HORNSBY GIRLS' HIGH SCHOOL

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question on a fresh sheet of paper

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value

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Total Marks – 120

Attempt Questions 1 – 8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find $\displaystyle\int \frac{dx}{x^2 - 4x + 40}$ **2**

(b) Evaluate $\displaystyle\int_0^2 x^3 e^{x^2}\, dx$. **3**

(c) Find $\displaystyle\int \sin^3 x\, dx$ **2**

(d) Evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{4-x}}\, dx$ **3**

(e) (i) Find the real numbers a, b and c such that **3**

$$\frac{3x^2 + 2x + 11}{(x^2+3)(1-x)} \equiv \frac{ax+b}{x^2+3} + \frac{c}{1-x}.$$

(ii) Hence find $\displaystyle\int \frac{3x^2 + 2x + 11}{(x^2+3)(1-x)}\, dx$. **2**

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Given z is a complex number such that $z = 1 + i$

(i) Write z in mod-arg form **2**

(ii) Evaluate z^{12} **2**

(b) If $P(z) = z^4 - 30z^2 + 289$

(i) Show that $z = 4 + i$ is a zero of $P(z)$ **2**

(ii) Find all zeros of $P(z)$ over the complex field **5**

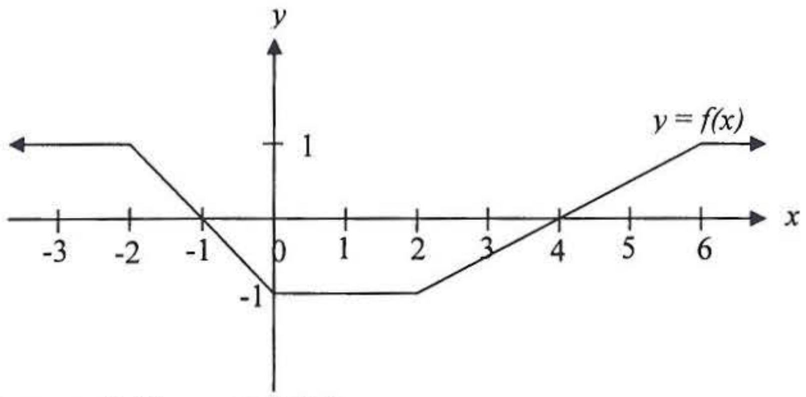
(c) $P(z)$ is a point on the argand diagram such that **4**

$$\arg \frac{z-i}{z+2} = \frac{\pi}{2}$$

Draw and describe the locus of $P(z)$.

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The diagram below is a sketch of the function $y = f(x)$

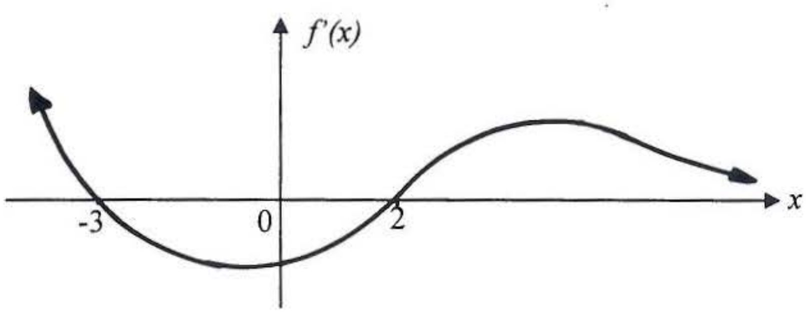


On separate diagrams sketch

(i) $y = |f(x)|$ **2**

(ii) $y = f(|x|)$ **1**

(b) The graph below represents the derivative $f'(x)$ of a certain function $f(x)$. **3**
Given that $f'(x) \to 0$ as $x \to \infty$, $f(0) = 0$ and $f(1) < 0$, sketch the graph of $f(x)$, noting the behaviour as $x \to \infty$.



(c) (i) Sketch the curve $y = \dfrac{x^3 + 4}{x^2}$, showing any stationary points and asymptotic behaviour. **2**

(ii) Hence or otherwise, deduce the values of k, for which the equation $x^3 - kx^2 + 4 = 0$ may have one real root. **1**

(d) (i) If $x = a$ is a multiple root of the polynomial equation $P(x)$ such that $P(x) = 0$, prove that $P'(a) = 0$. **3**

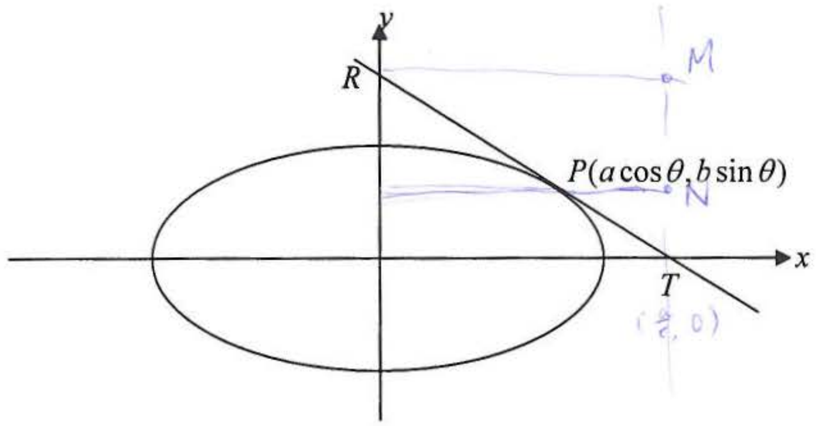
(ii) Find all roots of $P(x) = 16x^3 - 12x^2 + 1$ given that two of the roots are equal. **3**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) An ellipse has parametric equations $x = \sqrt{2}\cos\theta$ and $y = 3\sin\theta$. **2**

Find the Cartesian equation and the eccentricity of the ellipse.

(b)



The ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ shown in the diagram above has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles or otherwise, show that $RP = e^2 RT$. **3**

(c) The area between the curve $y=\ln(x+1)$ and the x-axis, between $x=0$ and $x=1$ is rotated about the y-axis. **4**

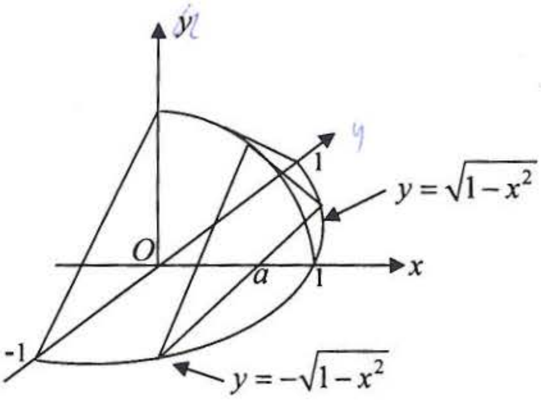
Find the volume of the solid of revolution formed using the method of cylindrical shells.

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) (i) Write down the value of $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$. **1**

(ii) Explain why $\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$ is equal to zero. **1**

(b)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.

Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal side lengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}\left(1 - a^2\right)$. **2**

(ii) Hence find the volume of the solid. **2**

(c) The point $T\left(ct, \frac{c}{t}\right)$ lies on the hyperbola $xy = c^2$. The tangent at T meets the x-axis at P and the y-axis at Q. The normal at T meets the line $y=x$ at R.

(i) Prove that the tangent at T has equation $x + t^2 y = 2ct$. **2**

(ii) Find the coordinates of P and Q. **2**

(iii) Write down the equation of the normal at T. **1**

(iv) Show that the x coordinate of R is $x = \frac{c}{t}\left(t^2 + 1\right)$. **2**

(v) Prove that ΔPQR is isosceles. **2**

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ.

Find a polynomial equation in x whose roots are:

(i) $-\alpha, -\beta, -\gamma$ **1**

(ii) $\alpha^2, \beta^2, \gamma^2$ **2**

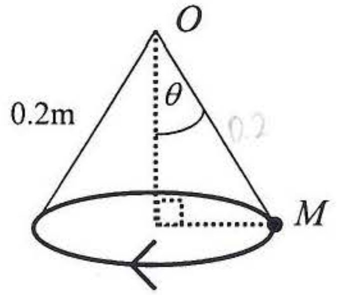
(iii) $\pm\alpha, \pm\beta, \pm\gamma$ **2**

(b) Find a and b if $(1+i)$ is a root of $x^2 + (a+2i)x + 5 - ib = 0$ **3**

(c) A body M, of mass 650g, is fixed to point O by a light wire 0.2m long.

The body rotates in a horizontal plane at 72 revolutions per minute.

Taking $g = 10 m/s^2$,



(i) Prove that $\tan\theta = \dfrac{72\pi^2 \sin\theta}{625}$. **3**

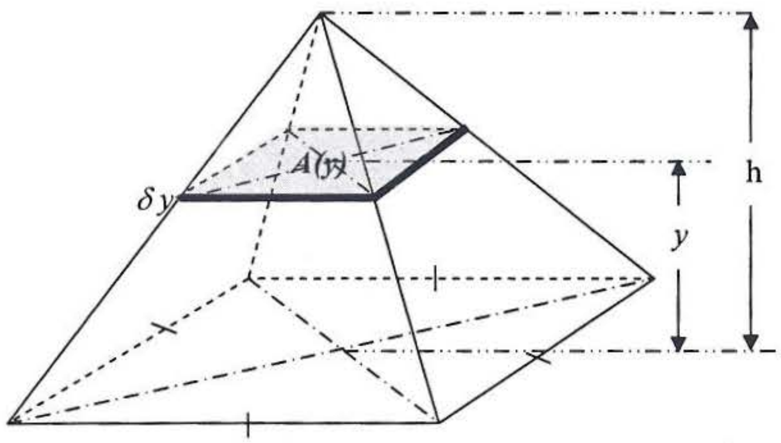
(ii) Find θ to the nearest minute. **2**

rad.?

(iii) The mass of the body is to be doubled but the speed of rotation is to remain the same. What will happen to the value of θ? **2**

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The great pyramid of Cheops at Giza in Egypt is approximately 150 m high and its base is a square with an area of approximately 5 hectares.



(i) Show that the area of the cross section $A(y)$, at y is given by **1**

$$A(y) = (5\times10^4)\times\left(\frac{h-y}{h}\right)^2$$

check (ii) Find the volume of the pyramid by using the slicing technique. **4**

(b) A particle of mass 1 kg is projected vertically upwards with an initial velocity of 100 m/s in a medium in which the resistance force is equal to 0.01 times the square of the body's velocity, i.e. $0.01v^2$. Use $g = 10m/s^2$.

(i) Show that the maximum height reached by the particle is $50\log_e 11$ metres. **4**

(ii) Will the downward velocity of the particle on its return to the point of projection be greater than, less than, or equal to 100m/s? Justify your answer. **2**

(iii) Calculate the actual downward velocity of the particle on its return to the point of projection. **4**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Use the following identity to answer the following questions.

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

(i) Solve $x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$. **3**

(ii) Hence show that

(1) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$ **1**

(2) $\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$ **1**

(iii) Find the polynomial of least degree that has zeros **3**

$$\left(\cot\frac{\pi}{24}\right)^2, \left(\cot\frac{7\pi}{24}\right)^2, \left(\cot\frac{13\pi}{24}\right)^2, \left(\cot\frac{19\pi}{24}\right)^2.$$

(b) Let $I_n = \int_0^1 x(x^2-1)^n \, dx$ for $n = 0,1,2,...$

(i) Use integration by parts to show that **3**

$$I_n = \frac{-n}{n+1} I_{n-1} \text{ for } n \geq 1.$$

(ii) Hence or otherwise show that **2**

$$I_n = \frac{(-1)^n}{2(n+1)} \text{ for } n \geq 0.$$

(iii) Explain why $I_{2n} > I_{2n+1}$ for $n \geq 0$ **1**

(iv) Explain whether or not $I_n > I_{n+2}$ for all $n \geq 0$. **1**

End of Examination

2008 Extension 2 Trial Solutions

Q1 a) $\int \frac{dx}{x^2-4x+40} = \int \frac{dx}{(x-2)^2+36}$

$= \frac{1}{6}\tan^{-1}\left(\frac{x-2}{6}\right)$

b) $\int_0^2 x^3 e^{x^2}\,dx = \left[\frac{x^2 e^{x^2}}{2}\right]_0^2 - \int_0^2 2x\frac{e^{x^2}}{2}\,dx$ $\quad u = x^2 \quad v = \frac{1}{2}e^{x^2}$

$= \left[4\frac{e^4}{2} - 0\right] - \frac{1}{2}\left[e^{x^2}\right]_0^2$ $\quad du = 2x \quad dv = xe^{x^2}$

$= 2e^4 - \frac{1}{2}(e^4-1)$

$= \frac{1}{2}(3e^4+1)$

c) $\int \sin^3 x\,dx = \int \sin^2 x.\sin x\,dx$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int \sin x - \cos^2 x.\sin x\,dx$

$= -\cos x + \frac{\cos^3 x}{3}$

d) $\int_0^1 \frac{x}{\sqrt{4-x}}\,dx$ $\quad$ let $u = 4-x \Rightarrow x = 4-u$

$du = -1$

$= \int_4^3 \frac{4-u}{\sqrt{u}}.-1\,du$ $\quad x=0,\ u=4$

$= \int_3^4 (4u^{-1/2} - u^{1/2})\,du$ $\quad x=1,\ u=3$

$= \left[8u^{1/2} - \frac{2}{3}u^{3/2}\right]_3^4$

$= \left(8\sqrt{4} - \frac{2}{3}.4\sqrt{4} - (8\sqrt{3} - \frac{2}{3}.3\sqrt{3})\right)$

$= 16 - \frac{16}{3} - 6\sqrt{3}$

$= \frac{1}{3}(32 - 18\sqrt{3})$

e) i) $\frac{3x^2+2x+11}{(x^2+3)(1-x)} = \frac{(ax+b)(1-x)+c(x^2+3)}{(x^2+3)(1-x)}$

$ax - ax^2 + b - bx + cx^2 + 3c = 3x^2 + 2x + 11$

$c - a = 3 \rightarrow a = c-3$ ①

$a - b = 2 \rightarrow c-3-b = 2 \therefore c-b = 5$ — ②

$3c + b = 11$ — ③

②+③ : $4c = 16$, $c = 4$ $\quad$ ie $a=1,\ b=-1,\ c=4$

in ② $4 - b = 5$, $b = -1$

in ① $a = 4 - 3 = 1$

ii) $\int \frac{3x^2+2x+11}{(x^2+3)(1-x)}\,dx = \int \frac{x-1}{x^2+3} + \frac{4}{1-x}\,dx$

$= \int \frac{1}{2}\cdot\frac{2x}{x^2+3} - \frac{1}{x^2+3} + \frac{4}{1-x}$

$= \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 4\ln(1-x)$ $\quad , x<1$

SOLUTIONS QUESTION 2

(a) (i) Given $z = 1 + i$

$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\arg z = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$

$$\therefore z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad (2)$$

(ii)

$$\begin{aligned} z^{12} &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{12} \\ &= 2^6\left(\cos 3\pi + i\sin 3\pi\right) \qquad \text{De Moivre's Thm} \\ &= 64(-1+0) \\ &= -64 \qquad (2) \end{aligned}$$

(b) $P(z) = z^4 - 30z^2 + 289$

(i) If $P(4+i) = 0$ then $4+i$ is a zero of $P(z)$.

$$\begin{aligned} P(4+i) &= (4+i)^4 - 30(4+i)^2 + 289 \\ &= ((4+i)^2)^2 - 30(15+8i) + 289 \qquad \text{substitution \& partial simp.} \\ &= (15+8i)^2 - 450 - 240i + 289 \qquad (1) \\ &= 225 + 240i + 64i^2 - 450 - 240i + 289 \\ &= 225 + 240i - 64 - 450 - 240i + 289 \qquad (1) \\ &= 0 \end{aligned}$$

$\therefore (4+i)$ is a zero of $P(z)$

(ii) Since $4+i$ is a zero, so also is $4-i$ (complex conj.)

i.e. $[z-(4+i)][z-(4-i)]$ is a factor (1)

$= [(z-4)-i][(z-4)+i]$ is a fact

$= (z-4)^2 - i^2$ " " "

$= z^2 - 8z + 17$ " " " (1)

(ii) (continued)

$$\begin{array}{r|l} & z^2 + 8z + 17 \\ \hline z^2 - 8z + 17 & z^4 \qquad - 30z^2 \qquad + 289 \\ & \underline{z^4 - 8z^3 + 17z^2} \\ & 8z^3 - 47z^2 \qquad + 289 \\ & \underline{8z^3 - 64z^2 + 136z} \\ & 17z^2 - 136z + 289 \\ & \underline{17z^2 - 136z + 289} \\ & 0 \end{array} \qquad (1)$$

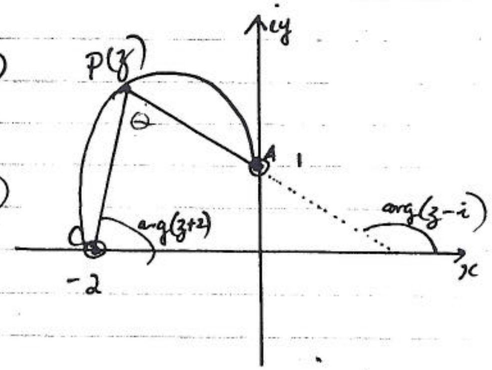
$\therefore z^2 + 8z + 17$ is a factor of $P(z)$

$$\begin{aligned} \therefore z^2 + 8z + 17 &= (z^2 + 8z + 16) + 1 \qquad \text{(Alt: use quad. form)} \\ &= (z+4)^2 - i^2 \\ &= (z+4-i)(z+4+i) \\ &= (z-(-4+i))(z-(-4-i)) \qquad (1) \end{aligned}$$

$\therefore$ remaining zero's are $-4+i$, $-4-i$

$\therefore$ zeros of $P(z)$ are $4+i$, $4-i$, $-4+i$, $-4-i$. (1)

(c)



(2)

Geometrically

$$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{2}$$

$$\arg(z-i) - \arg(z+2) = \frac{\pi}{2}$$

NB $\arg(z-i) > \arg(z+2)$

From diagram

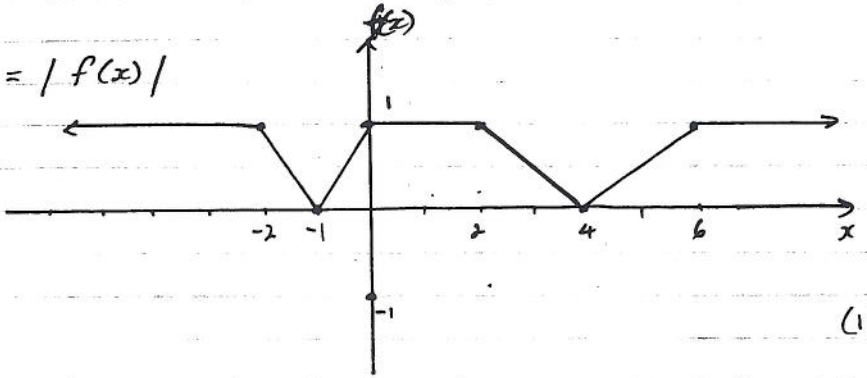
Now $\theta + \arg(z+2) = \arg(z-i)$ (ext. angle of $\triangle$)

$$\therefore \theta = \arg(z-i) - \arg(z+2) = \frac{\pi}{2}$$

$\therefore P(z)$ is a semicircle with AC as diameter and P above AC. Points A, C are excluded from the locus. C ... $(-1, \frac{1}{2})$ radius $\sqrt{5}/2$ (2)

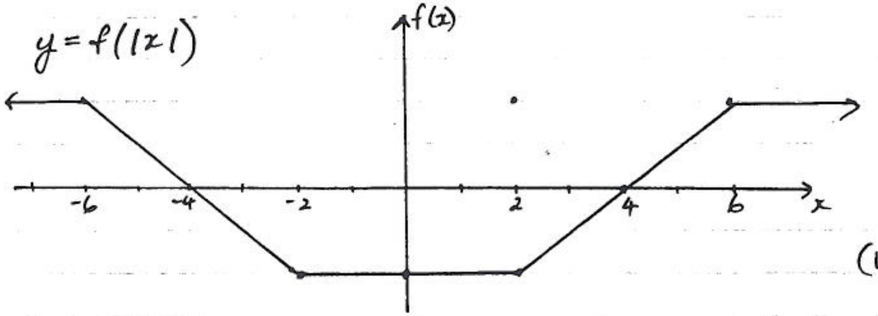
SOLUTIONS QUESTION 3

(a) (i) $y = |f(x)|$



(1)

(ii) $y = f(|x|)$



(1)

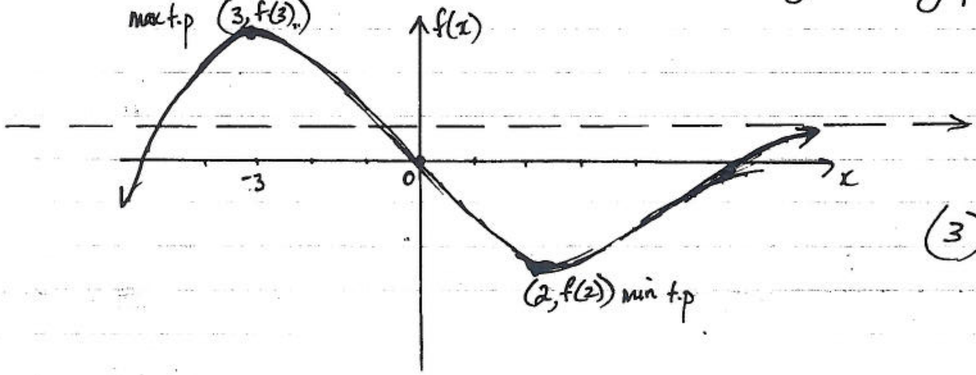
(b) $f'(2) = 0 \quad f'(2^-) < 0 \quad f'(2^+) > 0 \quad - \searrow \_ \nearrow + \quad \therefore x = 2$ is min

$f'(-3) = 0 \quad f'(-3^-) > 0 \quad f'(-3^+) < 0 \quad + \nearrow \_ \searrow - \quad \therefore x = -3$ is max

Curve passes through $(0,0)$

Curve below x axis when $x = 1$

as $x \to \infty$, $f'(x) \to 0$ ie flattens out to horizontal asymptote.



(3)

(c) (i) $y = \dfrac{x^3 + 4}{x^2}$

$= x + 4x^{-2}$

zero's $x^3 = -4$

$x = \sqrt[3]{-4}$

T.P's. $\left(\frac{dy}{dx} = 0\right)$

$\dfrac{dy}{dx} = 1 - \dfrac{8}{x^3}$

$\dfrac{dy}{dx} = 0$ when $x = 2$

Inflexions $\left(\frac{d^2y}{dx^2} = 0\right)$

$\dfrac{d^2y}{dx^2} = \dfrac{24}{x^4} > 0 \therefore$ no inflexions

test t.p.

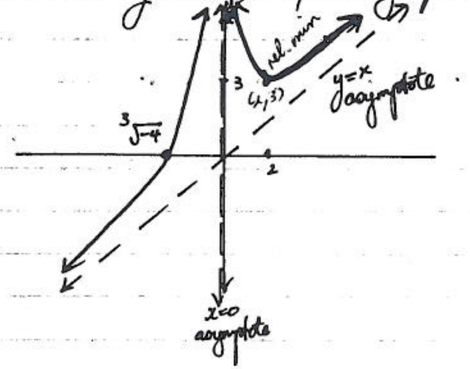
$\dfrac{d^2y}{dx^2} > 0$ when $x = 2$ $\therefore$ min t.p. at $(2,3)$

Asymptotic Behaviour

vertical $x = 0$ $\quad x \to 0^+ \; y \to \infty$

$x \to 0^- \; y \to \infty$

other as $x \to \infty \therefore y = x$ (oblique asymptote)



(3)

(ii) From the graph $x^3 - kx^2 + 4 = 0 \Rightarrow k = \dfrac{x^3 + 4}{x^2}$ what horizontal line $y = k$ will only have one solution? any $k < 3$. (1)

(d) (i) If 'a' is a mult. root of $P(x)$ then

$$P(x) = (x-a)^r \cdot Q(x)$$
$$P'(x) = (x-a)^r \cdot Q'(x) + Q(x) \cdot r(x-a)^{r-1}$$
$$= (x-a)^{r-1}\left[(x-a)Q'(x) + rQ(x)\right]$$
$$= (x-a)^{r-1} S(a) \quad \text{where } S(a) = (x-a)Q'(x) + rQ(x)$$

$$\therefore P'(a) = (a-a)^{r-1} S(a)$$
$$= 0 \times S(a)$$
$$= 0 \quad \text{as required.}$$ (3)

(ii) $P(x) = 16x^3 - 12x^2 + 1$

$P'(x) = 48x^2 - 24x$

when $P'(x) = 0$

$48x^2 - 24x = 0$

$24x(2x-1) = 0$

$\therefore x = 0, \frac{1}{2}$

Since $P(0) = 1$

and $P(\frac{1}{2}) = 0$ $x = \frac{1}{2}$ is double root

Now $\alpha + \beta + \gamma = \frac{12}{16}$ but $\alpha = \beta = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{2} + \gamma = \frac{12}{16}$

$\gamma = -\frac{1}{4}$

$\therefore$ 3 roots are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}$ (3)

Q4. a) $x = \sqrt{2}\cos\theta$ $y = 3\sin\theta$

$\cos\theta = \frac{x}{\sqrt{2}}$ $\sin\theta = \frac{y}{3}$

$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ $(\cos^2\theta + \sin^2\theta = 1)$

$\therefore \frac{x^2}{2} + \frac{y^2}{9} = 1$

$a^2 = b^2(1-e^2)$, $a < b$

$2 = 9(1-e^2)$

$1 - e^2 = \frac{2}{9}$

$e^2 = \frac{7}{9}$

$\therefore e = \frac{\sqrt{7}}{3}$

b) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{a^2} \times \frac{b^2}{2y}$

$= \frac{-b^2 x}{a^2 y}$

at $P(a\cos\theta, b\sin\theta)$, $\frac{dy}{dx} = \frac{-b^2 . a\cos\theta}{a^2 b\sin\theta}$

$= -\frac{b\cos\theta}{a\sin\theta}$

$\therefore$ equation of tangent is $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$

$b\cos\theta x + a\sin\theta y = ab(\sin^2\theta + \cos^2\theta)$

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ — ①

ii) at T, $y = 0$ & T lies on tangt $\therefore$ $\frac{x\cos\theta}{a} = 1$

ie $x = \frac{a}{\cos\theta}$

$\therefore T\left(\frac{a}{\cos\theta}, 0\right)$

equation of directrix is $x = \frac{a}{e}$

$\therefore \frac{a}{\cos\theta} = \frac{a}{e}$

ie $\cos\theta = e$

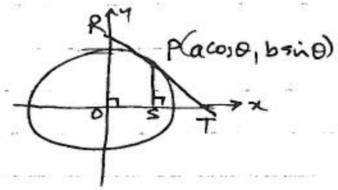
4b) iii) • since $\cos\theta = e$, the x-coordinate of P is ae

The focus has coordinates $S(ae, 0)$

$\therefore$ The focal chord makes an angle of $90°$ with x-axis

iv) $\triangle TOR \;|||\; \triangle TSP$ (equiangular)

$\therefore \dfrac{RT}{OT} = \dfrac{RP}{OS}$ (ratio of intercepts)



$T(\frac{a}{e}, 0)$ and $S(ae, 0)$

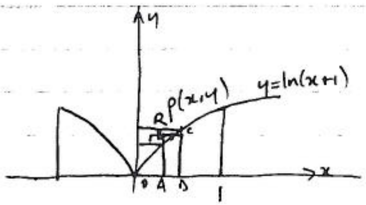
$\therefore OT = \frac{a}{e}$ and $OS = ae$

$\therefore \dfrac{RT}{\frac{a}{e}} = \dfrac{RP}{ae}$

$RT = \dfrac{RP}{ae} \cdot \dfrac{a}{e}$

$\therefore RP = e^2 RT$

c)



let ABCD be rectangle of height y and width δx where $P(x,y)$ is the midpt of the rectangle

let R be outer radius & r be inner radius

$\delta V = \pi(R^2 - r^2) \times h$

$= \pi(R+r)(R-r) \times y$

$= 2\pi\left(\frac{R+r}{2}\right)(R-r) \times y$

$= 2\pi \cdot x \cdot \delta x \cdot y$

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x y\, \delta x, \quad y = \ln(x+1)$

$= \int_0^1 2\pi x \cdot \ln(x+1)\, dx \qquad u = \ln(x+1) \quad v = \frac{x^2}{2}, \quad du = \frac{1}{x+1} \quad dv = x$

$= 2\pi\left(\left[\frac{x^2}{2}\ln(x+1)\right]_0^1 - \int \frac{x^2}{2(x+1)}\, dx\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\int_0^1 \left(x - 1 + \frac{1}{x+1}\right) dx\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1\right)$

$= 2\pi\left(\frac{1}{2}\ln 2 - \frac{1}{2}\left[\frac{1}{2} - 1 + \ln 2 - (0 - 0 + 0)\right]\right)$

$\therefore$ Volume $= \frac{\pi}{2}$ units^3

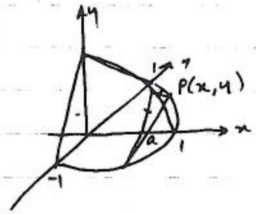
Q5 a) i) $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2} \cdot \pi \cdot a^2$, area of semicircle centre $(0,0)$, radius a

$= \dfrac{\pi a^2}{2}$

ii) $f(x) = x\sqrt{a^2 - x^2}$ is an odd f^n $\therefore \int_{-a}^{0} f(x)\, dx = -\int_0^a f(x)\, dx$

$\therefore \int_{-a}^{a} f(x)\, dx = 0$

b) i)



$\frac{3}{4} \times 2\sqrt{1-a^2} = \dfrac{3\sqrt{1-a^2}}{2}$

$2y = 2\sqrt{1-a^2}$

$\frac{3}{2}\sqrt{1-a^2}$

$\sqrt{1-a^2}$

height $= \sqrt{\frac{9}{4}(1-a^2) - (1-a^2)}$

$= \sqrt{\frac{5}{4}(1-a^2)}$

$= \frac{\sqrt{5}}{2}\sqrt{1-a^2}$

$\therefore$ Area of $\triangle = \frac{1}{2} \times 2\sqrt{1-a^2} \times \frac{\sqrt{5}}{2}\sqrt{1-a^2}$

$= \frac{\sqrt{5}}{2}(1 - a^2)$

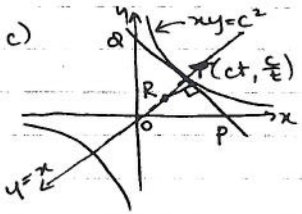
ii) $V = \int_0^1 \frac{\sqrt{5}}{2}(1 - x^2)\, dx$

$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$

$= \frac{\sqrt{5}}{2}\left(1 - \frac{1}{3}\right)$

$\therefore$ Vol $= \frac{\sqrt{5}}{3}$ units^3

c)



i) $xy = c^2 \Rightarrow y = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

at $T(ct, \frac{c}{t})$ $M_T = \dfrac{-c^2}{(ct)^2} = -\dfrac{1}{t^2}$

$\therefore$ eqn of tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$

$x + t^2 y = 2ct$

ii) at P $y = 0$ $\therefore x = 2ct$ $\therefore P(2ct, 0)$

at Q, $x = 0$ $\therefore t^2 y = 2ct$ $\therefore Q(0, \frac{2c}{t})$

iii) $M_N = t^2$ $\therefore$ eqn is $y - \frac{c}{t} = t^2(x - ct)$

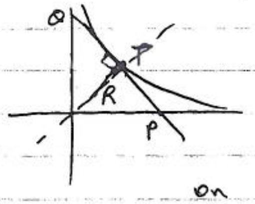
ie $ty - c = t^3 x - ct^4$

$t^3 x - ty = ct^4 - c$

iv) R lies on $y=x$ and the normal at T, $t^3x - ty = ct^4 - c$

$$\therefore t^3x - tx = c(t^4-1)$$
$$tx(t^2-1) = c(t^2-1)(t^2+1)$$
$$x = \frac{c}{t}(t^2+1)$$

v) R lies on $y=x$ $\therefore$ coordinates of R $\left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right)$



$P(2ct, 0)$ $Q\left(0, \frac{2c}{t}\right)$

Midpt of PQ is $\left(\frac{2ct+0}{2}, \frac{0+\frac{2c}{t}}{2}\right)$

$= \left(ct, \frac{c}{t}\right)$, which is the point T.

$\therefore$ Any point on the normal at T will lie on the perpendicular bisector of PQ.

$\therefore \triangle PRQ$ is isosceles

Question 6 Solutions

(a)(i) let $x = -X$

$$(-X)^3 + 2(-X) - 1 = 0$$
$$-X^3 - 2X - 1 = 0$$
$$\therefore x^3 + 2x + 1 = 0 \quad (1)$$

(ii) let $x = \sqrt{X}$

$$(\sqrt{X})^3 + 2\sqrt{X} - 1 = 0$$
$$X\sqrt{X} + 2\sqrt{X} = 1 \quad (\text{sq. b.s})$$
$$X^3 + 4X^2 + 4X = 1$$
$$\therefore x^3 + 4x^2 + 4x - 1 = 0 \quad (2)$$

(iii) let $x = \pm X$ (Note $x = -X$) in part (i) and $x = X$ is original equation

$\therefore (X^3+2X-1)(X^3+2X+1) = 0$ must have $x = \pm X$

$$\therefore (x^3+2x-1)(x^3+2x+1) = 0$$
$$x^6 + 4x^4 + 4x^2 - 1 = 0 \quad (2)$$

(b) Note $1-i$ is <u>NOT</u> a root of $x^2 + (a+2i)x + 5 - ib = 0$ since all coefficients are NOT real

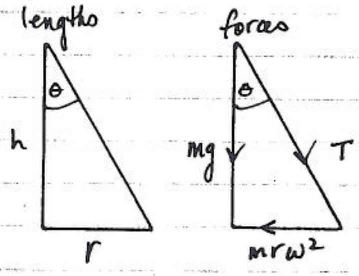
$1+i$ is a root of $x^2 + (a+2i)x + 5 - ib = 0$ (given)

$$\therefore (1+i)^2 + (a+2i)(1+i) + 5 - ib = 0$$
$$\therefore 1 + 2i - 1 + a + 2i + ai - 2 + 5 - ib = 0$$
$$(a+3) + (4+a-b)i = 0$$

$\therefore a+3 = 0$ as $Re = 0$ $\Rightarrow a = -3$

$4+a-b = 0$ as $Im = 0$ $\Rightarrow b = 1$ (3)

(c)



(i) $T\sin\theta = mr\omega^2$ (hor.)

$T\cos\theta = mg$ (vert)

$$\therefore \tan\theta = \frac{r\omega^2}{g}$$
$$\therefore \tan\theta = \frac{\sin\theta}{5} \times \frac{144\pi^2}{25} \times \frac{1}{10}$$
$$= \frac{72\pi^2 \sin\theta}{625} \text{ as req.}$$

(3)

$$r = 0.2\sin\theta$$
$$= \frac{\sin\theta}{5}$$
$$\omega = \frac{72 \times 2\pi}{60} = \frac{12\pi}{5} \text{ rad/s}$$

(ii) $\tan\theta = \dfrac{72\pi^2\sin\theta}{625}$ from (i)

$$\frac{\cancel{\sin\theta}}{\cos\theta} = \frac{72\pi^2\cancel{\sin\theta}}{625}$$

$$\therefore \cos\theta = \frac{625}{72\pi^2}$$

$$\theta = 28^\circ 25' \qquad (2)$$

(iii) from (i) $\tan\theta = \dfrac{r\omega^2}{g}$

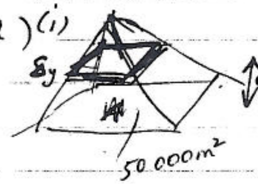
$$\frac{r}{h} = \frac{r\omega^2}{g}$$

$h = \dfrac{g}{\omega^2}$ as can be seen h depends on g a constant and ω which is the same as before $\therefore$ h remains the same $\therefore$ double mass $\Rightarrow$ no effect on θ.

(2)

Question 7 Solutions

(a)(i)



Since a square pyramid

If $A_1 : A_2$ then $l_1^2 : l_2^2$

$$\frac{A(y)}{A} = \frac{(h-y)^2}{h^2}$$

similar areas (all □'s are similar)

$$\therefore A(y) = 5\times10^4 \times \frac{(h-y)^2}{h^2} \qquad (1)$$

(ii) Volume $= \sum_0^h A(y)\,\delta y$ ✓ 1

$$= \int_0^h \frac{A(h-y)^2}{h^2}\,dy$$

$$= \left[-\frac{A}{3h^2}(h-y)^3\right]_0^h \quad ✓\ 1$$

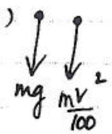
$$= -\frac{A}{3h^2}\left(0-h^3\right)$$

$$= \frac{Ah}{3} \quad ✓\ 1$$

$$= \frac{50\,000\times150}{3}$$

$$= 2\,500\,000\ m^3 \quad ✓\ 1$$

(b) (i)



note $m=1kg$

$$m\ddot{x} = -10m - \frac{mv^2}{100} \qquad (m = 1\ kg)$$

$$\therefore \ddot{x} = -\left(10+\frac{v^2}{100}\right)$$

$$v\frac{dv}{dx} = -\left(\frac{1000+v^2}{100}\right)$$

$$\frac{dv}{dx} = -\left(\frac{1000+v^2}{100v}\right)$$

$$\therefore \frac{dx}{dv} = -\frac{100v}{1000+v^2}$$

$\therefore x = -50\ln(1000+v^2)$

when $x=0$ $v=100$

$\therefore c = 50\ln 11000$

$\therefore x = 50\ln\left(\dfrac{11000}{1000+v^2}\right)$

when $v=0$ (highest pt)

$x = 50\ln 11$ (4)

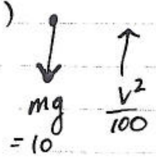
(ii) The downward velocity will less than 100 m/s (its original). For example consider a speargun being fired upwards in water from low in the ocean then after reaching its highest point, slowly drifting back downwards.

OR more everyday

A football being kicked from a boot directly upwards high velocity at impact, highest point zero velocity. person is able to catch it on return (lower velocity at return, otherwise unable to catch it.)

(2)

(iii)



$$m\ddot{x} = 10 - \frac{v^2}{100} \qquad (m=1)$$

$$\ddot{x} = \frac{1000-v^2}{100}$$

$$v\frac{dv}{dx} = \frac{1000-v^2}{100}$$

$$\frac{dv}{dx} = \frac{1000-v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{1000-v^2}$$

$$x = -50\ln(1000-v^2)+c \qquad (\text{at top } x=0\ v=0)$$

$$\therefore C = 50\ln 1000$$

$$x = 50\ln\frac{1000}{1000-v^2}$$

when $x = 50\ln 11$ (distance of return from highest pt)

$$11 = \frac{1000}{1000-v^2}$$

$$11\,000 - 11v^2 = 1000$$

$$10\,000 = 11v^2$$

$$v^2 = \frac{10\,000}{11} \qquad v = \frac{100}{\sqrt{11}}\text{ m/s} = \frac{100\sqrt{11}}{11}\text{ m/s.}$$

(4)

Q8. a) i)

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$

let $x = \tan\theta$ and $\tan 4\theta = \frac{1}{\sqrt{3}}$

$$\therefore \frac{4x-4x^3}{1-6x^2+x^4} = \frac{1}{\sqrt{3}}$$

$$\therefore x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$$

$$\tan 4\theta = \frac{1}{\sqrt{3}}$$

$$4\theta = n\pi + \frac{\pi}{6}$$

$$\theta = \frac{n\pi}{4} + \frac{\pi}{24}, \quad n = 0,1,2,3$$

$n=0$, $\theta = \frac{\pi}{24}$ so $x = \tan\frac{\pi}{24}$

$n=1$, $\theta = \frac{\pi}{4}+\frac{\pi}{24} = \frac{7\pi}{24}$, $x = \tan\frac{7\pi}{24}$

$n=2$, $\theta = \frac{\pi}{2}+\frac{\pi}{24} = \frac{13\pi}{24}$, $x = \tan\frac{13\pi}{24}$ (or $-\tan\frac{11\pi}{24}$)

$n=3$, $\theta = \frac{3\pi}{4}+\frac{\pi}{24} = \frac{19\pi}{24}$, $x = \tan\frac{19\pi}{24}$ (or $-\tan\frac{5\pi}{24}$)

ii) (1) from (i) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} - \tan\frac{5\pi}{24} - \tan\frac{11\pi}{24} = -4\sqrt{3}$ (sum of roots)

$$\therefore \tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$$

(2) $\left(\tan\frac{\pi}{24}\right)\left(\tan\frac{7\pi}{24}\right)\left(-\tan\frac{5\pi}{24}\right)\left(-\tan\frac{11\pi}{24}\right) = 1$ (product of roots)

$$\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \frac{1}{\tan\frac{7\pi}{24}\cdot\tan\frac{11\pi}{24}}$$

$$\therefore \tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$$

iii) let $\alpha = \tan\frac{\pi}{24}$

$$\frac{1}{\alpha^2} = \left(\cot\frac{\pi}{24}\right)^2$$

let $\frac{1}{\alpha^2} = x$ so $\alpha = \frac{1}{\sqrt{x}}$

$$\left(\frac{1}{\sqrt{x}}\right)^4 + 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right)^3 - 6\left(\frac{1}{\sqrt{x}}\right)^2 - 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right) + 1 = 0$$

$$\frac{1}{x^2} + \frac{4\sqrt{3}}{x\sqrt{x}} - \frac{6}{x} - \frac{4\sqrt{3}}{\sqrt{x}} + 1 = 0$$

$$x^2 - 4\sqrt{3}x\sqrt{x} - 6x + 4\sqrt{3}\sqrt{x} + 1 = 0$$

$$4\sqrt{3}\sqrt{x}(x-1) = x^2 - 6x + 1$$

$$48x(x-1)^2 = (x^2-6x+1)^2$$

$$48x^3 - 96x^2 + 48x = x^4 - 12x^3 + 2x^2 + 36x^2 - 12x + 1$$

ie $x^4 - 60x^3 + 134x^2 - 60x + 1 = 0$

18 b) i) $I_n = \int_0^1 x(x^2-1)^n\,dx$

$= \left[\frac{x^2}{2}(x^2-1)^n\right]_0^1 - \int_0^1 \frac{x^2}{2}\cdot n(x^2-1)^{n-1}\cdot 2x\,dx$

$= [\frac{1}{2}\times 0 - 0] - n\int_0^1 x^3(x^2-1)^{n-1}\,dx$

$= -n\int_0^1 \frac{x^3(x^2-1)^n}{x^2-1}\,dx$

$= -n\int_0^1 \left(x + \frac{x}{x^2-1}\right)(x^2-1)^n\,dx$

$= -n\left[\int_0^1 x(x^2-1)^n\,dx + \int_0^1 \frac{x}{x^2-1}(x^2-1)^n\,dx\right]$

$= -n\int_0^1 x(x^2-1)^n\,dx - n\int_0^1 x(x^2-1)^{n-1}\,dx$

$\therefore I_n = -nI_n - nI_{n-1}$

$(1+n)I_n = -nI_{n-1}$

$I_n = \frac{-n}{(n+1)} I_{n-1}$ for $n \geq 1$

ii) "Hence": $I_n = \frac{-n}{n+1} I_{n-1}$

$= \frac{-n}{n+1}\cdot\frac{-n+1}{n}\cdot\frac{-n+2}{n-1}\cdot\ldots\frac{-3}{4}\cdot\frac{-2}{3}\cdot\frac{-1}{2} I_0$

$I_0 = \int_0^1 x(x^2-1)^0\,dx$

$= \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}$

So $I_n = (-1)^n \frac{n}{n+1}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdots\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$= \frac{(-1)^n}{2(n+1)}$, $n \geq 0$

"Otherwise": $I_n = \int_0^1 x(x^2-1)^n\,dx$

$= \frac{1}{2}\int_0^1 2x(x^2-1)^n\,dx$

$= \frac{1}{2}\left[\frac{(x^2-1)^{n+1}}{n+1}\right]_0^1$

$= \frac{1}{2(n+1)}\left(0-(-1)^{n+1}\right)$

$= \frac{(-1)^n}{2(n+1)}$, $n \geq 0$

iii) $I_0 = \frac{1}{2}$, $I_1 = \frac{-1}{4}$, $I_2 = \frac{1}{6}$, $I_3 = \frac{-1}{8}$, $I_4 = \frac{1}{10}$, $I_5 = \frac{-1}{12}$

ie $I_{2n} > 0$ and $I_{2n+1} < 0$

so $I_{2n} > I_{2n+1}$

OR: $I_{2n} = \frac{(-1)^{2n}}{2(2n+1)}$

$= \frac{1}{2(2n+1)}$

> 0

$I_{2n+1} = \frac{(-1)^{2n+1}}{2((2n+1)+1)}$

$= \frac{-1}{4(n+1)}$

< 0

So $I_{2n} > I_{2n+1}$

iv) from (iii), if n is even, $I_n > I_{n+2}$ ie $I_2 = \frac{1}{6} > I_4 = \frac{1}{10} > I_6 = \frac{1}{14}$, etc

if n is odd, $I_1 = \frac{-1}{4} < I_3 = \frac{-1}{8} < I_5 = \frac{-1}{12}$

$\therefore I_n \not> I_{n+2}$ for all $n \geq 0$