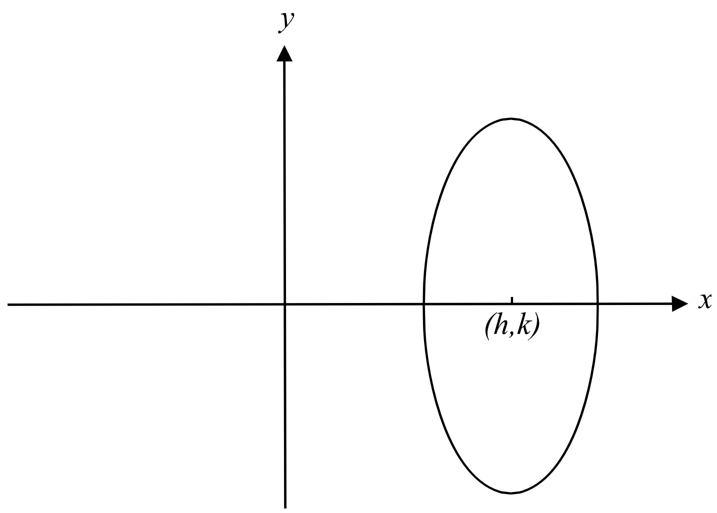
A. $x - \sin^{-1} x$

B. $x - \cos^{-1} x$

C. $x - 2\sqrt{1-x^2}$

D. $\frac{-2x^3}{3}\sqrt{1-x^2}$

9. An ellipse is shown on the set of axes below. The scale on the x and y axes is the same. The centre of the ellipse is the point (h, k).



The general form of the equation of an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

For the ellipse shown above

A. $h < k$ and $a < b$
B. $h < k$ and $a = b$
C. $h = k$ and $a > b$
D. $h > k$ and $a < b$

10. The eccentricity of a hyperbola with parametric equations $x = 3\sec\theta$ and $y = 4\tan\theta$ is:

A. $\frac{5}{3}$

B. $\frac{3}{5}$

C. $\frac{5}{4}$

D. $\frac{4}{3}$

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 40 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet

a) Find $\displaystyle\int \frac{dt}{\sqrt{7+6t-t^2}}$ 2

b) i) Show that $\dfrac{1}{(x^2+3)(x^2+1)} = \dfrac{1}{2}\left[\dfrac{1}{(x^2+1)} - \dfrac{1}{(x^2+3)}\right]$ 2

ii) Hence evaluate $\displaystyle\int_0^1 \frac{1}{(x^2+3)(x^2+1)}dx$ 2

c) The equation $z^2 - (a+ib)z - 6i = 0$, where a and b are real, has roots α and β such that $\alpha^2 + \beta^2 = 5$.

i) Show that $a^2 - b^2 = 5$ and ab= -6 2

ii) Hence find the values of a and b. 3

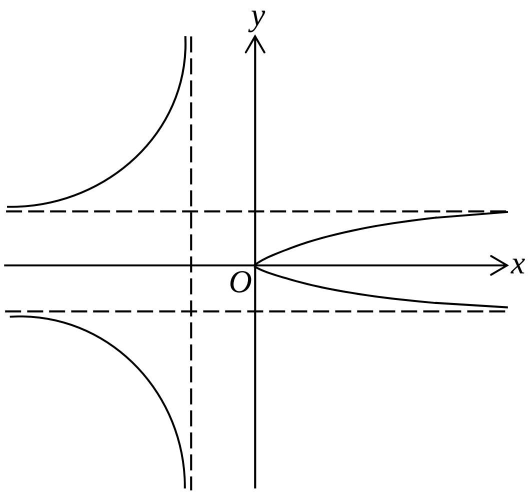
d) If $I_m = \int_0^k \left(k^2 - x^2\right)^m dx$, for $m \geq 1$, show that

$$I_m = \frac{2k^2 m}{2m+1} . I_{m-1}$$ 4

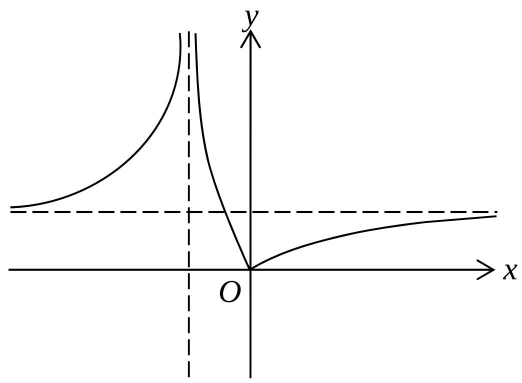
$$\left[\text{Hint } \frac{x^2}{k^2 - x^2} = \frac{k^2}{k^2 - x^2} - 1\right]$$

Question 12 (15 marks) Use a SEPARATE writing booklet

a) Given the graphs of $y^2 = f\left(x\right)$ and $y = \left| f\left(x\right) \right|$ as shown.



$y^2 = f(x)$



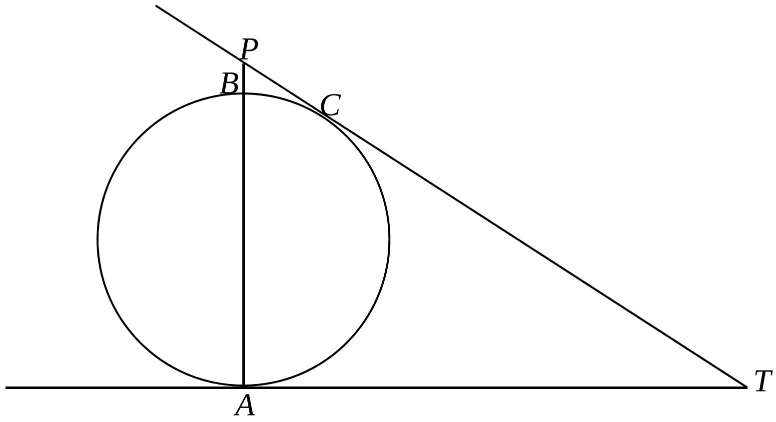
$y = \left|f\left(x\right)\right|$

i) Explain the shape and position of $y = f\left(x\right)$ in relation to the above graphs. 2

ii) Hence, sketch the grapgh of $y = f\left(x\right)$. 2

iii) Hence sketch the graph of $y = f\left|\left(x\right)\right|$. 2

b) *AB* is a diameter of a circle *ABC*. The tangent at *A* and *C* meet at *T*. The lines *TC* and *AB* are produced to meet at *P*. Join *AC* and *CB*.



i) Prove that $\angle BCP = 90^\circ - \angle CAT$. 2

ii) Explain why $ATCB$ could never be a cyclic quadrilateral. 2
You must supoort your answers with geometrical reasons.

c) i) Prove that if a polynomial $P(x)$ has a root α of multiplicity *r*, then $P'(x)$ has a root of multiplicity $(r-1)$. 2

ii) Given that $x=1$ is a double root of the equation $x^4 - 5x^3 + 16x^2 - 21x + 9 = 0$, and using the result of ci), or otherwise, find the other roots. 3

Question 13 (15 marks) Use a SEPARATE writing booklet

a) i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. 2

ii) Hence, or otherwise, solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. 3

b) A hyperbola has eccentricity $\frac{3}{2}$ and directrices $x = -4$ and $x = 4$.
Find the Cartesian equation of this hyperbola. 3

c) The point $P\left(a\cos\theta, b\sin\theta\right)$ lies on the ellipse, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to the tangent through P.

i) Show that $ST=\dfrac{\left|e\cos\theta-1\right|}{\sqrt{\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}}}$ 2

ii) Hence prove $ST.S'T'=b^2$. 3

d) A corner on a race track is an arc of a circle of radius 100m. The track is banked such that there is no tendency for a vehicle to move sideways when cornering at 100km/h. Find the angle, to the nearest minute, of banking. Take g to be $10\,\text{m/s}^2$. 2

Question 14 (15 marks) Use a SEPARATE writing booklet

a) If $z=\cos\theta+i\sin\theta$,

i) Show that $z^z+z^{-z}=2\cos n\ \theta$ for $n=1,2,3,.....$ 2

ii) Hence show that $4\cos\theta\cos2\theta\cos3\theta=1+\cos2\theta+\cos4\theta+\cos6\theta$ 3

iii) Hence find the general solution of $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta=1$. 3

b) The region bounded by the graphs of $y=x^2$ and $y=x+2$ is revolved around the line $x=3$. Express the volume of the resulting solid as a definite integral. Do not calculate the value of this integral. 3

c) A solid has , as its base, the circular region in the *xy* plane bounded by the graph of $x^2+y^2=a^2$, where $a>0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base. 4

Question 15 (15 marks) Use a SEPARATE writing booklet

a) An ellipse can be described as the locus of a point moving so that the sum of its distance from two fixed points (foci) is a constant.

i) If the two fixed points are $A(-4,0)$ and $B(4,0)$ and the sum of the distances of $P(x,y)$ from these points is 10units, show that the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$. 2

ii) Show that the ellipse can be represented parametrically by the equations $x=5\cos\theta$ and $y=3\sin\theta$ and find the equation, in general form of the tangent to the ellipse at $\theta=\frac{\pi}{6}$. 4

b) The roots of $x^3+3px+q=0$ are α,β and γ none of which are equal to 0.

i) Find the monic equation with roots $\frac{\beta\gamma}{\alpha},\frac{\alpha\gamma}{\beta},\frac{\alpha\beta}{\gamma}$, giving the coefficients in terms of p and q. 4

ii) Deduce if $\alpha\beta=\gamma$, then $\left(3p-q^2\right)+q=0$. 2

c) Given that $\sin^{-1}x,\cos^{-1}x$ and $\sin^{-1}\sqrt{\left(1-x^2\right)}$ are acute, show that:

$\sin\left(\sin^{-1}x-\cos^{-1}x\right)=2x^2-1$. 3

Question 16 (15 marks) Use a SEPARATE writing booklet

a) A particle of mass m is projected vertically upward under gravity with a speed v in a medium where resistance is mk times the speed, where k is a positive constant. If the particle reaches its greatest height H, in time T, show that $v = gT + kH$. 4

b)

Given that $\sin(x+y)+\sin(x-y)=2\sin x\cos y$ and $\cos(x+y)+\cos(x-y)=2\cos x\cos y$ and by using the fact that $A+B+C=\pi$ for triangle ABC.

Show that $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\cot\dfrac{C}{2}$ 4

c) If $\cos\alpha=\tan x\cot y$, show that $\tan^2\dfrac{1}{2}\alpha=\sin(y-x)\operatorname{cosec}(y+x)$ 3

d) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}}\frac{1}{9-10\sin^2 x}dx$ 4

End of Assessment Task

1 2 3 4 5 6 7 8 9 10
C A A A D B D A D A

Trials 2004 Soln EXT2

Q1. Let $y = x^{\frac{1}{2}}$

$y^{3/2} + 3y = -4$

$\Rightarrow y^{3/2} = -3y - 4$

expand + square

$\Rightarrow y^3 - 9y^2 - 24y - 16 = 0$

$\therefore$ C

C

A

2. C ~~as P(x)~~ has real ~~coefficients~~, complex roots occur in conjugate pairs.

Q2 $w = \sqrt{3}\,\text{cis}\,\pi$

$w^4 = (\sqrt{3})^4\,\text{cis}(4\pi)$

$= 9\,\text{cis}\,0 \quad \therefore$ A

3. $\int \frac{3x^2}{\sqrt{x^3-2}}\,dx = \int \frac{du}{dx}\, u^{-\frac{1}{2}}\,dx$

let $u = x^3 - 2$, $\frac{du}{dx} = 3x^2$

$= \int u^{-\frac{1}{2}}\,du$

$= 2u^{\frac{1}{2}} + c$ A

$= 2\sqrt{x^3-2} + c$

4. $y = \tan 2x$

$\frac{dy}{dx} = 2\sec^2 2x$

$= 2(\cos 2x)^{-2}$

$\frac{d^2y}{dx^2} = -4(\cos 2x)^{-3} \cdot -2\sin 2x$

$= \frac{8\sin 2x}{\cos^3 2x}$ A

5.

5.

$T = kx$

$= 0.25gx$

$T = 0.5g \quad \therefore x = 2$

Since x is the extension in the spring beyond its natural length, the length of the string is $2+2 = 4$m $\therefore$ D

6. no of oscillations per sec is f

$\therefore f = \frac{n}{2\pi}$, $n = 3$. from $x = 2\cos(3t)$

$\therefore f = \frac{3}{2\pi}$ B

7. $(2-i)z + (2+i)\bar{z} = 1$

$2z - iz + 2\bar{z} + i\bar{z} = 1$

If $z = x + yi$

$\Rightarrow 2(x+iy) - i(x+iy) + 2(x-iy) + i(x-iy) = 1$

$2x + 2iy - ix + y + 2x - 2iy + ix + y = 1$

$4x + 2y = 1$

$2y = -4x + 1$

$y = -2x + \frac{1}{2}$

m y int $\therefore$ D

8. $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 1$

$\frac{dy}{dx} = 1 - \frac{1}{\sqrt{1-x^2}}$

$y = \int 1 - \frac{1}{\sqrt{1-x^2}}\,dx$

$= x - \sin^{-1}x + C$ A

9. For centre of ellipse, $k = 0$, $h > 0$ so $h > k$. The length of semi-major axis is b, This runs ∥ to y axis. semi minor is a; ∥ to x axis.

$\therefore a < b$ D

10. A

$e = \sqrt{\frac{a^2+b^2}{a^2}}$

$= \sqrt{\frac{9+16}{9}}$

$= 5/3$

Question 11.

(a) $\int \frac{dt}{\sqrt{7+6t-t^2}}$

$= \int \frac{dt}{\sqrt{16-(t-3)^2}}$

$= \sin^{-1}\left(\frac{t-3}{4}\right) + c$ (2)

(b) (i) Let $\frac{1}{(x^2+3)(x^2+1)} = \frac{A}{x^2+3} + \frac{B}{x^2+1}$

$\therefore 1 = A(x^2+1) + B(x^2+3)$

Let $x^2 = -1$ $\therefore$ $1 = 2B$

$B = \frac{1}{2}$

Let $x^2 = -3$ $\therefore$ $1 = A - 2$ (2)

$A = -\frac{1}{2}$

$\therefore \frac{1}{(x^2+3)(x^2+1)} = \frac{1}{2}\left[\frac{1}{(x^2+1)} - \frac{1}{(x^2+3)}\right]$

(ii) $\frac{1}{2}\int_0^1 \left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right) dx$

$= \frac{1}{2}\left[\tan^{-1}x - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1$

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6}\right]$

$= \frac{\pi}{4}\left[\frac{1}{2} - \frac{\sqrt{3}}{9}\right]$ (2)

OR students can expand RHS to become LHS.

11(c) (i) $\alpha + \beta = a + ib$

$\alpha\beta = -ib$

$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\therefore (a+ib)^2 = 5 - 12i$

$a^2 - b^2 + 2iab = 5 - 12i$

equate real + imag. (2)

$a^2 - b^2 = 5$

$ab = -6$.

(ii) $a^2 - b^2 = 5$

$a^4 - a^2b^2 = 5a^2$

$a^4 - 5a^2 - 36 = 0$ 1

$(a^2-9)(a^2-4) = 0$

$a = 3,\ b = -2$ 1

or $a = -3,\ b = 2$ (3)

11(d) $I_m = \int_0^k (k^2-x^2)^m \cdot 1\, dx$

Let $u = (k^2-x^2)^m$ $\quad v' = 1$

$du = m(k^2-x)^{m+1} \cdot 2x$ $\quad v = x$ 1

$\therefore I_m = \left[x(k^2-x^2)^m\right]_0^k + \left[\int_0^k 2mx^2(k^2-x^2)^{m-1} dx\right]$ 1

$= [k(k^2-k^2)^m - 0] + 2m\int_0^k x^2(k^2-x^2)^{m-1} dx$

$= 2m\int_0^k \frac{x^2(k^2-x^2)^m}{k^2-x^2} dx$

$= 2mk^2\int_0^k (k^2-x^2)^{m-1} dx - 2m\int (k^2-m^2)^m dx$

$\therefore I_m = 2k^2m\, I_{m-n} = 2m\, I_m$

$I_m(1+2m) = 2k^2m\, I_{m-1}$ (4)

$I_m = \frac{2k^2m}{2m+1} I_{m-1}$

50% of students could not start out correctly, then could not ans this question

Q12

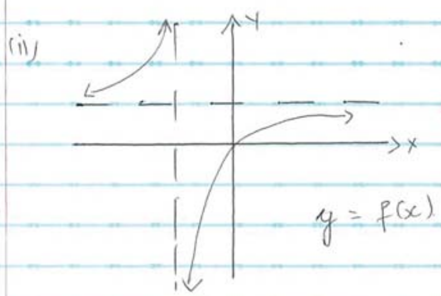
(a) i From the graph $y^2 = f(x)$, value of $f(x)$ between vertical asymptote + y-axis is negative. (1)

$\therefore$ the part of the graph $y = |f(x)|$ must be obtained by reflection about x-axis. (1) 2

graphs answered v well.

✓

(ii)



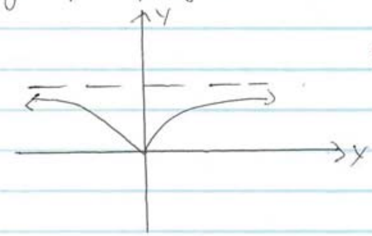
2

position wrt asymp (1)
shape (1)

(iii) $y = |f(x)|$ is symmetrical about y-axis.

since $f(|x|) = f(|x|)$ for $x > 0$

$\therefore$ graph of $y = f(|x|)$ is



2.

position wrt asymp (1)
shape (1)

12 (b)(i) $\angle CAT = \angle ABC$ ($\angle$ bet tan + chord)

and $\angle BCP = \angle CAB$ (" ")

but $\angle BAT = 90°$ (diam ⊥ radius)

$\therefore \angle BCP = 90° - \angle CAT$. (2)

Reasons were poorly stated.

✓

(ii) Since $\angle BCP$ is ext $\angle$ and its $= \angle BAC$ which is part of int opp $\angle$, 1

ATBC cannot be cyclic quad as theorem ext$\angle$ = int opp $\angle$ does not hold. 1 (2)

✓

(c) i) Let $P(x) = (x-\alpha)^r Q(x)$

$\therefore P'(x) = Q(x)\left[r(r-\alpha)^{r-1}\right] + \left[(x-\alpha)^r Q'(x)\right]$ 1

ie $P'(x) = (x-\alpha)^{r-1}\left[rQ(x) + (x-\alpha)Q'(x)\right]$

$\therefore P'(x)$ has a root 1

$x = \alpha$ of multiplicity $(r-1)$

Since $P(\alpha) = P'(\alpha) = 0$ by subst. (2)

Some students $P'(x)$ but did not factorise $(x-\alpha)^{r-1}$ + show that $P'(\alpha) = P(\alpha) = 0$.

(ii) Let $P(x) = x^4 - 5x^3 + 16x^2 - 21x + 9$

$P'(x) = 4x^3 - 15x^2 + 32x - 21$

we know $P(1) = P'(1) = 0$

$P'(1) = 4 - 15 + 32 - 21 = 0 = P(1)$ (1)

To find other roots, several approaches

NB $P(x) = (x-1)^2 Q(x)$

by inspect $Q(x) = x^2 - 3x + 9$

OR Divide $(x-1)^2$ into $P(x)$ + use long ÷ to find $P(x)$

OR $1, 1, \alpha, \bar{\alpha}$ are roots since in $Q(x)$, $\Delta < 0$ 2.

$x^2 - 3x + 9 = 0$, $x = \dfrac{3 \pm \sqrt{-27}}{2} = \dfrac{3 \pm 3\sqrt{3}i}{2}$ (3)

✓

Q13 (a)(i) $\sin x + \sin 3x$

$= \sin(2x - x) + \sin(2x + x)$ 1

$= \sin 2x \cos x - \cos 2x \sin x$

$+ \sin 2x \cos x + \cos 2x \sin x$ 1

$= 2 \sin 2x \cos x$ (2)

Alt. soln

$\sin S + \sin T = 2\sin\left(\frac{S+T}{2}\right)\cos\left(\frac{S-T}{2}\right)$

$LHS = 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)$

$= 2\sin 2x \cos(-x)$

$= 2\sin 2x \cos x$

$= RHS$ QED

(ii) $\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x \cos x + \sin 2x = 0$

$\sin 2x (2\cos x + 1) = 0$

$\sin 2x = 0 \quad \cos x = -\frac{1}{2}$ ✓

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$ ✓ $\quad x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ 3

b) $e = \frac{3}{2}$, directrices $x = \pm 4$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ equdicet

$x = \pm \frac{a}{e}$

$\therefore a = 4 \times \frac{3}{2} = 6$ ✓

and $b^2 = a^2(e^2 - 1)$

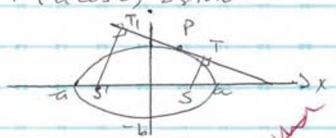
$= 36\left(\frac{9}{4} - 1\right)$

$= 45$ ✓ 3

$\therefore$ eqn $\frac{x^2}{36} - \frac{y^2}{45} = 1$ ✓

• incorrect formula $a^2 = b^2(e^2 - 1)$ ✗

c(i) The tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has eqn $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ at $P(a\cos\theta, b\sin\theta)$



S has co-ord $(ae, 0)$ ✓ 2

$ST = \frac{|e\cos\theta - 1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$ ($\perp$ dist from line)

→ long way

$* \; b\cos\theta\, x + a\sin\theta\, y - ab = 0$

eqn tan P: $y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$

$-a\sin\theta\, y + ab\sin^2\theta = b\cos\theta\, x - ab\cos^2\theta$

$*$

$ST = \frac{|b\cos\theta(ae) + a\sin\theta(0) - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$= \frac{|abe\cos\theta - ab|}{ab\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}, \; a, b > 0$

$ST = \frac{ab\,|e\cos\theta - 1|}{ab\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$

$= \frac{|e\cos\theta - 1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$ QED

13 c(ii)

$ST \cdot S'T' = \frac{1 - e^2\cos^2\theta}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$

For ellipse $b^2 = a^2(1 - e^2)$

$\Rightarrow e^2 = 1 - \frac{b^2}{a^2}$ ✓

$ST \cdot S'T' = \frac{1 - \cos^2\theta + \frac{b^2}{a^2}\cos^2\theta}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$

$= \frac{1 - \cos^2\theta\left(1 - \frac{b^2}{a^2}\right)}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}$ ✓

$= \frac{a^2b^2(1 - e^2\cos^2\theta)}{b^2\cos^2\theta + a^2\sin^2\theta}$

$= \frac{a^2b^2(1 - e^2\cos^2\theta)}{a^2(1 - e^2)\cos^2\theta + a^2\sin^2\theta}$

$= \frac{a^2b^2(1 - e^2\cos^2\theta)}{a^2(\cos^2\theta + \sin^2\theta) - a^2e^2\cos^2\theta}$ ✓

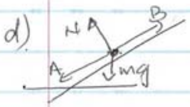
$= \frac{a^2b^2(1 - e^2\cos^2\theta)}{a^2(1 - e^2\cos^2\theta)}$ 3

$= b^2$

$= RHS$ QED.

• students lost in Alg simplification

d)



$A = mg\sin\theta \quad B = \frac{mv^2}{r}\cos\theta$

$A - B = 0$

$\therefore \frac{mv^2}{r}\cos\theta = mg\sin\theta$

$\therefore \tan\theta = \frac{v^2}{gr}$ 2

$\therefore V = \frac{100 \times 1000}{3600}$ ✓ $\quad \therefore \tan\theta = \frac{\left(\frac{250}{9}\right)^2}{10 \times 100}$ ✓

$= \frac{250}{9}$ m/s $\quad \theta = 37°39'$

• a few covered not resolve forces eg $A - B = 0$.

Q14

(a) (i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta$
$+ \cos(-n\theta) + i\sin(-n\theta)$ ✓

$= \cos n\theta + i\sin n\theta - i\sin n\theta$ (note cos even + sin odd)

$= 2\cos n\theta$. 2. ✓

(ii) $z + z^{-1} = 2\cos\theta$, $z^2 + z^{-2} = 2\cos 2\theta$
$z^3 + z^{-3} = 2\cos 3\theta$

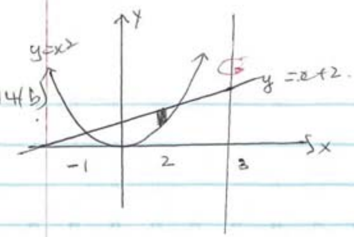
$8\cos\theta\cos 2\theta\cos 3\theta = (z + z^{-1})(z^2 + z^{-2})(z^3 + z^{-3})$ ✓
$= z^6 + 1 + z^2 + z^4 + z^{-4} + z^{-2} + 1 + z^{-6}$
$= 2 + z^2 + z^{-2} + z^4 + z^{-4} + z^6 + z^{-6}$ ✓
$= 2 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta$
$\therefore 4\cos\theta\cos 2\theta\cos 3\theta$ ✓ 3.
$= 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$

• insufficient working.

(iii) $2\cos^2\theta + 2\cos^2 2\theta + 2\cos^2 3\theta = 2$
$1 - \cos 2\theta + \cos 4\theta + \cos 6\theta = 0$ ✓
$4\cos\theta\cos 2\theta\cos 3\theta = 0$
$\theta = 2k\pi \pm \pi/2$ ✓
or $\theta = \frac{2k\pi}{2} \pm \pi/4$ ✓ 3.
or $\theta = \frac{2k\pi}{3} \pm \pi/6$.

• v. poorly answered

14(b)



Pts of intersection $x = -1, 2$.

δx, $y_1 - y_2$. Rotate strip to form a shell.

$\therefore$ Vol shell $\delta V = 2\pi r h\,\delta x$ ✓ $r = 3 - x$
$h = y_1 - y_2$
$= x + 2 - x^2$

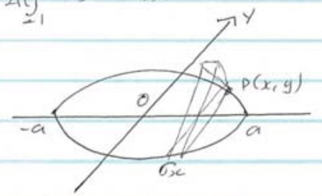
$\therefore V \doteq \sum_{x=-1}^{x=2} 2\pi(3-x)(x+2-x^2)\,\delta x$ ✓ 3

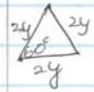
$V = \lim_{\delta x \to 0} \sum_{x=-1}^{x=2} 2\pi(3-x)(x+2-x^2)\,\delta x$ ✓

$V = 2\pi\int_{-1}^{2} (3-x)(x+2-x^2)\,dx$

• answered v poorly
• could not obtain volume eqn in terms of rotation around $x=3$

(c)



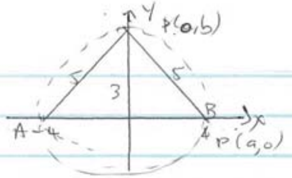


$\delta V = \frac{1}{2}\,2y\cdot 2y\cdot\sin 60°\,\delta x$
$= y^2\sqrt{3}\,\delta x$ ✓
$V \doteq \sum_{-a}^{x=a} y^2\sqrt{3}\,\delta x$
$V = \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} y^2\sqrt{3}\,\delta x$ ✓
$= \lim_{\delta x \to 0} \sum_{x=-a}^{x=a} (a^2 - x^2)\sqrt{3}\,\delta x$
$= \sqrt{3}\int_{-a}^{a} (a^2 - x^2)\,dx$
$= 2\sqrt{3}\int_{-a}^{a} (a^2 - x^2)\,dx$ ✓ 4
$= 2\sqrt{3}\left[a^2 x - \frac{x^3}{3}\right]_0^a$
$= \frac{4\sqrt{3}}{3}a^3$ ✓

• answered well.

Q15

16 (c)



$OP = 3$ (by Pyth)

$\therefore b = 3$ ✓

when P @ $B(a,0)$, $a=5$ ✓

Sub $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

2

eqn $\frac{x^2}{25}+\frac{y^2}{9}=1.$ ✓

(ii) $\frac{x^2}{25}+\frac{y^2}{9} = \frac{5^2\cos^2\theta}{25} + \frac{3^2\sin^2\theta}{9}$

$= \cos^2\theta + \sin^2\theta$ ✓

$= 1 = RHS.$ QED.

$x = 5\cos\theta$

$\frac{dx}{d\theta} = -5\sin\theta$

$y = 3\sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta$ ✓

$\frac{dy}{dx} = \frac{dx}{d\theta}\times\frac{d\theta}{dy} = 3\cos\theta \cdot \frac{-1}{5\sin\theta}$ ✓

• Students used eqn of tan rather than finding it

when $\theta = \frac{\pi}{6}$

$\frac{dy}{dx} = \frac{-3\times\frac{\sqrt{3}}{2}}{5\times\frac{1}{2}} = \frac{-3\sqrt{3}}{5}.$

• messy algebra.

$x = 5\times\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$y = 3\times\frac{1}{2} = 3/2$

4

$\therefore y-\frac{3}{2} = \frac{-3\sqrt{3}}{5}\left(x-\frac{5\sqrt{3}}{2}\right)$

$3\sqrt{3}x + 5y - 30 = 0$ ✓

16(b)(i) $\alpha+\beta+\gamma = 0$

$\alpha\beta+\alpha\gamma+\beta\gamma = 3b$ ✓

$\therefore \frac{\beta\gamma}{\alpha}+\frac{\alpha\beta}{\gamma}+\frac{\alpha\gamma}{\beta} = \frac{(\beta\gamma)^2+(\alpha\beta)^2+(\alpha\gamma)^2}{\alpha\beta\gamma}$

• very poorly answered

$= \frac{(\beta\gamma+\alpha\gamma+\alpha\beta)^2-2(\alpha\beta\gamma^2+\alpha\beta^2\gamma+\beta\gamma\alpha)}{\alpha\beta\gamma}$

• students failed to set up step 1.

$= \frac{(\beta\gamma+\alpha\gamma+\alpha\beta)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)}{\alpha\beta\gamma}$ ✓

$= \frac{(3p)^2-2q(0)}{-q}$

$= \frac{9p^2}{-q}$ ✓ 4

$\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta}+\frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\beta}{\gamma} = \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0-2.3p = \underline{\underline{-6p}}$

$\therefore \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} = \alpha\beta\gamma = -q.$ ✓

$\therefore$ reqd eqn $x^3+\frac{9p^2x^2}{q}-6px+q=0$

(ii) $\frac{\alpha\beta}{\gamma}=1$ is a root ✓

• Hence could not answer (ii)

$\therefore 1+\frac{9p^2}{q^2}-6p+q=0$ ✓

$q+9p^2-6pq+q^2=0$

2

$\therefore \underline{(3p-q)^2+q=0}.$

15d) Let $A = \sin^{-1} x$ & $B = \cos^{-1} x$

$x = \sin A$ $\qquad$ $x = \cos B$

15d)

$\sqrt{1-x^2} = \cos A$ and $\sqrt{1-x^2} = \sin B$

LHS $= \sin(\sin^{-1} x - \cos^{-1} x)$

$= \sin(A-B)$

$= \sin A \cos B - \cos A \sin B$

$= x \cdot x - \sqrt{1-x^2}\sqrt{1-x^2}$

$= x^2 - (1-x^2) = 2x^2 - 1$

3

Q16(a) initial position @ origin, + initial direct^n up

$\dot{v} = -g - kv$

initial conditions: $t=0, x=0, v=u$

reln bet x + v

$v\frac{dv}{dx} = -(g+kv)$

$-dx = \frac{v\,dv}{g+kv}$

$-k\,dx = \frac{kv\,dv}{g+kv}$, ✓

$-k\,dx = \left[1 - \frac{g}{g+kv}\right]dv$

$-kx + c = kv - g\ln|g+kv|$

$x=0, v=u \Rightarrow c = ku - g\ln|g+ku|$

answered v. poorly only 1 stud. got full marks

$\therefore x = \frac{u-v}{k} + \frac{g}{k^2}\ln\left|\frac{g+kv}{g+ku}\right|$ ✓ - ①

If particle reaches greatest height H at a time T, its speed $v=0$.

$\therefore$ from ①

$\boxed{H = \frac{u}{k} + \frac{g}{k^2}\ln\left|\frac{g}{g+ku}\right|}$

Reln bet v + t ✓

$\frac{dv}{dt} = -(g+kv)$

$-dt = \frac{dv}{g+kv}$, $-k\,dt = \frac{k\,dv}{g+kv}$

$-kt = \frac{k\,dv}{g+kv}$ $\therefore -kt + c = \ln|g+kv|$

$\therefore t = -\frac{1}{k}\ln\left|\frac{g+kv}{g+ku}\right|$ ... ②

$T = -\frac{1}{k}\ln\left|\frac{g}{g+ku}\right|$ ✓ from ②

$\therefore \ln\left|\frac{g}{g+ku}\right| = -kT$

$\Rightarrow H = \frac{u}{k} - \frac{gkT}{k^2}$ 4

$\therefore \boxed{u = kH + gT}$

16b) Let $A = x+y$

$B = x-y$

$\therefore A+B = 2x$

$X = \frac{(A+B)}{2}$

$Y = \frac{(A-B)}{2}$

$\therefore \sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ • students failed to apply sum to prod formulas

$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}$ ✓

$= \tan\left(\frac{A+B}{2}\right)$ ✓

Now $A+B+C = \pi$

$\therefore A+B = \pi - C$ 4 • could not work out $\cot\frac{C}{2}$

LHS $\tan\frac{\pi - C}{2}$

$= \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$ ✓

$= \cot\frac{C}{2}$

$=$ RHS. QED.

(16c) Let $t = \tan\frac{\alpha}{2}$

$\Rightarrow \cos\alpha = \frac{1-t^2}{1+t^2}$ ✓ • Not attempted by many

$\therefore (1+t^2)\cos\alpha = 1-t^2$

$\therefore t^2 = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1-\tan x\cot y}{1+\tan x\cot y}$

$t^2 = \frac{1-\tan x\cot y}{1+\tan x\cot y} \times \frac{\sin y\cos x}{\sin x\cos y}$

$= \frac{\sin y\cos x - \sin x\cos y}{\sin y\cos x + \sin x\cos y}$

$= \sin(y-x)\operatorname{cosec}(y+x)$

$\therefore \tan^2\frac{1}{2}\alpha = \sin(y-x)\operatorname{cosec}(y+x)$

1(d) $\frac{1}{9-10\sin^2 x} = \frac{\sec^2 x}{9-\tan^2 x}$

Since $\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}$, we get

$$\frac{1}{9-10\sin^2 x} = \left[9 - 10\frac{\tan^2 x}{1+\tan^2 x}\right]^{-1}$$

$$= \left[\frac{9-\tan^2 x}{1+\tan^2 x}\right]^{-1}$$

$$= \frac{1+\tan^2 x}{9-\tan^2 x}$$

$$= \frac{\sec^2 x}{9-\tan^2 x}.$$

Hence using sub $u = \tan x$

$du = \sec^2 x\, dx$

$x=0 \quad u=0$

$x=\frac{\pi}{3} \quad u=\sqrt{3}$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{1}{9-10\sin^2 x}dx$$

$$= \int_0^{\sqrt{3}} \frac{1}{9-u^2}du$$

$$= \frac{1}{6}\left[\ln\left|\frac{3+u}{3-u}\right|\right]_0^{\sqrt{3}}$$

4.

$$= \frac{1}{6}\ln\frac{3+\sqrt{3}}{3-\sqrt{3}}$$

$$= \frac{1}{6}\ln\frac{(3+\sqrt{3})^2}{3^2-(\sqrt{3})^2} = \frac{1}{6}\ln(2+\sqrt{3})$$

$$= \frac{1}{6}\ln\left(\frac{12+6\sqrt{3}}{6}\right)$$

$$= \frac{1}{6}\ln(2+\sqrt{3})$$

OR see students' attemp.

• Very few got 2/4.
• Many sub $\sin^2$ ~~inverse~~ but got lost in Alg.
• 2 ~~some~~ good attempts.

Start here

Students attempt.

d. $\int_0^{\pi/3} \frac{1}{9-10\sin^2 x}dx.$

$$= \int_0^{\pi/3} \frac{1}{9-10(1-\cos^2 x)}dx$$

$$= \int_0^{\pi/3} \frac{1}{-1+10\cos^2 x}dx$$

$$= \int_0^{\pi/3} \frac{1}{10\cos^2 x - 1}dx.$$

let $t = \tan x$ | $x = \pi/3,\ t = \sqrt{3}$

$dt = \sec^2 x\, dx$ | $x=0,\ t=0$

$dt = (1+t^2)dx$ $\qquad 10\cos^2 x = \frac{10}{1+t^2}$

$dx = \frac{dt}{1+t^2}$

$$I = \int_0^{\sqrt{3}} \frac{1}{\frac{10}{1+t^2} - 1}\cdot\frac{dt}{1+t^2}$$

$$= \int_0^{\sqrt{3}} \frac{1}{\frac{10-(1+t^2)}{1+t^2}}\cdot\frac{dt}{1+t^2}$$

$$= \int_0^{\sqrt{3}} \frac{1+t^2}{9-t^2}\cdot\frac{dt}{1+t^2}$$

$$= \int_0^{\sqrt{3}} \frac{1}{9-t^2}dt \rightarrow = \frac{1}{6}\int \frac{1}{(3-t)} + \frac{1}{(3+t)}dt$$

$$= \left.\frac{1}{t}\right]_0^{\sqrt{3}} \qquad = \frac{1}{6}\left[-\ln(3-t) + \ln(3+t)\right]_0^{\sqrt{3}}$$

$$= \frac{1}{6}\left[-\ln(3-\sqrt{3}) + \ln(3+\sqrt{3})\right]$$

$$= \frac{1}{\sqrt{3}} \qquad = \frac{1}{6}\ln\frac{3+\sqrt{3}}{3-\sqrt{3}}$$