HORNSBY GIRLS' HIGH SCHOOL

2010
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (120)

- Attempt Questions 1– 8
- All questions are of equal value

Total Marks – 120

Attempt Questions 1-8
All Questions are of equal value

Begin each question on a NEW SHEET of paper, writing your student number and question number at the top of the page. Extra paper is available.

Question 1 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Find

(i) $\displaystyle\int \frac{dx}{x^2+2x+2}$ 2

(ii) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$ 2

(iii) $\displaystyle\int \frac{x^3+1}{x^2+1}\,dx$ 3

(iv) $\displaystyle\int \frac{dx}{(x-2)(x+1)}$ 2

(v) $\displaystyle\int \sin^3 x\,dx$ 2

(b) Given that $\int \sec x dx = \ln(\sec x + \tan x)$, evaluate $\displaystyle\int_0^2 \sqrt{x^2+4}\,dx$. 4

Give your answer as a decimal correct to 4 decimal places.

Question 2 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) Let $z = 2+i$. Find, in the form $x+iy$,

(i) $z\overline{z}$ **1**

(ii) $z-\overline{z}$ **1**

(b) By evaluating, or otherwise, show that $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$ is a real number. **2**

(c) (i) Write $2+2i$ in the form $r(\cos\theta + i\sin\theta)$ **2**

(ii) Hence, or otherwise, find $(2+2i)^5$ in the form $a+ib$, where a and b are integers. **2**

(d) (i) Find all pairs of integers a and b such that $(a+ib)^2 = 8+6i$. **3**

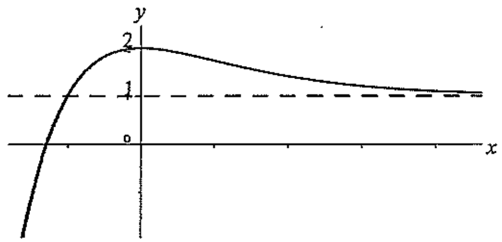
(ii) Hence solve: $z^2+2z(1+2i)-(11+2i)=0$. **2**

(e) Sketch the region in the complex plane where the two inequalities $|z+1-i|\le 3$ and $\dfrac{\pi}{4}\le \arg(z+1-i)\le \dfrac{3\pi}{4}$ both hold. **2**

Question 3 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) The diagram below shows the graph of $y = 1+(x+1)e^{-x}$.

The line $y = 1$ is an asymptote.



Draw separate one-third page sketches of the graphs of the following

(i) $y = f(-x)$ **2**

(ii) $y^2 = f(x)$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(b) If $1+i$ is a complex root of $ax^3-bx+2=0$, where a and b are real numbers.

(i) Find the other two roots. **2**

(ii) Hence, or otherwise, find a and b. **2**

(c) Show the equation $x^2-4y^2-6x-8y+1=0$ represents a hyperbola. **3**

Hence, or otherwise, determine its eccentricity.

(d) The asymptotes of a hyperbola meet at the origin and are inclined at an angle of 60° to the x-axis. If $(4,0)$ is a focus of the hyperbola, find the equation of the hyperbola. **2**

Question 4 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) If α, β and γ are the roots of $x^3 - px + q = 0$, find in terms of p and q a cubic equation with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ **2**

(ii) $\alpha^3, \beta^3, \gamma^3$ **2**

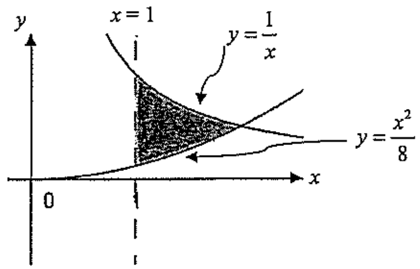
(b) Find the coordinates of the points where the tangent to the curve $x^2 - 2xy + 3y^2 = 8$ is horizontal. **3**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$ and the x-axis. **3**

Sections perpendicular to the base and the x-axis are squares.

Use the slicing technique to find the volume of the solid

(d) The region bounded by $y = \frac{1}{x}$, $y = \frac{x^2}{8}$ and $x = 1$ is rotated about the line $x = 1$.



(i) Use the method of cylindrical shells to find an integral which gives the volume of the resulting solid of revolution. **3**

(ii) Find the volume of this solid of revolution. **2**

Question 5 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) A car of mass one tonne is travelling at 40 km/h on a horizontal road and is turning a corner which is in the form of an arc of a circle of radius 155m. Find:

(i) The angular velocity of the car as it travels through the curve. **1**

(ii) The frictional force, in Newtons, required to keep the car from sliding off the road. **1**

(b) A bullet is fired vertically into the air, from the origin O, with a speed of 850 m/s. The bullet experiences air resistance. Let x be the displacement of the bullet above O at time t seconds after the bullet is fired, so that the equation of motion is $\ddot{x} = -g - \frac{v}{5}$, where g ms^{-2} is the acceleration due to gravity.

(i) Find the greatest height reached, to the nearest metre. (Use $g = 10$ ms^{-2}) **3**

(ii) The time taken to reach this height. **3**

(iii) As the bullet returns to the ground it is subject to the same forces. Find its terminal velocity. **1**

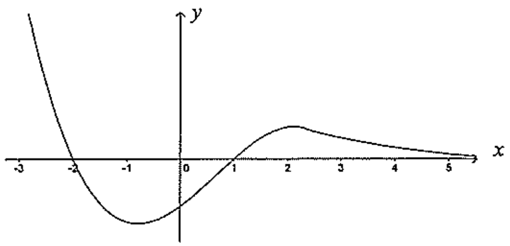
(c) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ **1**

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \geq 0$, show that $I_n = \frac{n}{n+2} . I_{n-1}$, for $n \geq 1$ **3**

(iii) Hence, or otherwise, evaluate I_4. **2**

Question 8 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a)



The function $f(x)$ has derivative $f'(x)$ whose graph appears above. You are given that $f'(-2)=f'(1)=0$, $f'(x)\to\infty$ as $x\to-\infty$ and $f'(x)\to 0$ as $x\to\infty$.
Sketch the graph of $y=f(x)$ showing its behaviour at its stationary points, as $x\to\pm\infty$ and any asymptotes, given that $f(0)=0$, $f(2)=0$ and $f(3)>f(-2)$. **4**

(b) The line through O perpendicular to the tangent at $P(cp,\frac{c}{p})$ on the rectangular hyperbola $xy=c^2$ meets the tangent at N.

(i) Show that the coordinates of N are $\left(\dfrac{2cp}{1+p^4},\dfrac{2cp^3}{1+p^4}\right)$. **2**

(ii) Hence, or otherwise, find the locus of N as p varies. **2**

(c) A body of mass 5 kg slides on a horizontal surface, pulled by a rope inclined at 20° to the horizontal, with a constant tension in the rope of 30N.
Two resistance forces act horizontally on the body. One is a constant force of magnitude 0.2R, where R is the reaction force the surface exerts on the mass. The other resistance force is due to air resistance and has a magnitude of $3kv$, where k is a constant and v is the speed of the body. (Use $g=10ms^{-2}$).

(i) Show that the equation of motion of the body is: **2**

$a=b-\dfrac{3kv}{5}$, where a is the acceleration and $b\approx 4.05$.

(ii) Explain why this equation implies that the body has a terminal velocity of $\dfrac{5b}{3k}$. **1**

(iii) Initially the body is travelling with half its terminal velocity. The body is observed to have attained 90% of its terminal velocity after 2 seconds. Find the value of k, and the distance travelled during these first 2 seconds, correct to 1 decimal place. **4**

End of paper

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq-1;\quad x\neq0,\text{ if } n<0$

$\int \frac{1}{x}\,dx = \ln x,\quad x>0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq0$

$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x=\log_e x,\quad x>0$

Question 6 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a) $P(a\cos\theta, b\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$, is a point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a > b > 0$. The normal to the ellipse at P has equation:

$$ax\sin\theta - by\cos\theta = \left(a^2-b^2\right)\sin\theta\cos\theta. \text{ (DO NOT PROVE THIS)}$$

This normal cuts the x-axis at A and the y-axis at B.

(i) Show that ΔOAB has an area of $\dfrac{\left(a^2-b^2\right)^2}{2ab}\sin\theta\cos\theta$. **3**

(ii) Find the maximum area of ΔOAB and the coordinates of P when this maximum occurs. **3**

(b) If $P(x) = 4x^3+4x^2+x+k$ for some real value of k, find the values of x for which $P'(x)=0$. **3**
Hence find the values of k for which the equation $P(x)=4x^3+4x^2+x+k$ has more than one real root.

(c) Suppose that $z^5 = 1$ where $z \neq 1$.

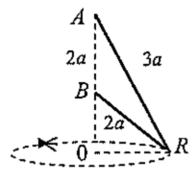
(i) Deduce that $z^2+z+1+\dfrac{1}{z}+\dfrac{1}{z^2}=0$. **2**

(ii) By letting $x = z+\dfrac{1}{z}$, reduce the equation in (i) to a quadratic equation in x. **2**

(iii) Hence deduce that $\cos\dfrac{2\pi}{5}\cos\dfrac{4\pi}{5} = -\dfrac{1}{4}$. **2**

Question 7 (15 marks) Use a SEPARATE sheet of paper. **Marks**

(a)



NOT TO SCALE

Two ends of a light inextensible string of length $5a$ are attached to two fixed points A and B (B is vertically below A) where $AB = 2a$. A smooth ring R of mass m is threaded on the string, and the system rotates about AB with constant angular velocity. The ring moves in a horizontal circle (whose plane is below the level of B), so that $AR = 3a$ and $RB = 2a$.

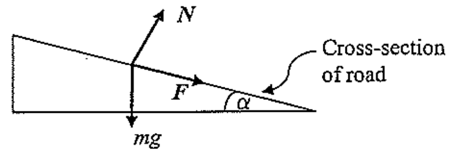
(i) Prove that the tension in the string is $\dfrac{8mg}{7}$. **3**

(ii) Find the angular velocity of the system. **2**

(b) (i) On a number plane, shade the region representing $(x-2R)^2+y^2 \le R^2$. **1**

(i) The region in part (i) is rotated about the y-axis to form a torus. Show that the volume of the torus is given by $V = 4\pi^2R^3$. **3**

(c)



NOT TO SCALE

A road contains a bend that is part of a circle radius r. At the bend, the road is banked at an angle α to the horizontal. A car travels around the bend at constant speed v. Assume that the car is represented by a point of mass m, and that the forces acting on the car are the gravitational force mg, a sideways friction force F (acting down the road as drawn) and a normal reaction N to the road.

(i) By resolving the horizontal and vertical components of force, find expressions for $F\cos\alpha$ and $F\sin\alpha$. **2**

(ii) Show that $F = \dfrac{m\left(v^2-gr\tan\alpha\right)\cos\alpha}{r}$. **2**

(iii) Suppose that the radius of the bend is 300m and that the road is banked to allow cars to travel at 80 kilometres per hour with no sideways friction force. Find the value of α, to the nearest degree. (Use $g = 10\,\text{m/s}^2$). **2**

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Q1(a) i) $\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1}$

$= \tan^{-1}(x+1)$

ii) $\int \frac{x}{\sqrt{x+1}}\,dx$ let $u = x+1$, $x = u-1$

$\frac{du}{dx} = 1$

$= \int \frac{u-1}{\sqrt{u}}\,du$

$= \int u^{1/2} - u^{-1/2}\,du$

$= \frac{2}{3}u^{3/2} - 2u^{1/2}$

$= \frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1}$

iii) $\int \frac{x^3+1}{x^2+1}\,dx$

$= \int \frac{(x^3+x)-x+1}{x^2+1}\,dx$

$= \int x - \frac{x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x$

iv) $\int \frac{dx}{(x-2)(x+1)}$ $\qquad \frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$\therefore A(x+1) + B(x-2) = 1$

$x = 2$, $3A = 1$, $A = 1/3$

$x = -1$, $-3B = 1$, $B = -1/3$

$= \frac{1}{3}\int \frac{1}{x-2} - \frac{1}{x+1}\,dx$

$= \frac{1}{3}\ln\left(\frac{x-2}{x+1}\right)$

v) $\int \sin^3 x\,dx$

$= \int (1-\cos^2 x)\sin x\,dx$

$= \int(\sin x - \cos^2 x\sin x)\,dx$

$= -\cos x + \frac{\cos^3 x}{3}$

b) $\int \sec x\,dx = \ln(\sec x + \tan x)$

$\int_0^2 \sqrt{x^2+4}\,dx$ let $x = 2\tan\theta$ $\qquad x=0,\ \theta=0$

$\frac{dx}{d\theta} = 2\sec^2\theta$ $\qquad x = 2,\ \theta = \frac{\pi}{4}$

$= \int_0^{\pi/4} \sqrt{4\tan^2\theta+4}\,.\,2\sec^2\theta\,d\theta$

$= 4\int_0^{\pi/4} \sec\theta\,.\,\sec^2\theta\,d\theta$ $\qquad u = \sec\theta \quad v = \tan\theta$

$= \left(4\int_0^{\pi/4} \sec^3\theta\,d\theta\right)$ $\qquad du = \sec\theta\tan\theta \quad dv = \sec^2\theta$

$= 4\left[[\sec\theta\tan\theta]_0^{\frac{\pi}{4}} - \int \sec\theta\tan^2\theta\,d\theta\right]$

$= 4\left[\sec 2\tan 2 - 0 - \int \sec\theta(\sec^2\theta - 1)\,d\theta\right]$

$= 4\sec 2\tan 2 - 4\int \sec^3\theta\,d\theta + 4\int \sec\theta\,d\theta$

$\therefore 8\int_0^{\pi/4} \sec^3\theta\,d\theta = 4\sec\frac{\pi}{4}\tan\frac{\pi}{4} + 4[\ln(\sec x + \tan x)]_0^{\frac{\pi}{4}}$

$\therefore 4\int_0^{\pi/4} \sec^3\theta\,d\theta = 2\sec\frac{\pi}{4}\tan\frac{\pi}{4} + 2[\ln(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}) - \ln(1+0)]$

$= 2\sqrt{2} + 2[\ln(\sqrt{2}+1) - 0]$

$= 4.59117..$

Q2 a) $z = 2+i$

(i) $z\bar{z} = (2+i)(2-i)$

$= 4+1$

$= 5$

(ii) $z - \bar{z} = 2+i-(2-i)$

$= 2i$

b) $\dfrac{1+2i}{3-4i} \times \dfrac{3+4i}{3+4i} + \dfrac{2-i}{5i} \times \dfrac{i}{i}$

$= \dfrac{3+4i+6i-8}{25} + \dfrac{2i+1}{-5}$

$= \dfrac{3+4i+6i-8-10i-5}{25}$

$= \dfrac{-10}{25} = -\dfrac{2}{5}$ which is real

c) i) $2+2i = \sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

ii) $(2+2i)^5 = (2\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= 128\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= -128 - 128i$

d) $(a+ib)^2 = 8+6i$

$a^2 - b^2 = 8$ —① & $2abi = 6i$ —②

$(a^2-b^2)^2 = 64$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$= 64 + 36$

$= 100$

$a^2+b^2 = 10$ —③ $(a^2+b^2>0)$

①+③ $2a^2 = 18$

$a^2 = 9$

$a = \pm 3$

in② $a=3, b=1$; $a=-3, b=-1$

b) ii) $z^2 + 2z(1+2i) - (11+2i) = 0$ $\qquad (1+2i)^2 = 1+4i-4$

$z = \dfrac{-2(1+2i) \pm \sqrt{4(1+2i)^2 + 4(11+2i)}}{2}$ $\qquad = -3+4i$

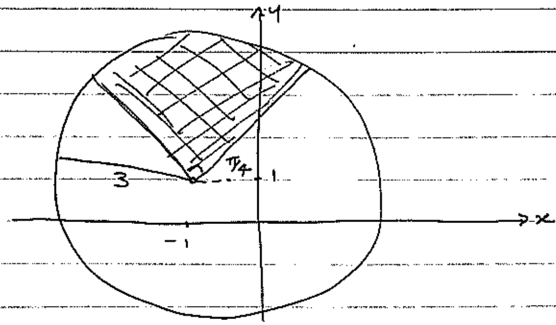
$= \dfrac{-2(1+2i) \pm 2\sqrt{-3+4i+11+2i}}{2}$

$= -1-2i \pm \sqrt{8+6i}$

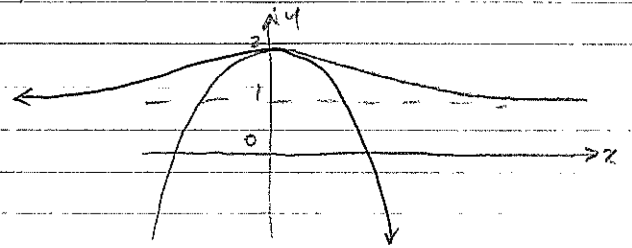
$\therefore z = -1-2i \pm (3+i)$

ie $z = 2-i$ or $z = -4-3i$

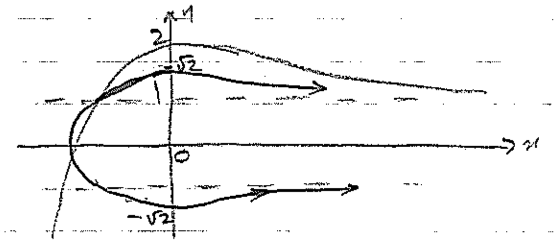
e)



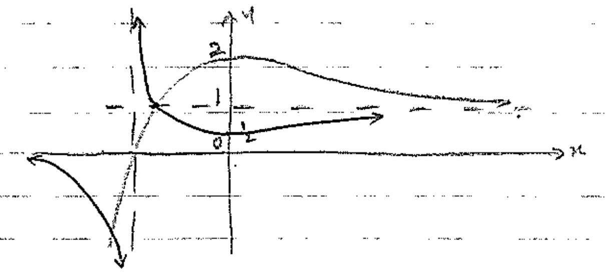
Q3. a) i)



ii)



iii)



b) i) $ax^3 - bx + 2 = 0$ $\quad 1+i$ a root

a & b real $\therefore$ $1-i$ is also a root

Sum of roots: $1+i+1-i+\alpha = 0$

$\therefore \alpha = -2$

$\therefore$ other roots are $1-i$ & -2

ii) Product roots: $(1+i)(1-i).-2 = -\frac{2}{a}$

$-4a = -2 \therefore a = \frac{1}{2}$

$P(-2) = 0 \therefore a(-2)^3 - b(-2) + 2 = 0$

$-4 + 2b + 2 = 0$

$\therefore b = 1 \qquad$ ie $a = \frac{1}{2}, b = 1$

Q3 c) $x^2 - 6x + 9 - 4y^2 - 8y = -1 + 9$

$(x-3)^2 - 4(y^2 + 2y + 1) = 8 - 4$

$(x-3)^2 - 4(y+1)^2 = 4$

ie $\frac{(x-3)^2}{4} - (y+1)^2 = 1$

eccentricity: $b^2 = a^2(e^2 - 1)$

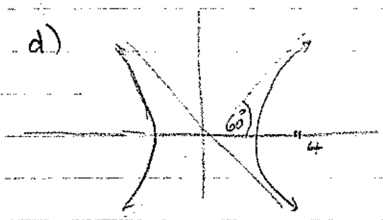
$1 = 4(e^2 - 1)$

$e^2 - 1 = \frac{1}{4}$

$e^2 = \frac{5}{4}$

$e = \frac{\sqrt{5}}{2}$

d)



$S(4,0)$ ie $ae = 4$ — ①

$\tan 60° = \sqrt{3}$ $\therefore$ asymptotes are $y = \sqrt{3}x$

ie $\frac{b}{a} = \sqrt{3}$ — ②

$\frac{b^2}{a^2} = e^2 - 1$

from ② $\therefore e^2 - 1 = (\sqrt{3})^2 = 3$

$e^2 = 4$

$e = 2$

from ①: $ae = 4 \therefore a = 2$

in ②: $\therefore b = 2\sqrt{3}$

eqn is $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Q4 a) $x^3 - px + q = 0 \rightarrow$ roots α, β, γ

i) roots $\frac{1}{\gamma}, \frac{1}{\beta}, \frac{1}{\gamma}$ let $y = \frac{1}{x}$ ie $x = \frac{1}{y}$

$$\left(\frac{1}{y^3}\right) - p\left(\frac{1}{y}\right) + q = 0$$

$$1 - py^2 + qy^3 = 0$$

ie eqn is $qx^3 - px^2 + 1 = 0$

ii) roots $\alpha^3, \beta^3, \gamma^3$ let $y = x^3$ $\therefore x = \sqrt[3]{y}$

$$(\sqrt[3]{y})^3 - p(\sqrt[3]{y}) + q = 0$$

$$y - p\sqrt[3]{y} + q = 0$$

$$p\sqrt[3]{y} = y + q$$

$$p^3 y = (y+q)^3 = y^3 + 3y^2q + 3yq^2 + q^3$$

$\therefore$ eqn is $x^3 + 3x^2q + 3xq^2 - p^3x + q^3 = 0$

or $x^3 + 3qx^2 + (3q^2 - p^3)x + q^3 = 0$

b) $x^2 - 2xy + 3y^2 = 8$

$$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{y-x}{3y-x}$$

$\frac{dy}{dx} = 0$ ie $y - x = 0$

$\therefore x = y$

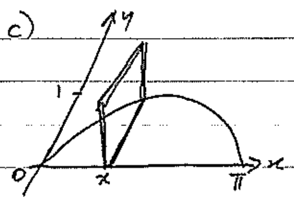
$$\therefore x^2 - 2x^2 + 3x^2 = 8$$

$$2x^2 = 8$$

$x = \pm 2$, $y = \pm 2$

$\therefore$ pts are $(2, 2)$ or $(-2, -2)$

c)



$$\delta V = y^2 \delta x$$

$$= \sin^2 x \, \delta x$$

$$\text{Vol} = \lim_{\delta x \to 0} \sum_{x=0}^{\pi} \sin^2 x \, \delta x$$

$$= \int_0^{\pi} \sin^2 x \, dx$$

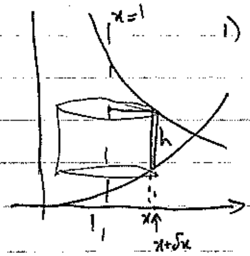
$$= \frac{1}{2}\int_0^{\pi} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2}[\pi - 0 - (0 - 0)]$$

$\therefore$ Vol $= \frac{\pi}{2}$ units^3

d)



i) $R = x - 1 + \delta x$ $r = x - 1$

$$h = \frac{1}{x} - \frac{x^2}{8}$$

Area of annulus $= \pi(R^2 - r^2)$

$$= 2\pi\frac{(R+r)}{2}(R-r)$$

$$= 2\pi(x-1)\,\delta x$$

$$\delta V = 2\pi(x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right)\delta x$$

point of intersection: $\frac{1}{x} = \frac{x^2}{8}$ $x^3 = 8$ $x = \pm 2$

$$\therefore V = 2\pi\int_1^2 (x-1)\left(\frac{1}{x} - \frac{x^2}{8}\right)dx$$

ii) $V = 2\pi\int_1^2 1 - \frac{x^3}{8} - \frac{1}{x} + \frac{x^2}{8} \, dx$

$$= 2\pi\left[x - \frac{x^4}{32} - \ln x + \frac{x^3}{24}\right]_1^2$$

$$= 2\pi\left[2 - \frac{16}{32} - \ln 2 + \frac{8}{24} - \left(1 - \frac{1}{32} - 0 + \frac{1}{24}\right)\right]$$

$$= 2\pi\left[\frac{79}{4 \times 24} - \ln 2\right]$$

$$= \frac{79\pi}{48} - 2\pi \ln 2$$

QUESTION 5

a) i) $V = 40$ km/h

$= \frac{100}{9}$ m/s

$\omega = \frac{V}{r}$

$= \frac{100}{9} \times \frac{1}{155}$

$= \frac{20}{279}$ rad/s

ii) $F = m\omega^2 r$

$= 1000 \times \frac{20^2}{279^2} \times 155$

$= 796$

b) i) $\ddot{x} = -g - \frac{V}{5}$

$\frac{v\,dv}{dx} = -g - \frac{V}{5}$

$\frac{dx}{dV} = \frac{-5V}{5g+V}$

$x = -5\int \frac{5g+V}{5g+V} - \frac{5g}{5g+V}\, dV$

$x = -5\left(V - 5g\ln(5g+V)\right) + C$

when $x = 0$, $V = 850$

$C = 4250 - 25g\ln(5g+850)$

$\therefore x = -5V + 4250 + 25g\ln\left(\frac{5g+V}{5g+850}\right)$

when $V = 0$, $x = 3527$

ii) $\frac{dv}{dt} = -g - \frac{V}{5}$

$\frac{dt}{dV} = \frac{-5}{5g+V}$

$t = -5\ln(5g+V) + C$

when $t = 0$ $V = 850$

$C = 5\ln(5g+850)$

$\therefore t = 5\ln\left(\frac{5g+850}{5g+V}\right)$

when $V = 0$, $t = 5\ln 18$

$= 14.5$ s

iii) $\ddot{x} = -g + \frac{V}{5}$

Terminal velocity when $\ddot{x} = 0$

i.e. $g = \frac{V}{5}$

$V = 5g$

$V = 50$ m/s

QUESTION 5 (continued)

c) i) R.H.S. $= (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$= (1-\sqrt{x})^{n-1}\left[1 - 1 + \sqrt{x}\right]$

$= (1-\sqrt{x})^{n-1}.\sqrt{x}$

$=$ L.H.S.

ii) $I_n = \int_0^1 (1-\sqrt{x})^n\, dx$

let $u = (1-\sqrt{x})^n$ $\quad dv = dx$

$du = n(1-\sqrt{x})^{n-1} \cdot \frac{-x^{-1/2}}{2}$ $\quad v = x$

$= \frac{-n}{2\sqrt{x}}(1-\sqrt{x})^{n-1}$

$I_n = \left[x(1-\sqrt{x})^n\right]_0^1 + \frac{n}{2}\int_0^1 \sqrt{x}(1-\sqrt{x})^{n-1}$

$= 0 + \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$I_n = \frac{n}{2}\left(I_{n-1} - I_n\right)$

$I_n + \frac{nI_n}{2} = \frac{nI_{n-1}}{2}$

$I_n\left(1 + \frac{n}{2}\right) = \frac{nI_{n-1}}{2}$

$I_n = \frac{nI_{n-1}}{2} \times \frac{2}{2+n}$

$= \frac{n}{n+2} I_{n-1}$

iii) $I_0 = \int_0^1 dx$

$= [x]_0^1$

$= 1$

$I_1 = \frac{1}{3} \times I_0$

$= \frac{1}{3}$

$I_2 = \frac{1}{2} \times I_1$

$= \frac{1}{6}$

$I_3 = \frac{3}{5} \times I_2$

$= \frac{1}{10}$

$I_4 = \frac{2}{3} \times I_3$

$= \frac{1}{15}$

QUESTION 6

a) $ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

i) when $x=0$, $-by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

$$y = \frac{(a^2-b^2)\sin\theta\cos\theta}{-b\cos\theta}$$

$$= \frac{(a^2-b^2)}{-b}\sin\theta$$

when $y=0$, $ax\sin\theta = (a^2-b^2)\sin\theta\cos\theta$

$$x = \frac{a^2-b^2}{a}\cdot\cos\theta$$

$\therefore$ Area $= \left|\frac{1}{2}\frac{(a^2-b^2)}{-b}\cdot\sin\theta\,\frac{(a^2-b^2)}{a}\cos\theta\right|$

$$= \frac{(a^2-b^2)^2}{2ab}\sin\theta\cos\theta$$

ii) $A = \frac{(a^2-b^2)^2}{4ab}\sin 2\theta$

max area when $\sin 2\theta = 1$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

$\therefore A = \frac{(a^2-b^2)^2}{4ab}$

and $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

b) $P(x) = 4x^3+4x^2+x+k$

$P'(x) = 12x^2+8x+1 = 0$

$$x = \frac{-8\pm\sqrt{64-48}}{24}$$

$$= -1/6,\ -1/2$$

$P(-\frac{1}{6})\cdot P(-1/2) \le 0$

$k(k-2/27) \le 0$

$\therefore 0 \le k \le \frac{2}{27}$

QUESTION 6 (continued)

c) i) $z^5-1 = (z-1)(z^4+z^3+z^2+z+1)$

$\therefore$ if $z^5-1=0$ and $z\ne 1$

$z^4+z^3+z^2+z+1 = 0$

$\therefore\quad z^2+z+1+\frac{1}{z}+\frac{1}{z^2} = 0$

ii) $\left(\frac{1}{z}+z\right)^2 = \frac{1}{z^2}+2+z^2$

$\frac{1}{z^2}+z^2 = \left(\frac{1}{z}+z\right)^2 - 2$

$\therefore \left(\frac{1}{z^2}+z^2\right)+\left(\frac{1}{z}+z\right)+1 = 0$

$\therefore \left(\frac{1}{z}+z\right)^2 - 2 + \left(\frac{1}{z}+z\right)+1 = 0$

$\therefore\quad x^2+x-1 = 0$

iii) $\frac{1}{z}+z = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$$= 2\cos\theta$$

$$= 2\cos\tfrac{2\pi}{5} \text{ and } 2\cos\tfrac{4\pi}{5} \quad \text{(see below)}$$

$\therefore$ Product of the roots $= 2\cos\frac{2\pi}{5}\times 2\cos\frac{4\pi}{5} = -1$

$$4\cos\tfrac{2\pi}{5}\cos\tfrac{4\pi}{5} = -1$$

$$\cos\tfrac{2\pi}{5}\cos\tfrac{4\pi}{5} = -\tfrac{1}{4}$$

$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

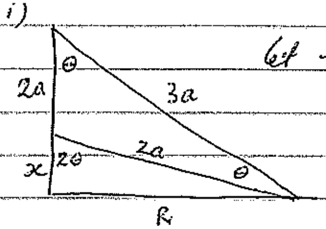
$\frac{1}{z_1} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$\frac{1}{z_2} = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

QUESTION 7

a) i)



Let tension in string = T

$4a^2 = R^2 + x^2$ ——— (1)

$9a^2 = (2a+x)^2 + R^2$ ——— (2)

(1) $\Rightarrow R^2 = 4a^2 - x^2$

$\therefore\ 9a^2 = 4a^2 + 4ax + x^2 + 4a^2 - x^2$

$a^2 = 4ax$

$x = \frac{a}{4}$

$\cos\theta = \frac{9a}{4} \times \frac{1}{3a}$ $\qquad \sin\theta = \frac{R}{3a}$

$= \frac{3}{4}$

$\cos 2\theta = \frac{a}{4} \times \frac{1}{2a}$ $\qquad \sin 2\theta = \frac{R}{2a}$

$= \frac{1}{8}$

now $T\cos\theta + T\cos 2\theta = mg$

$\frac{3T}{4} + \frac{T}{8} = mg$

$\frac{7T}{8} = mg$

$T = \frac{8mg}{7}$

ii) $T\sin\theta + T\sin 2\theta = m\omega^2 R$

$\frac{RT}{3a} + \frac{RT}{2a} = m\omega^2 R$

$\frac{5T}{6a} = m\omega^2$

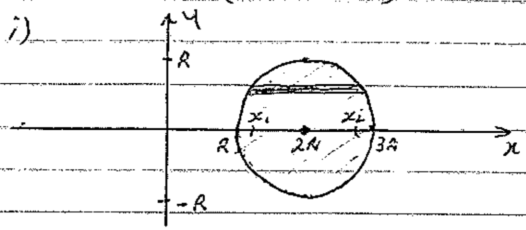
$\omega^2 = \frac{5}{6am} \times \frac{8mg}{7}$

$\omega^2 = \frac{20g}{21a}$

$\omega = \sqrt{\frac{20g}{21a}}$

QUESTION 7 (continued)

b) i)



ii) $(x-2R)^2 + y^2 = R^2$

$x - 2R = \pm\sqrt{R^2 - y^2}$

$x = 2R \pm \sqrt{R^2 - y^2}$

$\therefore\ x_1^2 = 4R^2 - 4R\sqrt{R^2-y^2} + R^2 - y^2$

$x_2^2 = 4R^2 + 4R\sqrt{R^2-y^2} + R^2 - y^2$

$x_2^2 - x_1^2 = 8R\sqrt{R^2-y^2}$

$V = \pi\int_{-R}^{R} x_2^2 - x_1^2 \, dy$

$= 8R\pi \int_{-R}^{R} \sqrt{R^2-y^2}\, dy$

$= 8R\pi \times \frac{\pi R^2}{2}$ (area semi-circle)

$= 4\pi^2 R^3$

c) i) $F\sin\alpha = N\cos\alpha - mg$ ——— (1)

$F\cos\alpha = \frac{mv^2}{r} - N\sin\alpha$ ——— (2)

ii) (1) $\Rightarrow F\sin^2\alpha = N\sin\alpha\cos\alpha - mg\sin\alpha$ ——— (3)

(2) $\Rightarrow F\cos^2\alpha = \frac{mv^2\cos\alpha}{r} - N\sin\alpha\cos\alpha$ ——— (4)

(3) + (4) $\Rightarrow F(\sin^2\alpha + \cos^2\alpha) = \frac{mv^2\cos\alpha}{r} - mg\sin\alpha$

$= m\left(\frac{v^2\cos\alpha - gr\sin\alpha}{r}\right)$

$= m\left(\frac{v^2 - gr\tan\alpha}{r}\right)\cos\alpha$

QUESTION 7 (continued)

c) iii) $80 \text{ km/h} = \frac{200}{9} \text{ m/s}$

when $F=0$, $v^2 - gr\tan\alpha = 0$

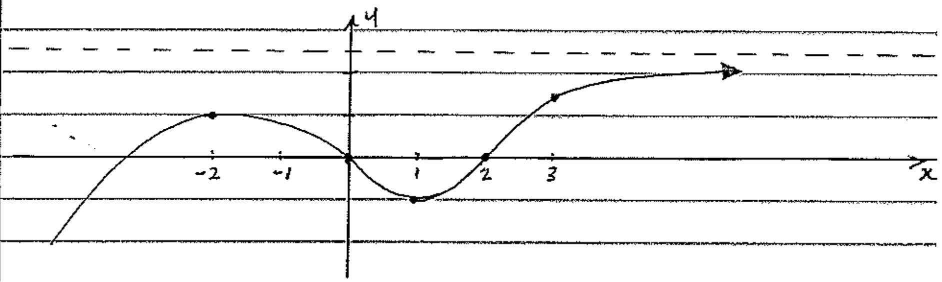
$$\tan\alpha = \frac{v^2}{rg}$$

$$\tan\alpha = \frac{40000}{81} \times \frac{1}{10 \times 300}$$

$$\alpha = 9^\circ$$

QUESTION 8

a)



b) $y = c^2 x^{-1}$

$\frac{dy}{dx} = -c^2 x^{-2}$

when $x = cp$

$\frac{dy}{dx} = \frac{-c^2}{c^2p^2}$

$= \frac{-1}{p^2}$

$\therefore$ grad line $= p^2$

$\therefore$ eqn line $\Rightarrow y = p^2 x$ —— ①

$\therefore$ eqn tangent

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$x + p^2 y = 2cp$ —— ②

subst ① into ②

$x + p^4 x = 2cp$

$x(1+p^4) = 2cp$

$x = \frac{2cp}{1+p^4}$ —— ③

subst ③ into ①

$y = p^2 \cdot \frac{2cp}{1+p^4}$

$= \frac{2cp^3}{1+p^4}$

QUESTION 8 (continued)

ii) $y = p^2 x$

$\therefore p^2 = \frac{y}{x}$

now $x = \frac{2cp}{1+p^4}$

$x(1+p^4) = 2cp$

$x^2(1+p^4)^2 = 4c^2p^2$

$x^2\left(1 + \frac{y^2}{x^2}\right)^2 = 4c^2 \cdot \frac{y}{x}$

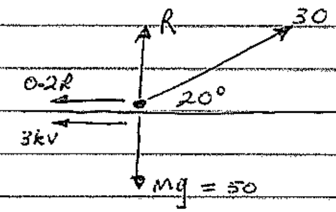
$x^2\left(\frac{x^2+y^2}{x^2}\right)^2 = \frac{4c^2 y}{x}$

$\frac{x^4(x^2+y^2)^2}{x^4} = 4c^2 y \; x$

$(x^2+y^2)^2 = 4c^2 xy$

c)

i)



$R + 30\sin 20 = 50$

$R = 39.7394$

$ma = 30\cos 20 - 0.2R - 3kv$

$a = 6\cos 20 - 1.5896 - \frac{3kv}{5}$

$a = b - \frac{3kv}{5}$

where $b = 6\cos 20 - 1.5896$

$= 4.05$.

ii) Terminal velocity when $a = 0$

i.e. $\frac{3kv}{5} = b$

$v = \frac{5b}{3k}$

iii) $\frac{dv}{dt} = \frac{5b - 3kv}{5}$

$\frac{dt}{dv} = \frac{5}{5b - 3kv}$

$t = \frac{-5}{3k}\int \frac{-3k}{5b - 3kv}\,dv$

$= \frac{-5}{3k}\ln(5b - 3kv) + C$

when $t = 0$, $v = \frac{5b}{6k}$

$C = \frac{5}{3k}\ln\left(\frac{5b}{2}\right)$

$\therefore t = \frac{5}{3k}\ln\left(\frac{5b}{10b - 6kv}\right)$

when $t = 2$, $v = \frac{3b}{2k}$

$2 = \frac{5}{3k}\ln\left(\frac{5b}{10b - 9b}\right)$

$k = \frac{5}{6}\ln 5 = 1.3412$ *

now $t = \frac{2}{\ln 5}\ln\left(\frac{5b}{10b - 5v\ln 5}\right)$

$\frac{t\ln 5}{2} = \ln\left(\frac{5b}{10b - 5v\ln 5}\right)$

$e^{\frac{t\ln 5}{2}} = \frac{5b}{10b - 5v\ln 5}$

$e^{-\frac{t\ln 5}{2}} = \frac{10b - 5v\ln 5}{5b}$

QUESTION 8 (continued)

$$5be^{-\frac{t\ln 5}{2}} = 10b - 5v\ln 5$$

$$5v\ln 5 = 10b - 5be^{-\frac{t\ln 5}{2}}$$

$$v = \frac{2b}{\ln 5} - \frac{be^{-\frac{t\ln 5}{2}}}{\ln 5}$$

$$x = \frac{2bt}{\ln 5} + \frac{2b}{(\ln 5)^2}\int -\frac{\ln 5}{2} e^{-\frac{t\ln 5}{2}}\, dt$$

$$= \frac{2bt}{\ln 5} + \frac{2b}{(\ln 5)^2} e^{-\frac{t\ln 5}{2}} + c$$

when $x=0,\ t=0,\quad c = \frac{-2b}{(\ln 5)^2}$

$$\therefore x = \frac{2bt}{\ln 5} + \frac{2b}{(\ln 5)^2} e^{-\frac{t\ln 5}{2}} - \frac{2b}{(\ln 5)^2}$$

when $t=2,\quad x = \frac{4b}{\ln 5} + \frac{2b}{(\ln 5)^2} e^{-\ln 5} - \frac{2b}{(\ln 5)^2}$

$$= \frac{20b\ln 5 - 8b}{5(\ln 5)^2}$$

$$= 7.564$$