HORNSBY GIRLS' HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- **Reading Time- 5 minutes**
- **Working Time – 3 hours**
- **Write using a black or blue pen**
- **Approved calculators may be used**
- **A table of standard integrals is provided at the back of this paper.**
- **All necessary working should be shown for every question.**
- **Begin each question on a fresh sheet of paper.**

Total marks (120)

- **Attempt Questions 1– 8**
- **All questions are of equal value**

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Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find

(i) $\int x^2 e^{-x}\,dx$ **3**

(ii) $\int \sqrt{9-x^2}\,dx$ **3**

(b) Evaluate, correct to 3 decimal places

(i) $\int_0^2 \sqrt{\frac{6-x}{6+x}}\,dx$ **3**

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$, using the substitution $t=\tan\frac{x}{2}$ **3**

(c) Find the real numbers A and B such that

$$\frac{5}{(x+3)(2x+1)} \equiv \frac{A}{x+3}+\frac{B}{2x+1}$$ **2**

Hence find $\int \frac{5}{(x+3)(2x+1)}dx$ **1**

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Let $z=2-5i$, find

(i) $\left|z-\bar{z}\right|$ **1**

(ii) $\arg\left(z-\bar{z}\right)$ **1**

(b) Find the square roots of , $1-4i\sqrt{3}$, in the form $a+ib$ **3**

(c) Sketch on the Argand diagram the locus of z given by:

(i) $2 \le |z+2-i| \le 4$ **1**

(ii) $\frac{-\pi}{2} \le \arg z < \frac{\pi}{6}$ **1**

(d) Simplify $i^{2000}+i^{2001}+i^{2002}$ **1**

(e) Simplify $\sqrt{1+i\sqrt{3}}+\sqrt{1-i\sqrt{3}}$ **1**

(f) Find the locus of z if $z=\frac{\omega+2i}{\omega-1}$ and $|\omega|=3$ **3**

(g) Express $z=\frac{\sqrt{2}}{1-i}$ in modulus-argument form and hence express z^6 in the form $x+iy$. **3**

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The equation of an ellipse is given by $3x^2 + 4y^2 = 12$.

(i) Find S and S' the foci of the ellipse **1**

(ii) Find the equations of the directrices M and M' **1**

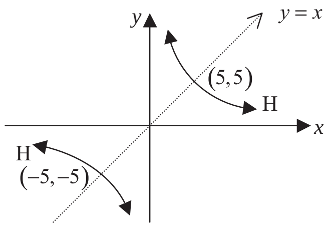
(iii) Sketch the ellipse showing foci, directrices and axial intercepts. **2**

(iv) Let P be any point on the ellipse. **2**

Show $SP + SP' = 4$

(v) Find the equation of the chord of contact from $(3,2)$. **1**

(b) The Hyperbola H drawn below has equation $xy = 25$



(i) $P\left(5p, \frac{5}{p}\right)$ and $Q\left(5q, \frac{5}{q}\right)$, where P and Q are two distinct arbitrary points on H, p and q are both greater than zero. Find the equation of the chord PQ. **2**

(ii) Prove the equation of the tangent at P is $x + p^2 y = 10p$ **2**

(iii) The tangents at P and Q intersect at T. Find the co-ordinates of T **2**

(iv) The chord PQ produced passes through the point $N(0,10)$. Find and describe the locus of T. **2**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) If $f(x) = e^x - 2$ draw large (half page) separate, neat and accurate sketches of:

(i) $y = -f(x)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $|y| = f(x)$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = \dfrac{1}{f(x)}$ **3**

(b) If $g(x) = \dfrac{1}{f(x)}$, find $g'(x)$ in terms of $f(x)$ and $f'(x)$ and deduce that the x-coordinates of the stationary points of $y = g(x)$ are the x-coordinates of the stationary points of $y = f(x)$ for which $f(x)$ is non-zero. **2**

(c) Find the set of values of x for which the limiting sum of the following series exists **2**

$$1 + \left(\frac{2x-3}{x+1}\right) + \left(\frac{2x-3}{x+1}\right)^2 + \left(\frac{2x-3}{x+1}\right)^3 + \ldots\ldots$$

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) A curve in a railway track is a sector of a circle of radius 200 metres. Trains rounding the curve always travel at 80 km/h. If the tracks are 1.82 metres apart, how many centimetres higher must the outer rail be than the inner rail if there is to be no lateral pressure on the rails? (Use g = 10 m/s^2) **3**

(b) A fluid provides a resistive force of $5mv^2$ Newtons to a body of mass m *kg* falling vertically through it at a velocity of *v m/s*. The body is released from rest at a point *O*. (Use g = 10 m/s^2)

(i) Show that, if the body has fallen x metres then, $v\dfrac{dv}{dx} = g - 5v^2$ **1**

(ii) Find the terminal velocity of the body. **1**

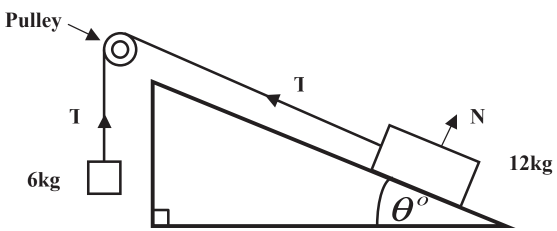
(iii) How far will the body have fallen from *O* when its velocity is 90% of the terminal velocity? **3**

(iv) How long will it take for the velocity to equal 90% of the terminal velocity? **3**

(c) A point *P* moves on the circle $x^2 + y^2 = 9$ with uniform angular velocity $\dfrac{\pi}{2}$ rad/s. **2**

Show that its angular velocity about the point *Q*(3, 0) is $\dfrac{\pi}{4}$ rad/s.

(d) Two masses, 12kg and 6kg are connected by a light inelastic string passing over a smooth pulley, as shown in the diagram. The 12kg mass rests on a smooth plane which makes an angle of θ^0 with the horizontal. The 6kg mass hangs in the air. The tension in the string is *T* and *N* is the normal reaction of the plane on the 12kg mass. If the masses are stationary, find the size of θ. **2**



Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find the equation whose roots are the square of those in **2**

$$x^3 + px^2 + qx + r = 0 .$$

(b) $2x^3 - 9x^2 + 12x + k = 0$ has two equal roots.

(i) Show that $k = -4$ is a possible value for k **2**

(ii) Solve $2x^3 - 9x^2 + 12x - 4 = 0$. **2**

(c) α, β and γ are the roots of $x^3 + px^2 + qx + r = 0$.

Find:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(iii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(d) The function $f(x)$ is given by $f(x) = \dfrac{4(2x-7)}{(x-3)(x+1)}$.

(i) express $f(x)$ in partial fractions **2**

(ii) show there are turning points at (2, 4) and (5, 1) (do not investigate their nature) **1**

(iii) sketch $y = f(x)$ showing x and y intercepts, turning points and asymptotes. **2**

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The area bounded by the curve $y = 2x - x^2$ and the x-axis is rotated about the. line $x = 1$.

(i) Show that the volume of the solid of revolution is given by **1**

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y$$

(ii) Hence find the volume of the solid of revolution **2**

(b) The area bounded by $y = \dfrac{1}{x^2+4}$ and the x-axis between $x = -2$ and $x = 2$ is rotated about the line x = -4. Use the technique of cylindrical shells to find the volume generated. **3**

(c) The base of a solid is an ellipse with equation $4x^2 + 9y^2 = 36$. **3**
Sections of the solid perpendicular to the major axis of the ellipse are sectors of a circle with the angle at the centre being 60^0 and whose radius lies on the plane of the base.
Find the volume of the solid.

(d) A truck of mass 5 tonnes travels around a curve of radius 100 metres which is banked at 8^0 at a constant speed of 45 km/h. Calculate the frictional force between the road surface and the wheels of the truck. (Use g = 10 m/s^2) **2**

(e) If $z = \cos\theta + i\sin\theta$ and using the expansion of $\left(z - \dfrac{1}{z}\right)^4$,

(i) show that $\sin^4\theta = \dfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$. **2**

(ii) Hence show that $\displaystyle\int_0^9 \frac{x\sqrt{x}}{\sqrt{9-x}}\,dx = \frac{243\pi}{8}$ by using the substitution $x = 9\sin^2\theta$. **2**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Let $I_n = \displaystyle\int \frac{x^n dx}{\sqrt{x^2-a^2}}$, show that $nI_n - (n-1)a^2 I_{n-2} = x^{n-1}\sqrt{x^2-a^2}$. **5**

Hence, evaluate $\displaystyle\int_{2a}^{3a} \frac{x^4 dx}{\sqrt{x^2-a^2}}$

(b) An open cone stands on its vertex. At a point O vertically above its vertex a particle of mass m kg is hung by a light string of length l so that when the particle describes a horizontal circle, touching the inner surface of the cone, with an angular velocity of ω, the angle α made by the string and the vertical is equal to the semi-vertical angle of the cone.
Assume there is no friction on the cone's inner surface.

(i) Show that the normal reaction by the cone's surface on the particle is given by **3**
$\dfrac{1}{2} m \tan 2\alpha (l\omega^2 - g\sec\alpha)$

(ii) Show that the tension in the string, so long as the mass remains in contact with the inner surface of the cone, is given by **3**
$\dfrac{m(g\cos\alpha - l\omega^2 \sin^2\alpha)}{\cos 2\alpha}$

(iii)Hence, find the condition for ω, if the particle is not to lose contact with the cone. **2**

(c) If n is an integer, prove that: **2**
$\left(1 + \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}\right)^n = -2^n \cos^n \dfrac{\pi}{n}$

End of paper

EXT 2 TRIAL 2009 SOLUTIONS

Q1 a) i) let $u = x^2$ $dv = e^{-x}dx$

$du = 2x\,dx$ $v = -e^{-x}$

$\therefore I = -x^2e^{-x} + \int 2xe^{-x}dx$

let $u = 2x$ $dv = e^{-x}dx$

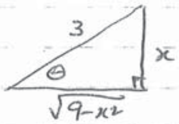
$du = 2\,dx$ $v = -e^{-x}$

$\therefore I = -x^2e^{-x} + (-2xe^{-x} + \int 2e^{-x}dx)$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$

ii) let $x = 3\sin\theta$ $\therefore \theta = \sin^{-1}(x/3)$

$dx = 3\cos\theta\, d\theta$



$\therefore I = \int \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta$

$= \int 9\cos^2\theta\, d\theta$

$= \frac{9}{2}\int 1+\cos 2\theta\; d\theta$

$= \frac{9\theta}{2} + \frac{9\sin 2\theta}{4}$

$= \frac{9\theta}{2} + \frac{9\sin\theta\cos\theta}{2}$

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2}$

b) i) $I = \int_0^2 \sqrt{\frac{6-x}{6+x}}\, dx$

$= \int_0^2 \frac{6-x}{\sqrt{36-x^2}}\, dx$

$= \int_0^2 \frac{6}{\sqrt{36-x^2}}\, dx + \frac{1}{2}\int_0^2 \frac{-2x}{\sqrt{36-x^2}}$

$= \left[6\sin^{-1}\left(\frac{x}{6}\right) + \sqrt{36-x^2}\right]_0^2$

$= 1.696$

ii) $I = \int_0^{\pi/2} \frac{dx}{1+\sin x}$

let $t = \tan\frac{x}{2}$ when $x=0$ $t=0$

$x = \pi/2$ $t = 1$

$\sin x = \frac{2t}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$\therefore I = \int_0^1 \frac{2dt}{1+t^2} \times \frac{1+t^2}{1+t^2+2t}$

$= \int_0^1 2(t+1)^{-2}\, dt$

$= \left[-2(t+1)^{-1}\right]_0^1$

$= 1$

c) $\frac{5}{(x+3)(2x+1)} \equiv \frac{A}{x+3} + \frac{B}{2x+1}$

$5 \equiv A(2x+1) + B(x+3)$

when $x = -3$, $A = -1$

$x = -\frac{1}{2}$, $B = 2$

$\therefore \int \frac{5}{(x+3)(2x+1)}\, dx = \int \frac{-dx}{x+3} + \int \frac{2}{2x+1}\, dx$

$= \ln\left(\frac{2x+1}{x+3}\right)$

Q2 a) i) $z - \bar{z} = -10i$

$|z-\bar{z}| = 10$

ii) $\arg(z-\bar{z}) = -\pi/2$

b) let $(a+ib)^2 = 1 - 4i\sqrt{3}$

$a^2 - b^2 + 2abi = 1 - 4i\sqrt{3}$

$\therefore a^2 - b^2 = 1$

$ab = -2\sqrt{3}$

$\therefore a = \frac{-2\sqrt{3}}{b}$

$\therefore \frac{12}{b^2} - b^2 = 1$

$b^4 + b^2 - 12 = 0$

$(b^2+4)(b^2-3) = 0$

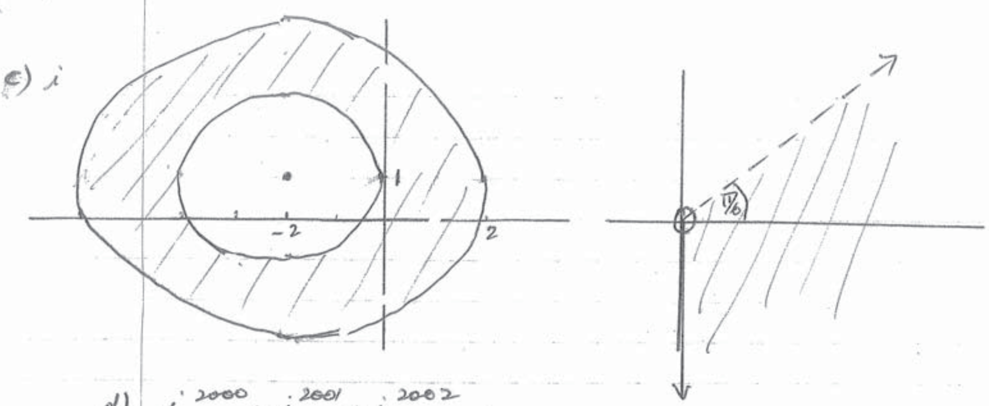
$\therefore b = \pm\sqrt{3}$

$a = \pm 2$

$\therefore$ roots are: $-2 + i\sqrt{3}$

$2 - i\sqrt{3}$

c) i



d) $i^{2000} + i^{2001} + i^{2002}$

$= i^{2000}(1 + i - 1)$

now $i^0 = 1$, $i^1 = 1$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$\therefore i^{2000} = 1$

$\therefore$ answer $= i$

e) $\left(\sqrt{1+i\sqrt3} + \sqrt{1-i\sqrt3}\right)^2 = 1 + i\sqrt3 + 2\sqrt{1+3} + 1 - i\sqrt3$

$= 6$

$\therefore \sqrt{1+i\sqrt3} + \sqrt{1-i\sqrt3} = \sqrt6$

f) $z = \dfrac{\omega + 2i}{\omega - 1}$

$z\omega - \omega = z + 2i$

$\omega(z-1) = z + 2i$

$3|z-1| = |z+2i|$

$3\sqrt{(x-1)^2 + y^2} = \sqrt{x^2 + (y+2)^2}$

$(x - \frac{9}{8})^2 + (y - \frac14)^2 = \frac{45}{64}$

g) $z = \dfrac{\sqrt2}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{\sqrt2 + i\sqrt2}{2}$

$|z| = \sqrt{\frac12 + \frac12}$

$= 1$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$\therefore z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$z^6 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$= \cos -\frac{\pi}{2} + i\sin -\frac{\pi}{2}$

$= -i$

Question 3

(a) $3x^2 + 4y^2 = 12 \Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$ $a^2 = 4$, $b^2 = 3$; $a = 2$, $b = \sqrt3$

$b^2 = a^2(1 - e^2)$

$3 = 4 - 4e^2$

$-1 = -4e^2$

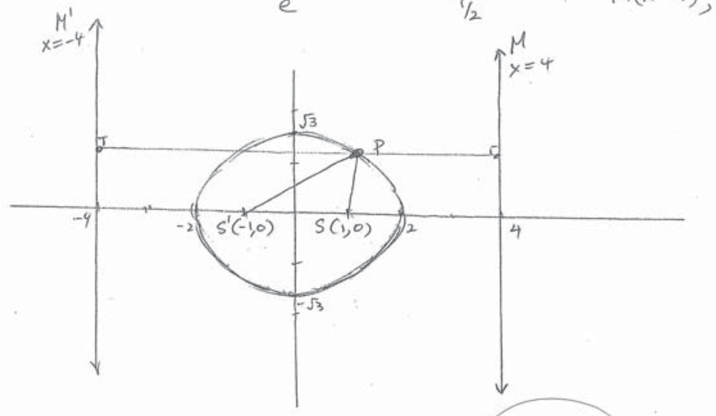
$e^2 = \frac14$

$e = \frac12$

(i) (α) Foci $(\pm ae, 0) = (\pm1, 0)$ $\therefore S(1,0)$ and $S'(-1,0)$ 1

(β) Directrices $x = \pm\dfrac{a}{e}$ $x = \pm\dfrac{2}{\frac12} = \pm4$ $M(x=4)$, $M'(x=-4)$ 1

(ii)



2

(iii) Using the focus-directrix definition of ellipse

$PS = ePM$ and $PS' = ePM'$

$\therefore PS + PS' = e(PM + PM')$

$= e \times 8$

$= \frac12 \times 8$

$= 4$ as required 2

(iv) The chord of contact has form $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$

where (x_0, y_0) is $(3,2)$

$\therefore \dfrac{3x}{4} + \dfrac{2y}{3} = 1$

OR $9x + 8y = 12$ 1

OR $9x + 8y - 12 = 0$

3(b) (i) $\dfrac{y-\frac{5}{p}}{x-5p} = \dfrac{\frac{5}{q}-\frac{5}{p}}{5q-5p}$ $\qquad p \neq q$

$$\frac{py-5}{p} \times 5(q-p) = (x-5p) \times \frac{5}{pq}(p-q)$$

$$5-py = \frac{x-5p}{q}$$

$$5q-pqy = x-5p$$

$$x+pqy-5(p+q)=0 \qquad 2$$

(ii) If $xy=25$ then $y=\frac{25}{x}$

$$\therefore \frac{dy}{dx} = \frac{-25}{x^2}$$

when $x=5p$

$$\frac{dy}{dx} = \frac{-1}{p^2}$$

$\therefore$ tangent is $y-\frac{5}{p} = \frac{-1}{p^2}(x-5p)$

$$p^2y-5p = -x+5p$$

$$x+p^2y = 10p \quad \text{as required} \qquad 2$$

(iii) $x+p^2y=10p$ —— ① $\Rightarrow x = 10p-p^2y$

$x+q^2y=10q$ —— ② $\Rightarrow x = 10q-q^2y$

$$\therefore 10p-p^2y = 10q-q^2y$$

$$q^2y-p^2y = 10q-10p$$

$$y(q^2-p^2) = 10(q-p)$$

$$y = \frac{10}{p+q}$$

$\therefore T\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$ 2

$$\therefore x = 10p-p^2 \cdot \frac{10}{p+q}$$

$$= \frac{10p^2+10pq-10p^2}{p+q} = \frac{10pq}{p+q}$$

(iv) $N(0,10)$ lies on PQ {Chord in (i)}

$$\therefore 10pq = 5(p+q)$$

$pq = \frac{p+q}{2}$ (relationship between p and q)

So $X = \dfrac{10pq}{p+q}$

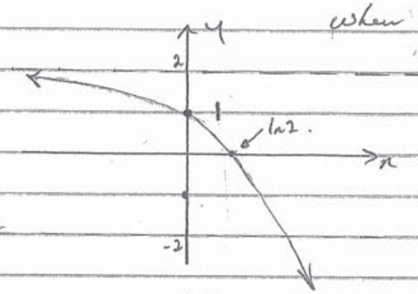
$$= \frac{10\left(\frac{p+q}{2}\right)}{p+q}$$

$$= 5$$

$\therefore$ locus is a straight vertical line passing through $x=5$. 2

Q4/

i)



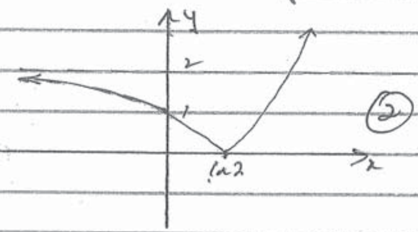
when $f(x) = 0$ $\quad e^x = 2$

$x = \ln 2$

$= 0.69$

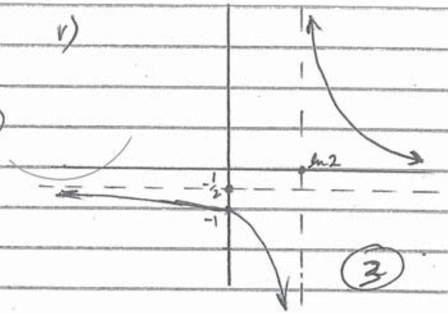
(2)

ii)



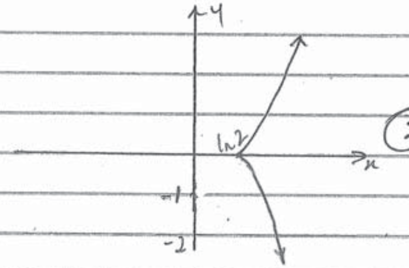
(2)

v)



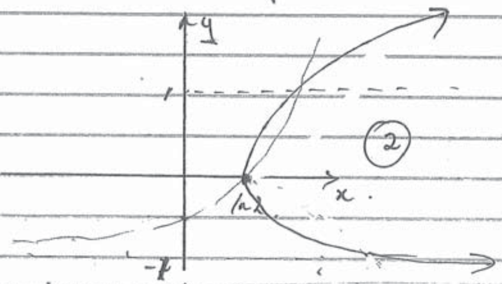
(3)

iii)



(2)

iv)



(2)

b) $g(x) = \dfrac{1}{f(x)}$

$= [f(x)]^{-1}$

$g'(x) = -[f(x)]^{-2} \cdot f'(x)$

$= \dfrac{-f'(x)}{(f(x))^2}$ (1)

Stat pts of $y = g(x)$ when $g'(x) = 0$

i.e. $f'(x) = 0$.

Stat pts of $y = f(x)$ when $f'(x) = 0$

$\therefore$ same pts (1)

but $g'(x)$ does not exist when $f(x) = 0$

c) $|v| < 1$

$\dfrac{|2x-3|}{|x+1|} < 1$

$|2x-3| < |x+1|$

when $x < -1$

$-2x+3 < -x-1$

$-x < -4$

$x > 4$

no soln.

when $-1 \leq x < 1\frac{1}{2}$

$-2x+3 < x+1$

$-3x < -2$

$x > \frac{2}{3}$.

$\therefore \frac{2}{3} < x < \frac{3}{2}$

when $x \geq 1\frac{1}{2}$.

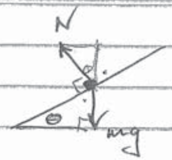
$2x-3 < x+1$

$x < 4$

$\therefore 1\frac{1}{2} \leq x < 4$

$\therefore \frac{2}{3} < x < 4$. (2)

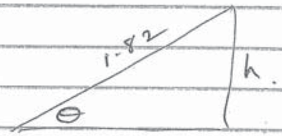
Q5/

a) 

$80 \text{ km/h} = \frac{80000}{3600} = \frac{200}{9} \text{ m/s}$

$N\cos\theta = mg$

$N\sin\theta = \frac{mv^2}{r}$



$\tan\theta = \frac{v^2}{rg}$ 1

$= \frac{200^2}{9^2 \times 200 \times 10}$

$= 0.2469$

$\theta = 0.2420717$ rad. 1

$\sin(0.2420717) = \frac{h}{1.82}$

$h = 0.43628$ m

$= 43.6$ cm

$= 44$ cm. 1 3

b) i) $F = ma = mg - 5mv^2$

$a = g - 5v^2$

$v\frac{dv}{dx} = g - 5v^2$ 1

ii) Terminal Velocity when $5mv^2 = mg$

$(a=0)$

$v^2 = g/5$

$v^2 = 2$ 1

$V = \sqrt{2}$ or 1.414 m/s.

(iii) $\frac{dv}{dx} = \frac{g - 5v}{v}$

$= \frac{g - 5v^2}{v}$

$\therefore \frac{dx}{dv} = \frac{v}{g - 5v^2}$

$\therefore x = -\frac{1}{10}\ln(g - 5v^2) + c$

$x=0$
$v=0$
$\therefore x = -\frac{1}{10}\ln g + c$
$\therefore c = \frac{1}{10}\ln g$

$\therefore x = \frac{1}{10}\ln g - \frac{1}{10}\ln(g - 5v^2)$

$x = \frac{1}{10}\ln\left(\frac{g}{g - 5v^2}\right)$

when $V = \frac{9\sqrt{2}}{10}$ $x = \frac{1}{10}\ln\left(\frac{10}{10 - 5\times\frac{162}{100}}\right)$

$= \frac{1}{10}\ln\left(\frac{1000}{1000 - 810}\right)$

$= \frac{1}{10}\ln\left(\frac{100\cancel{0}}{19\cancel{0}}\right)$

$= \frac{1}{10}\ln\left(\frac{100}{19}\right)$

$= 0.17$ m (2dp) (3)

(iv) $\frac{dv}{dt} = g - 5v^2$

$\frac{dt}{dv} = \frac{1}{g - 5v^2}$

$= \frac{1}{5(2 - v^2)}$

let $\frac{1}{2 - v^2} = \frac{A}{\sqrt{2} - v} + \frac{B}{\sqrt{2} + v}$

$1 = A(\sqrt{2}+v) + B(\sqrt{2}-v)$

when $V = \sqrt{2}$ $1 = 2A\sqrt{2}$

$A = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

sim $B = \frac{\sqrt{2}}{4}$

$\therefore t = \frac{\sqrt{2}}{20}\int \frac{1}{\sqrt{2} - v} + \frac{1}{\sqrt{2} + v}\, dv$

$= \frac{\sqrt{2}}{20}\left[-\ln(\sqrt{2} - v) + \ln(\sqrt{2} + v)\right] + c$

when $t=0$ $V=0$ $\therefore c = 0$

$= \frac{\sqrt{2}}{20}\left[\ln\left(\frac{\sqrt{2}+v}{\sqrt{2}-v}\right)\right]$

now when $V = \frac{9\sqrt{2}}{10}$

$$t = \frac{\sqrt{2}}{20} \ln\left(\frac{\sqrt{2} + \frac{9\sqrt{2}}{10}}{\sqrt{2} - \frac{9\sqrt{2}}{10}}\right)$$

$$= \frac{\sqrt{2}}{20} \ln\left(\frac{\frac{10\sqrt{2}+9\sqrt{2}}{10}}{\frac{10\sqrt{2}-9\sqrt{2}}{10}}\right)$$

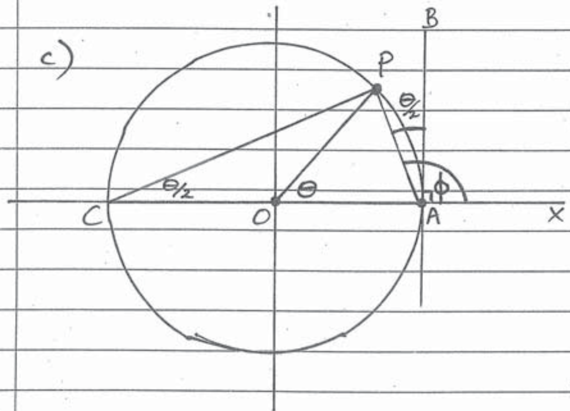
$$= \frac{\sqrt{2}}{20} \ln\left(\frac{19\sqrt{2}}{\sqrt{2}}\right)$$

$$= \frac{\sqrt{2}}{20} \ln 19$$

(3)

$$= 0.208\ldots \text{ seconds}$$

c)



given $\dot{\theta} = \frac{\pi}{2}$

$\angle OCP = \frac{1}{2} \angle AOP$ ($\angle$ at centre twice $\angle$ at the circumferen

$= \frac{\theta}{2}$

$\angle PAB = \frac{\theta}{2}$ ($\angle$ in alternate seg

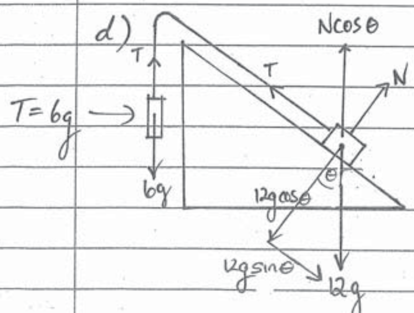
$\therefore \angle PAX = \phi = \frac{\pi}{2} + \frac{\theta}{2}$

$\therefore \dot{\phi} = \frac{\dot{\theta}}{2}$

$= \frac{\frac{\pi}{2}}{2}$ (2

$= \frac{\pi}{4}$ as requirec

d)



$N = 12g\cos\theta$ —— (1)

$T = 6g$ —— (2)

$T = 12g\sin\theta$ —— (3)

(2) into (3) $6g = 12g\sin\theta$

$\therefore \sin\theta = \frac{1}{2}$ $\quad \therefore \theta = 30° = \frac{\pi}{6}^c$ (2

Solutions Q6

(a) let $x^2 = y$ $\quad \therefore x = \sqrt{y}$

$\therefore (\sqrt{y})^3 + p(\sqrt{y})^2 + q(\sqrt{y}) + r = 0$ is the form of the required equation.

$y\sqrt{y} + yp + q\sqrt{y} + r = 0$ (not yet a polynomial

$\therefore \sqrt{y}(y+q) = -py - r$ (2) but OK as you were asked for the equation

$y(y+q)^2 = (py+r)^2$

$y(y^2 + 2qy + q^2) = p^2y^2 + 2pry + r^2$

$y^3 + 2qy^2 + q^2y - p^2y^2 - 2pry - r^2 = 0$

OR $y^3 + y^2(2q - p^2) + y(q^2 - 2pr) - r^2 = 0$ } These are in polynomial form

(b) let $P(x) = 2x^3 - 9x^2 + 12x + k$

(i) $\therefore P'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

$\therefore x = 1$ and $x = 2$ are two solutions that could be double roots

if $x = 1$ $\quad 2(1)^3 - 9(1)^2 + 12(1) + k = 0$

$2 - 9 + 12 + k = 0$

$\therefore k = -5$

if $x = 2$ $\quad 2(2)^3 - 9(2)^2 + 12(2) + k = 0$

$16 - 36 + 24 + k = 0$ (2)

$\therefore k = -4$ as required

(ii) $2x^3 - 9x^2 + 12x - 4 = (x-2)^2(ax+b)$

$= (x^2 - 4x + 4)(ax + b)$

$\therefore a = 2$ and $b = -1$ by inspection

$\therefore P(x) = (x-2)^2(2x-1)$ $\quad \therefore x = 2$ or $\frac{1}{2}$ (2)

(c) If α, β and γ are the roots $x^3+px^2+qx+r=0$

(i) then $\alpha+\beta+\gamma = -p$ 1

(ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-p)^2 - 2q$

$= p^2 - 2q$ 1

(iii) $\alpha^3+\beta^3+\gamma^3$: rewrite $x^3 = -px^2 - qx - r$

$\therefore \alpha^3 = -p\alpha^2 - q\alpha - r$

$\beta^3 = -p\beta^2 - q\beta - r$

$\gamma^3 = -p\gamma^2 - q\gamma - r$

$\therefore \alpha^3+\beta^3+\gamma^3 = -p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma) - 3r$

$= -p(p^2-2q) - q(-p) - 3r$

$= -p^3 + 3pq - 3r$ 2

(d) $f(x) = \dfrac{4(2x-7)}{(x-3)(x+1)}$

(i) $\dfrac{8x-28}{(x-3)(x+1)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x+1)}$

$8x-28 = A(x+1) + B(x-3)$

$x=-1$ $8(-1)-28 = B(-4)$

$-36 = -4B$

$B = 9$

$x=3$ $8(3)-28 = 4A$

$-4 = 4A$

$A = -1$

$\therefore \dfrac{4(2x-7)}{(x-3)(x+1)} = \dfrac{-1}{x-3} + \dfrac{9}{x+1}$ 2

(ii) $f'(x) = \dfrac{1}{(x-3)^2} - \dfrac{9}{(x+1)^2}$ 1

$= 0$ when $(x+1)^2 = 9(x-3)^2$

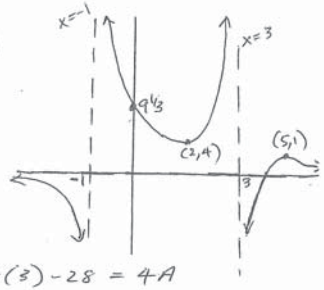
$x^2+2x+1 = 9x^2-54x+81$

$8x^2-56x+80=0$

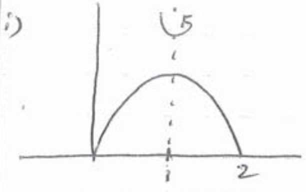
$x^2-7x+10=0$

$(x-2)(x-5)=0$ $\therefore x = 2, 5$
$y = 4, 1$

(iii)



Q7 a) i)



$\Delta V = \pi(1-x)^2 \Delta y$ or $\pi(x-1)^2\Delta y$

$y = 2x - x^2$

$(1-x)^2 = 1-2x+x^2$

$= 1-(2x-x^2)$

$= 1-y$

$\therefore \Delta V = \pi(1-y)\Delta y$ when $x=1$, $y=1$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y$

ii) $V = \pi\int_0^1 (1-y)\,dy$

$= \left[\pi\left(y - \frac{y^2}{2}\right)\right]_0^1$

$= \frac{\pi}{2}$

b) $\Delta V = 2\pi(4+x)\dfrac{1}{x^2+4}\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{-2}^{2} \Delta V$

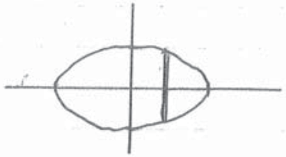
$= 2\pi \int_{-2}^{2} \dfrac{4+x}{x^2+4}\,dx$

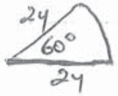
$= \int_{-2}^{2} \dfrac{8}{x^2+4} + \dfrac{2x}{x^2+4}\,dx$

$= \pi\left[4\tan^{-1}\left(\frac{x}{2}\right) + \ln(x^2+4)\right]_{-2}^{2}$

$= 2\pi^2$

c)





$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$y^2 = \frac{36-4x^2}{9}$

$\Delta V = \frac{1}{2}(2y)^2 \cdot \frac{\pi}{3} \;\Delta x$

$= \frac{2y^2\pi}{3} \cdot \Delta x$

$= \frac{2\pi}{3}\left(\frac{36-4x^2}{9}\right) \;\Delta x$

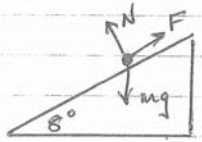
$V = \lim_{\Delta x \to 0} \sum_{-3}^{3} \Delta V$

$= \frac{2\pi}{27} \int_{-3}^{3} 36 - 4x^2 \; dx$

$= \frac{2\pi}{27}\left[36x - \frac{4x^3}{3}\right]_{-3}^{3}$

$= \frac{32\pi}{3}$

d)



$45 \text{ km/h} = 12.5 \text{ m/s}$

$N\cos 8° + F\sin 8° = mg$ —— ①

$N\sin 8° - F\cos 8° = \frac{mv^2}{r}$ —— ②

$N\sin 8°\cos 8° + F\sin^2 8° = mg\sin 8°$ —— ③

$N\sin 8°\cos 8° - F\cos^2 8° = \frac{mv^2}{r}\cos 8°$ —— ④

③ − ④ $\Rightarrow F = mg\sin 8° - \frac{mv^2}{r}\cos 8°$

$= 778 \text{ N}$

OR $F + mg\sin\theta = \frac{mv^2}{r}\cos\theta$

$\therefore F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

$= 778 \text{ N}$

e) i) $\left(z - \frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$

now $z^4 + \frac{1}{z^4} = \cos 4\theta + i\sin 4\theta + \cos 4\theta - i\sin 4\theta$

$= 2\cos 4\theta$

$\therefore z^2 + \frac{1}{z^2} = 2\cos 2\theta$

$\therefore \left(z - \frac{1}{z}\right)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$

now $\left(z - \frac{1}{z}\right)^4 = (\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)^4$

$= (2i\sin\theta)^4$

$= 16\sin^4\theta$

$\therefore 16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

ii) $I = \int_0^9 \frac{x\sqrt{x}}{\sqrt{9-x}}\, dx$ when $x = 0,\ \theta = 0$; $x = 9,\ \theta = \frac{\pi}{2}$

Let $x = 9\sin^2\theta$

$dx = 18\sin\theta\cos\theta \, d\theta$

$I = \int_0^{\pi/2} \frac{9\sin^2\theta \cdot \cancel{3}\sin\theta \cdot 18\sin\theta\cancel{\cos\theta}\, d\theta}{\cancel{3}\cancel{\cos\theta}}$

$= \int_0^{\pi/2} 162\sin^4\theta \, d\theta$

$= \frac{81}{4}\int_0^{\pi/2} \cos 4\theta - 4\cos 2\theta + 3 \;\; d\theta$

$= \frac{81}{4}\left[\frac{\sin 4\theta}{4} - 2\sin 2\theta + 3\theta\right]_0^{\pi/2}$

$= \frac{81}{4}\left(0 - 0 + \frac{3\pi}{2} - 0 + 0 - 0\right)$

$= \frac{243\pi}{8}$

Q8

$$I_n = \int \frac{x^n\,dx}{\sqrt{x^2-a^2}}$$

Let $u = x^{n-1}$ $\qquad dv = \frac{x}{\sqrt{x^2-a^2}}$

$du = (n-1)x^{n-2}\,dx$ $\qquad v = \sqrt{x^2-a^2}$

$$I_n = x^{n-1}\sqrt{x^2-a^2} - (n-1)\int \frac{x^{n-2}(x^2-a^2)}{\sqrt{x^2-a^2}}\,dx$$

$$= x^{n-1}\sqrt{x^2-a^2} - (n-1)\int \frac{x^n}{\sqrt{x^2-a^2}} - \frac{a^2x^{n-2}}{\sqrt{x^2-a^2}}\,dx$$

$$= x^{n-1}\sqrt{x^2-a^2} - (n-1)I_n + a^2(n-1)I_{n-2}$$

$$I_n + nI_n - I_n - (n-1)a^2 I_{n-2} = x^{n-1}\sqrt{x^2-a^2}$$

$$\therefore\ nI_n - (n-1)a^2 I_{n-2} = x^{n-1}\sqrt{x^2-a^2}$$

$\therefore$ when $n=4$, $\quad 4I_4 - 3a^2 I_2 = \left[x^3\sqrt{x^2-a^2}\right]_{2a}^{3a}$

$$= 27a^3\sqrt{8a^2} - 8a^3\sqrt{3a^2}$$

when $n=2$, $\quad 2I_2 - a^2 I_0 = \left[x\sqrt{x^2-a^2}\right]_{2a}^{3a}$

$$= 3a\sqrt{8a^2} - 2a\sqrt{3a^2}$$

$$I_0 = \int_{2a}^{3a} \frac{dx}{\sqrt{x^2-a^2}} = \left[\ln\left(x+\sqrt{x^2-a^2}\right)\right]_{2a}^{3a}$$

$$= \ln\left(3a+\sqrt{8a^2}\right) - \ln\left(2a+\sqrt{3a^2}\right)$$

$$= \ln\left(\frac{a(3+\sqrt{8})}{a(2+\sqrt{3})}\right)$$

$$= \ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)$$

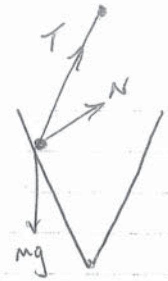
$$\therefore\ 2I_2 = 6a^2\sqrt{2} - 2a^2\sqrt{3} + a^2\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)$$

$$I_2 = 3a^2\sqrt{2} - a^2\sqrt{3} + \frac{a^2}{2}\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)$$

$$\therefore\ 4I_4 = 54a^4\sqrt{2} - 8a^4\sqrt{3} + 3a^2\left(3a^2\sqrt{2} - a^2\sqrt{3} + \frac{a^2}{2}\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)\right)$$

$$I_4 = \frac{27a^4\sqrt{2}}{2} - 2a^4\sqrt{3} + \frac{9a^4\sqrt{2}}{4} - \frac{3a^4\sqrt{3}}{4} + \frac{3a^4}{8}\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right)$$

b) i)



$$\sin\alpha = \frac{r}{l}$$

$$r = l\sin\alpha.$$

$T\cos\alpha + N\sin\alpha = mg$ —— ①

$T\sin\alpha + N\cos\alpha = m\omega^2 r = ml\omega^2\sin\alpha$ —— ②

① × $\sin\alpha \Rightarrow T\sin\alpha\cos\alpha + N\sin^2\alpha = mg\sin\alpha$ —— ③

② × $\cos\alpha \Rightarrow T\sin\alpha\cos\alpha + N\cos^2\alpha = ml\omega^2\sin\alpha\cos\alpha$ —— ④

③ − ④ $\Rightarrow N(\sin^2\alpha - \cos^2\alpha) = m(g\sin\alpha - l\omega^2\sin\alpha\cos\alpha)$

$$N = \frac{m\sin\alpha\cos\alpha(g\sec\alpha - l\omega^2)}{-\cos 2\alpha}$$

$$= \frac{\frac{1}{2}m\sin 2\alpha(l\omega^2 - g\sec\alpha)}{\cos 2\alpha}$$

$$= \frac{1}{2}m\tan 2\alpha(l\omega^2 - g\sec\alpha)$$

ii) ① × $\cos\alpha$ − ② × $\sin\alpha \Rightarrow T(\cos^2\alpha - \sin^2\alpha) = mg\cos\alpha - ml\omega^2\sin^2\alpha$

$$T = \frac{m(g\cos\alpha - l\omega^2\sin^2\alpha)}{\cos 2\alpha}$$

iii) For string to remain taut $T \geq 0$ & $N \geq 0$

$\therefore g\cos\alpha - l\omega^2\sin^2\alpha \geq 0$

$$\omega^2 \leq \frac{g\cos\alpha}{l\sin^2\alpha}$$

and $l\omega^2 - g\sec\alpha \geq 0$

$$\omega^2 \geq \frac{g\sec\alpha}{l}$$

$$\therefore \sqrt{\frac{g\sec\alpha}{l}} \leq \omega \leq \operatorname{cosec}\alpha\sqrt{\frac{g\cos\alpha}{l}}$$

c) $\left(1 + \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n$

$= \left(1 + 2\cos^2\frac{\pi}{n} - 1 + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)^n$

$= \left(2\cos^2\frac{\pi}{n} + 2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)^n$

$= 2^n\cos^n\frac{\pi}{n}\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^n$

$= 2^n\cos^n\frac{\pi}{n}\left(\cos\pi + i\sin\pi\right)$

$= -2^n\cos^n\frac{\pi}{n}$