HORNSBY GIRLS HIGH SCHOOL

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown for every question
- Begin each question in a new booklet

Total marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value

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Total Marks
Attempt Questions 1–8
All Questions are of equal value

Begin each question in a new booklet, writing your student number and question number in the boxes indicated. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x^2}{\sqrt{8-x^3}}\,dx$. **2**

(b) By completing the square, find $\int \frac{dx}{x^2-8x+20}$. **2**

(c) Evaluate $\int_0^{\pi} x\cos x\,dx$. **3**

(d) (i) Show that $\frac{2}{x^3+x^2+x+1}=\frac{1}{x+1}-\frac{x}{x^2+1}+\frac{1}{x^2+1}$. **2**

(ii) Hence, or otherwise, show $\int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}=\tan^{-1}2-\tan^{-1}\left(\frac{1}{2}\right)$. **3**

(e) Using the substitution $x=\tan\theta$, or otherwise, show **3**
$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx=\sqrt{2}-\frac{2}{\sqrt{3}}$.

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Write i^7 in the form $x+iy$ where x and y are real.
1

(b) Let $z=2+2i$ and $w=2-i$. Find in the form $x+iy$, where x and y are real,

(i) $z\overline{w}$ **1**

(ii) $\dfrac{8}{z}$ **1**

(c) It is given that $1+i$ is a root of $P(z)=2z^3-3z^2+rz+s$, where r and s are real.

(i) Explain why $1-i$ is also a root of the equation. **1**

(ii) Factorise $P(z)$ over the real field. **2**

(d) Find all the solutions of $z^4=16$. Express your solutions in the modulus-argument form. **2**

(e) Sketch the region in the complex plane where the inequalities $|z-\overline{z}|\leq 2$ and $|z-i|\leq 4$ hold. **3**

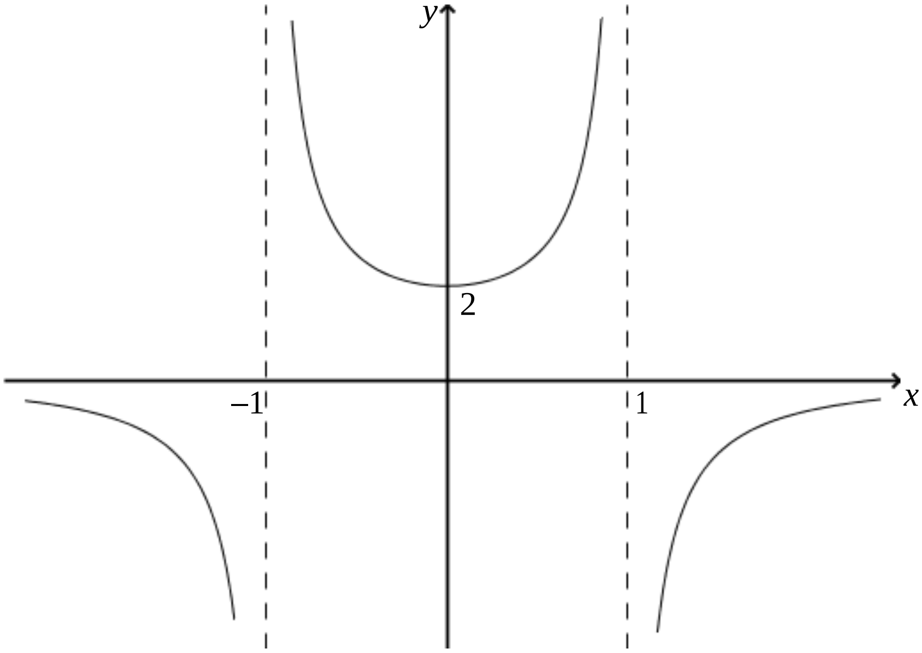
(f) (i) Prove, by Mathematical Induction, that for all integers n, **3**

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

(ii) Hence, find an expression for $\cos 3\theta$. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 0$ and vertical asymptotes at $x = \pm 1$.



NOT TO SCALE

Draw neat separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

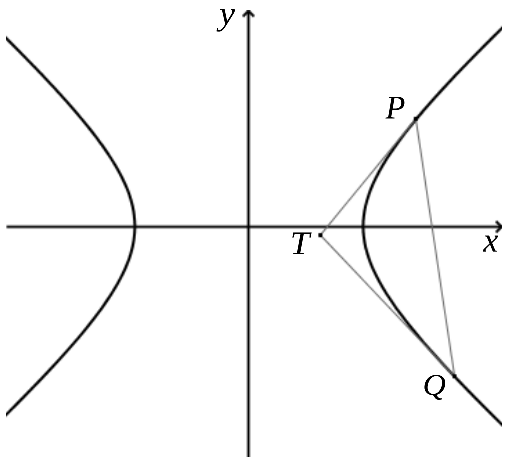
(ii) $y = f(x) + |f(x)|$ **2**

(iii) $y = e^{f(x)}$ **2**

Question 3 continues on page 6

Question 3 (continued)

(b)



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The points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the right branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The tangents at P and Q meet at $T(x_0, y_0)$.

(i) Show the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2**

(ii) Hence show the equation of the chord of contact is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. **2**

(iii) The chord PQ passes through the focus $S(ae, 0)$ where e is the eccentricity of the hyperbola. Prove T lies on the directrix of the parabola. **1**

(c) Let α, β, γ be the zeros of the polynomial $P(x) = 3x^3 + 7x^2 + 11x + 51$.

(i) Find $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(iii) Using part (ii), or otherwise, determine how many zeros of $P(x)$ are real. Justify your answer. **1**

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

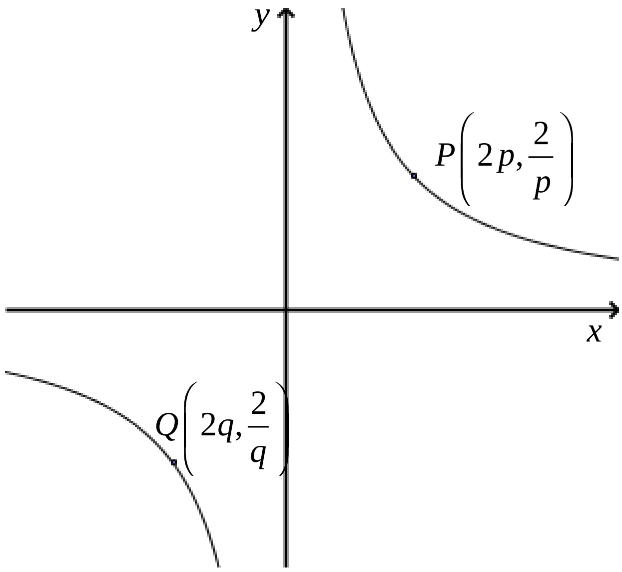
(a) A solid of height 2 metres rests on a horizontal surface. **3**

Every horizontal cross-section of the solid, x metres above the surface,

is a square of side $\sqrt{3x+1}$ metres.

Find the volume of the solid.

(b) Consider the rectangular hyperbola $xy = 4$, with points P and Q on different branches of the hyperbola.



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(i) Prove that the equation of the normal to $xy = 4$ at the point $P\left(2p, \frac{2}{p}\right)$ **3**

is $py - p^3x = 2(1 - p^4)$.

(ii) If this normal meets the hyperbola again at $Q\left(2q, \frac{2}{q}\right)$, prove that $q = \frac{-1}{p^3}$. **2**

(iii) Hence, show that there exists only one chord of the hyperbola **3**

which is normal to the hyperbola at P and Q, and find its equation.

(c) The equation $x^3 + 3x + 2 = 0$ has roots α, β and γ.

(i) Find the polynomial whose roots are α^2, β^2 and γ^2. **2**

(ii) Hence, or otherwise, find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let w be a complex root of unity (w is solution of $z^3 - 1 = 0$).

(i) Show that $(z-1)(z^2 + z + 1) = z^3 - 1$. **1**

(ii) Explain why $w^2 + w + 1 = 0$. **1**

(iii) Hence, other otherwise, show that $(1-w)(1-w^2)(1-w^4)(1-w^8) = 9$ **3**

(b) Consider $I = \int_1^{\infty} \frac{1}{x\sqrt{1+x^2}}\,dx$.

(i) By using a suitable substitution, show that $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin\theta}d\theta$. **2**

(ii) Hence, or otherwise, evaluate I. **3**

(c) (i) Find real numbers, a and b, such that **2**

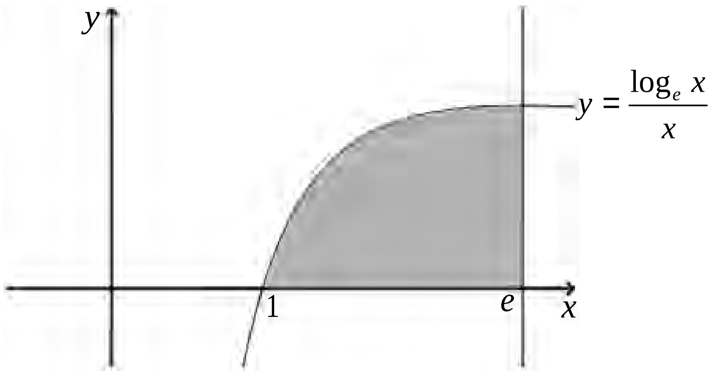
$x^4 + x^3 + x^2 + x + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

(ii) Given that $x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$ is a solution of $x^4 + x^3 + x^2 + x + 1 = 0$,

find the exact value of $\cos\frac{2\pi}{5}$. **3**

Question 6 (15 marks) Use a SEPERATE writing booklet. **Marks**

(a) Use the method of cylindrical shells to find the volume of the solid formed when the shaded region bounded by $y=0$, $y=\dfrac{\log_e x}{x}$ and $x=e$ is rotated about the y-axis. **4**



NOT TO SCALE

(b) (i) Show that $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$. **1**

(ii) Hence, or otherwise, solve the equation **3**

$\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta=0$ for $0\le\theta\le 2\pi$.

(c) A stone is projected vertically upwards in the air from a point h metres above the ground at a speed u and experiences a resistance equal to mkv^2, where m is the mass of the stone, v is the speed after time t and k is a constant. **3**

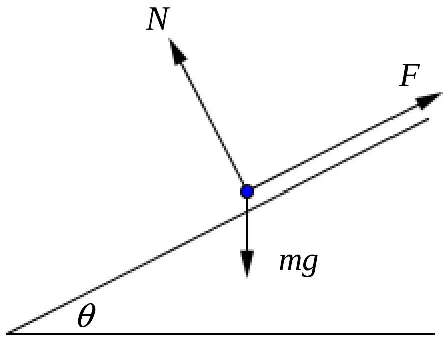
By considering the forces acting on the stone, show that the maximum height, H, the stone reaches above the ground is given by $H=h+\dfrac{1}{2k}\ln\left(1+\dfrac{ku^2}{g}\right)$, where g is acceleration due to gravity

(d) A group of n people are to be seated around a circular table. Find the number of possible arrangements if 3 particular people are to sit together. **2**

(e) Show that ${}^{n+2}C_r={}^{n}C_r+2\,{}^{n}C_{r-1}+{}^{n}C_{r-2}$ **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A particle of mass m is lying on an inclined plane and does not move. **2**
The plane is at an angle θ to the horizontal. The particle is subject to a gravitational force mg, a normal reaction force N, and a frictional force F parallel to the plane, as shown in the diagram above.

By resolving the forces acting on the particle parallel and perpendicular to the plane, find an expression for $\dfrac{F}{N}$ in terms of θ.

(b) The polynomial $P(x) = x^4 - 4x^3 + 3x^2 - 14x + 10$ has roots $a + ib$, $a - 2ib$, where a and b are real.

(i) Show that $a = 1$, and hence find the value(s) of b. **2**

(ii) Hence, factorise $P(x)$ over the rational field. **2**

(c) (i) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$, then show that $I_n = \dfrac{n-1}{n} I_{n-2}$. **3**

(ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx$. **2**

(d) Use Mathematical Induction to prove that for integer values of $n \geq 1$ **4**

$1\times 1! + 2\times 2! + 3\times 3! + \ldots + n\times n! = (n+1)! - 1$

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The function $y = f(x)$ is defined by $f(x) = \sqrt{3 - \sqrt{x}}$

(i) State the domain of the function $f(x)$. **1**

(ii) Show that $y = f(x)$ is a decreasing function and determine the range of $y = f(x)$. **2**

(iii) Sketch the graph of $y = f(x)$ for the domain and range determined above. **1**

(iv) Prove that $\int_0^9 \sqrt{3 - \sqrt{x}}\,dx = \frac{24\sqrt{3}}{5}$ **2**

(b) Show that $\frac{d}{du}\left(\sec u + \tan u\right) = \sec u(\sec u + \tan u)$ **2**

(c) Consider $f(x) = \cos\frac{x}{2}$.

(i) On the same set of axes, sketch the graph of $y = f(x)$, and hence the graph of $y = \frac{1}{f(x)}$ for the domain $0 \le x \le 2\pi$. **2**

(ii) By considering part (b), find the area bounded by the curve $y = \frac{1}{f(x)}$, the $x-$axis and the ordinates $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$, leaving your answer exact. **3**

(iii) The solid bounded by the curve $y = \frac{1}{f(x)}$, the $x-$axis and the ordinates $x = \frac{\pi}{3}$ and $x = \frac{\pi}{2}$ is rotated about the $y-$axis. **2**

By using the method of annular discs, find the volume as a **definite integral**.

DO NOT EVALUATE THIS INTEGRAL.

End of paper

HGHS : Ext 2 Trial Solutions 2011

Question 1

(a) $\int \frac{x^2}{\sqrt{8-x^3}}\,dx = \frac{1}{3}\int \frac{du}{u^{1/2}}$ $\quad u = 8-x^3$, $u' = -3x^2$

$= -\frac{1}{3}\left[2u^{1/2}\right]+c$

$= -\frac{2}{3}\sqrt{8-x^3} + c$

(b) $\int \frac{dx}{x^2-8x+20} = \int \frac{dx}{(x-4)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-4}{2}\right)+c$

(c) $\int_0^{\pi} x\cos x\,dx$ $\quad u=x$, $du=dx$, $dv=\cos x\,dx$, $v=\sin x$

$= \left[x\sin x\right]_0^{\pi} - \int_0^{\pi}\sin x\,dx$

$= 0-\left[-\cos x\right]_0^{\pi}$

$= 0-\left[1--1\right]$

$= -2$

(d)(i) RTS $\frac{2}{x^3+x^2+x+1} = \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$

RHS $= \frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}$

$= \frac{x^2+1-x^2-x+x+1}{(x+1)(x^2+1)}$

$= \frac{2}{x^3+x^2+x+1}$

$=$ LHS as req.

(ii) $\therefore \int_{\frac{1}{2}}^{2} \frac{2}{x^3+x^2+x+1}\,dx = \int_{\frac{1}{2}}^{2}\left(\frac{1}{x+1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$

$= \left[\ln(x+1) - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x\right]_{\frac{1}{2}}^{2}$

(d)(ii)(cont.)

$= \left[\ln 3 - \frac{1}{2}\ln 5 + \tan^{-1}2\right]$

$- \left[\ln\frac{3}{2} - \frac{1}{2}\ln\frac{5}{4} + \tan^{-1}\frac{1}{2}\right]$

$= \ln\left[3\times\frac{1}{\sqrt5}\times\frac{2}{3}\times\frac{\sqrt5}{2}\right] + \tan^{-1}2 + \tan^{-1}\frac{1}{2}$

$= \ln 1 + \tan^{-1}2 + \tan^{-1}\frac{1}{2}$

$= \tan^{-1}2 + \tan^{-1}\frac{1}{2}$ as req.

(e) $\int_1^{\sqrt3}\frac{dx}{x^2\sqrt{1+x^2}} = \int_{\frac{\pi}{4}}^{\pi/3}\frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}$

$x=\tan\theta$
$dx=\sec^2\theta\,d\theta$
$x=\sqrt3 \quad \theta=\frac{\pi}{3}$
$x=1 \quad \theta=\frac{\pi}{4}$

$= \int_{\frac{\pi}{4}}^{\pi/3}\frac{\sec^2\theta\,d\theta}{\frac{\sin^2\theta}{\cos^2\theta}\cdot\sec\theta}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\operatorname{cosec}\theta\cdot\cot\theta\,d\theta$

$= -\left[\operatorname{cosec}\theta\right]_{\pi/4}^{\pi/3}$

$= -\left(\frac{2}{\sqrt3} - \sqrt2\right)$

$= \sqrt2 - \frac{2}{\sqrt3}$

Question 2.

a) $i^7 = i^2 \cdot i^2 \cdot i^2 \cdot i$

$= (-1)\cdot(-1)(-1)\cdot i$

$= -i$

$= 0 - i$

) $z = 2+2i$

$w = 2-i$

i) $z\bar{w} = (2+2i)(2+i)$

$= 4+2i+4i+2i^2$

$= 2+6i$

i) $\frac{8}{z} = \frac{8\bar{z}}{z\bar{z}}$

$= \frac{8(2-2i)}{(2+2i)(2-2i)}$

$= \frac{8(2-2i)}{2^2+2^2}$

$= 2-2i$

) $P(z)$ has real coefficients, hence roots occur in conjugate pairs

i) $1+i+1-i+\alpha = \frac{3}{2}$

$\alpha = -\frac{1}{2}$.

$\therefore P(z) = 2(z-(1+i))(z+(1-i))(z+\frac{1}{2})$

$= (z^2-2z+2)(2z+1)$

d) $z^4 = 16$

$(z^2+4)(z^2-4) = 0$

$z = 2, -2$

$z^2 = -4$

$z = \pm\sqrt{-4}$

$z = \pm\sqrt{4i^2}$

$= \pm 2i$

$\therefore\ z = 2, -2, 2i, -2i$.

e) $|z-i| \le 4$.

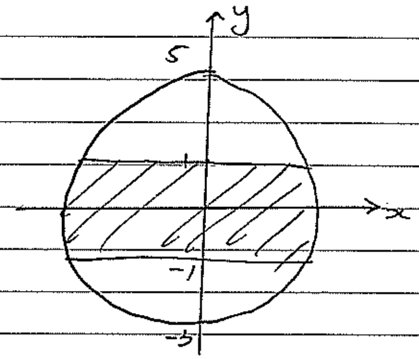
circle centre $(0,1)$

radius 4.

$|x+iy-(x-iy)| \le 2$

$|2iy| \le 2$

$-1 \le y \le 1$



f) $\cancel{(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

$\cancel{\text{Test } n=0}$

$\cancel{LHS = (\cos\theta+i\sin\theta)^0}$

$\cancel{= 1}$

$\cancel{RHS = \cos 0 + i\sin 0}$

$\cancel{= 1}$

$\cancel{\text{Assume true } n=k:}$

$\cancel{(\cos\theta+i\sin\theta)^k = \cos k\theta + i\sin k\theta}$

f)i) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

test $n=1$.

LHS $= \cos\theta + i\sin\theta$

RHS $= \cos\theta + i\sin\theta$

LHS = RHS.

$\therefore$ True for $n=1$.

assume true for $n=k$

i.e. $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Prove ~~true~~ for $n=k+1$

RTP: $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k \\
&= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta) \\
&= \cos\theta\cos k\theta + i\sin k\theta\cos\theta + i\sin\theta\cos k\theta \\
&\quad \cancel{\theta} - \cancel{\theta}\sin\theta\sin k\theta \\
&= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta) \\
&= \cos(\theta + k\theta) + i\sin(\theta + k\theta) \\
&= \cos\theta(k+1) + i\sin\theta(k+1).
\end{aligned}
$$

$\therefore$ if true for $n=1$, it must be true for $n=2$, $n=3$ and all other positive integers where $n \geq 1$

~~...~~ test $n=0$.

LHS $= 1$

RHS $= 1$

$\therefore$ true for $n=0$.

$\to$

Start here.

test for negative integers

RTP: $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{-n} &= ((\cos\theta + i\sin\theta)^n)^{-1} \\
&= (\cos n\theta + i\sin n\theta)^{-1} \\
&= \frac{1}{z} \qquad \text{where } z = \cos n\theta + i\sin n\theta \\
&= \frac{\bar{z}}{z\bar{z}} \\
&= \frac{\bar{z}}{|z|^2} \\
&= \cos n\theta - i\sin n\theta. \\
&= \cos(-n\theta) + i\sin(-n\theta).
\end{aligned}
$$

$\therefore$ true for all negative integers.

$\therefore (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ true for all integers

ii) ~~...~~

$(\cos\theta + i\sin\theta)^3 = (\cos 3\theta + i\sin 3\theta)$.

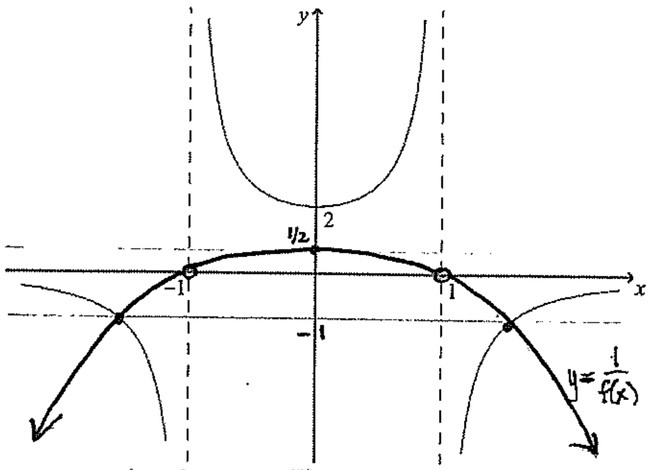
$\cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta \ldots$

equating real part.

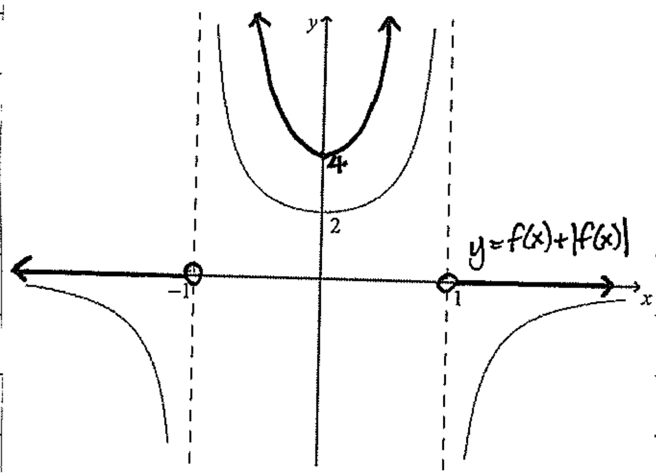
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

Question 3

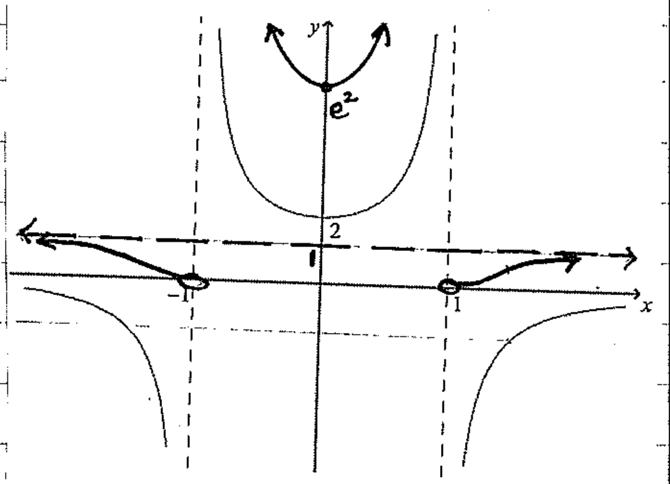
(a)(i)



(ii)



(iii)



(b) (i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\therefore \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$-\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

b(i) (cont.) $\therefore$ gradient at $P(x_1, y_1) = \dfrac{b^2x_1}{a^2y_1}$

$\therefore$ equation of tangent at P

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2yy_1 - a^2y_1^2 = b^2x_1x - b^2x_1^2$$

$$\therefore b^2x_1x - a^2y_1y = b^2x_1^2 - a^2y_1^2$$

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$$ since (x_1, y_1) lies on the hyperbola

(ii) Eq. of tangent at P $\dfrac{x_1x}{a^2} - \dfrac{y_1y}{b^2} = 1$

sim. Eq. of tangent at Q $\dfrac{x_2x}{a^2} - \dfrac{y_2y}{b^2} = 1$

$T(x_0, y_0)$ lies on both

$\therefore \dfrac{x_1x_0}{a^2} - \dfrac{y_1y_0}{b^2} = 1$ P thru' T

$\dfrac{x_2x_0}{a^2} - \dfrac{y_2y_0}{b^2} = 1$ Q thru' T

$\therefore$ Eq. of PQ (chord of contact) is

$\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$ as $P(x_1, y_1)$ and $Q(x_2, y_2)$ both satisfy this equation.

(iii) If PQ passes thru' $S(ae, 0)$ then

$$\frac{x_0ae}{a^2} - \frac{y(0)}{b^2} = 1$$

$$x_0 = \frac{a}{e}$$

$\therefore$ T lies on the directrix

Question 3

(c) $P(x) = 3x^3 + 7x^2 + 11x + 51$

(i) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2 = \alpha\beta\gamma(\alpha+\beta+\gamma)$

$= \frac{-51}{3} \times \frac{-7}{3}$

$= \frac{119}{3}$ OR $39\frac{2}{3}$

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= \left(-\frac{7}{3}\right)^2 - 2 \times \frac{11}{3}$

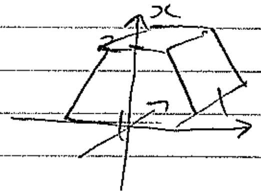
$= \frac{49}{9} - \frac{22}{3}$

$= -\frac{17}{9}$ OR $-1\frac{8}{9}$

(iii) $\alpha^2+\beta^2+\gamma^2 < 0$ $\therefore$ at least one root is unreal.
i.e. at least α^2 or β^2 or $\gamma^2 < 0$
also as the coefficients are all real, the roots occur in conjugates so if one root is unreal its conjugate is also unreal.
As $P(x)$ is degree 3 there is then only one more zero which must be real. Therefore there is exactly one root of $P(x)$ that is real.

Question 4.

(a)



Area $= (\sqrt{3x+1})^2$

$= 3x+1.$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} (3x+1)dx$

$= \int_0^2 (3x+1)dx$

$= \left[\frac{3x^2}{2} + x\right]_0^2$

$= \frac{3}{2}(2)^2 + 2$

$= 8m^3.$

(b)

(i) $y = \frac{4}{x}$

$\frac{dy}{dx} = \frac{-4}{x^2}$

At P,

$\frac{dy}{dx} = \frac{-1}{p^2}.$

$\therefore$ gradient of normal is p^2

$y - \frac{2}{p} = p^2(x-2p)$

$py - 2 = p^3x - 2p^4$

$py - p^3x = 2(1-p^4).$

(ii) $py - p^3x = 2 - 2p^4$

sub $Q(2q, \frac{2}{q})$

$p.\frac{2}{q} - p^3.2q = 2 - 2p^4$

$2p - 2p^3q^2 = 2q - 2qp^4$

$p - p^3q^2 - q + qp^4 = 0$

$qp^4 - p^3q^2 + p - q = 0$

$qp^3(p-q) + (p-q) = 0$

$\therefore \quad qp^3 + 1 = 0 \qquad (p \neq q)$

$qp^3 = -1$

$q = \frac{-1}{p^3}.$

(iii) $q = \frac{-1}{p^3}$

$p = \frac{-1}{q^3}$

$\therefore pq^3 = qp^3$

$pq^3 - qp^3 = 0$

$pq(q^2 - p^2) = 0$

$q^2 - p^2 = 0$

$(q-p)(q+p) = 0$

$q = p, \; q = -p.$

$\therefore$ when $q = p$

$p^4 = -1$

No sol.

$q = -p$

$-p^4 = -1$

$p = \pm 1.$

sub $p = 1,$

$y - x = 0$

$y = x$

sub $p = -1$

$-y + x = 0$

$y = x$

$\therefore$ Normal is $y = x$

Start here.

c) $x^3 + 3x + 2 = 0.$

$y = x^2, \quad x = \alpha, \beta, \gamma.$

$\sqrt{y} = x.$

$(\sqrt{x})^3 + 3(\sqrt{x}) + 2 = 0.$

$x\sqrt{x} + 3\sqrt{x} + 2 = 0.$

$\sqrt{x}(x+3) = -2.$

$x(x^2+6x+9) = 4.$

~~$x^4 + 6x^3 + 9x \ldots = 0$~~

$x^3 + 6x^2 + 9x - 4 = 0.$

ii) $\alpha^3 + \beta^3 + \gamma^3.$ from $x^3 + 3x + 2$

$\alpha^3 + 3\alpha + 2 = 0.$ ①

$\beta^3 + 3\beta + 2 = 0.$ ②

$\gamma^3 + 3\gamma + 2 = 0.$ ③ 2

① + ② + ③

$\alpha^3 + \beta^3 + \gamma^3 = -3(\alpha + \beta + \gamma) - 2.3.$

$= -3(0) \quad -6.$

$= -6.$

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Question 5

a) RTS: $(z-1)(z^2+z+1) = z^3-1$

LHS $= (z-1)(z^2+z+1)$

$= z^3+z^2+z-z^2-z-1$

$= z^3-1$

ii) ω is a complex root of unity

$\omega^3-1=0$

$\therefore (\omega-1)(\omega^2+\omega+1)=0$

But $\omega \neq 1$, $\omega \in \mathbb{C}$

$\therefore \omega^2+\omega+1=0$

ii) RTS: $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8) = 9.$

LHS $= (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

$\omega^4 = \omega^3.\omega$

$= \omega$

$\omega^8 = \omega^3.\omega^3.\omega^2$

$= \omega^2$

$\omega^2+\omega+1=0$

$\omega^2 = -\omega-1$

$\omega+1 = -\omega^2$

$\therefore$ LHS $= (1-\omega)^2(1-\omega^2)^2$

$= (1-2\omega+\omega^2)(1-2\omega^2+\omega^4)$

$= (1-2\omega-\omega-1)(1-2(-\omega-1)+\omega)$

$= (-3\omega)(1+2\omega+2+\omega)$

$= (-3\omega(3+3\omega)$

$= -3\omega(-3\omega^2)$

$= 9\omega^3$

$= 9$

(b)(i) $I = \int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$

Let $x = \tan\theta$ when $x=1$, $\theta = \pi/4$

$\frac{dx}{d\theta} = \sec^2\theta$ $x=\infty$, $\theta = \pi/2$.

$dx = \sec^2\theta\, d\theta$

$\therefore I = \int_{\pi/4}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\tan\theta\sqrt{1+\tan^2\theta}}$

$= \int_{\pi/4}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\tan\theta\sec\theta}$

$= \int_{\pi/4}^{\pi/2} \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$

$= \int_{\pi/4}^{\pi/2} \frac{1}{\sin\theta}\,d\theta$

(ii) $I = \int_{\pi/4}^{\pi/2} \operatorname{cosec}\theta\, d\theta$

$I = \left[-\ln(\operatorname{cosec}x+\cot x)\right]_{\pi/4}^{\pi/2}$

$= \left[\ln(\operatorname{cosec}x+\cot x)\right]_{\pi/2}^{\pi/4}$

$= \ln\left(\operatorname{cosec}\frac{\pi}{4}+\cot\frac{\pi}{4}\right) - \ln\left(\operatorname{cosec}\frac{\pi}{2}+\cot\frac{\pi}{2}\right)$

$= \ln(\sqrt{2}+1) - \ln(1+0)$

$= \ln(\sqrt{2}+1)$,

(c) (i) $x^4+x^3+x^2+x+1 = (x^2+ax+1)(x^2+bx+1)$

$= x^4+bx^3+x^2+ax^3+abx^2+ax+x^2+bx+1$

$= x^4+x^3(a+b)+x^2(2+ab)+x(a+b)+1$

$\therefore (a+b)=1 \quad \ldots (1)$

$2+ab=1 \quad \ldots (2)$

$ab=-1$

$b=\frac{-1}{a}$

$a-\frac{1}{a}=1$

$a^2-1=a$

$a^2-a-1=0$

$a=\frac{1\pm\sqrt{1-4.-1.1}}{2}$

$=\frac{1\pm\sqrt{5}}{2}$

$\therefore a=\frac{1+\sqrt{5}}{2}, \quad b=\frac{1-\sqrt{5}}{2}$ [by symmetry of equation]

(ii) $\therefore \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$ is a soln. of x^2+ax+1 or x^2+bx+1.

$\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^2+a\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)+1=0$

$\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+a\cos\frac{2\pi}{5}+ai\sin\frac{2\pi}{5}=-1$

equating real

$\cos\frac{4\pi}{5}+a\cos\frac{2\pi}{5}=-1. \qquad \cos2\theta=2\cos^2\theta-1.$

$2\cos^2\frac{2\pi}{5}-1+a\cos\frac{2\pi}{5}=-1$

$\cos\frac{2\pi}{5}\left(2\cos\frac{2\pi}{5}+a\right)=0$

$\therefore \cos\frac{2\pi}{5}=\frac{-a}{2}$ (or $-b$)

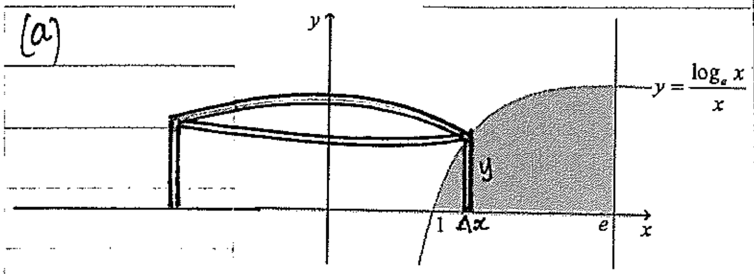
$\cos\frac{2\pi}{5}=-\frac{1-\sqrt{5}}{4}$ or $\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}.$

But $\cos\frac{2\pi}{5}>0$

$\therefore \cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$

Question 6

(a)



Area of Annulus $= \pi x^2 - \pi(x - \Delta x)^2$

$= \pi x^2 - \pi(x^2 - 2x\Delta x + (\Delta x)^2)$ ← too small

$= 2\pi x \Delta x$

$\therefore$ Volume of Shell $= 2\pi x \Delta x y$

$\therefore$ Volume of Solid $= \lim_{\Delta x \to 0} \pi \sum_1^e 2xy\Delta x$

$= 2\pi \int_1^e xy\,dx$

$= 2\pi \int_1^e \frac{x \log_e x}{x}\,dx$

$= 2\pi \int_1^e \log_e x\,dx$

$= 2\pi\ \mathrm{II}$

$= 2\pi \times 1$

$= 2\pi$ units^3

$\mathrm{II} = \int_1^e \ln x\,dx$ let $u = \ln x$ $u' = \frac{1}{x}$

$v' = 1$ $v = x$

$= [x \ln x]_1^e - \int_1^e x \times \frac{1}{x}\,dx$

$= e - [e - 1]$

$= 1$

* Note also $\int_1^e \ln x\,dx = 1$ from definition of area under $\log_e x$ from 1 to e

(b) (i) RTS $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

LHS $= \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= 2\sin\alpha\cos\beta$

$=$ RHS as required.

(ii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta$

$= \sin(2\theta - \theta) + \sin(3\theta - \theta) + \sin(2\theta + \theta) + \sin(3\theta + \theta)$

$= \sin(2\theta+\theta) + \sin(2\theta-\theta) + \sin(3\theta+\theta) + \sin(3\theta-\theta)$

$= 2\sin 2\theta\cos\theta + 2\sin 3\theta\cos\theta$

$= 2\cos\theta(\sin 2\theta + \sin 3\theta)$

$= 2\cos\theta\left(\sin\left(\frac{5\theta}{2} - \frac{\theta}{2}\right) + \sin\left(\frac{5\theta}{2} + \frac{\theta}{2}\right)\right)$

$= 2\cos\theta \times 2\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$

$= 4\cos\theta\sin\frac{5\theta}{2}\cos\frac{\theta}{2}$

$\therefore 4\cos\theta\sin\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$ $\quad 0 \le \theta \le 2\pi$, $0 \le \frac{\theta}{2} \le \pi$, $0 \le \frac{5\theta}{2} \le 5\pi$

When $\cos\theta = 0$ $\quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\cos\frac{\theta}{2} = 0$ $\quad \frac{\theta}{2} = \frac{\pi}{2}$ $\quad \theta = \pi$

$\sin\frac{5\theta}{2} = 0$

$\frac{5\theta}{2} = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

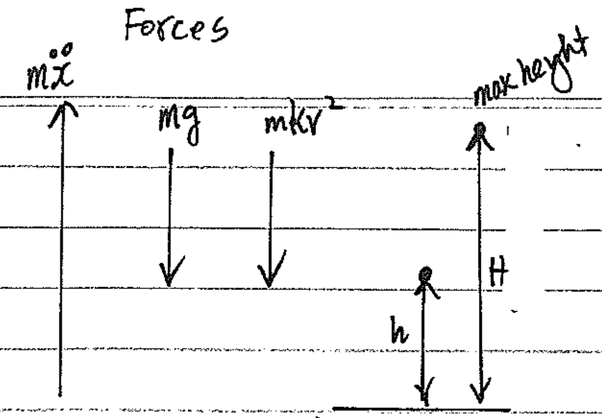
$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

$\therefore$ Solutions for θ are

$0, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, 2\pi$

(c)



$x=h$ $t=0$ speed $=u$

$x=H$ $t=$ final speed $=0$

$$m\ddot{x} = -mg - mkv^2$$

$$\therefore \ddot{x} = -g - kv^2$$

$$v\frac{dv}{dx} = -g - kv^2$$

$$\frac{v\,dv}{-g-kv^2} = dx$$

$$-\int_u^0 \frac{v\,dv}{g+kv^2} = \int_h^H dx$$

$$\int_h^H dx = \int_0^u \frac{v}{g+kv^2}\,dv$$

$$\therefore H - h = \left[\frac{1}{2k}\ln\left(g+kv^2\right)\right]_0^u$$

$$= \frac{1}{2k}\left[\ln(g+ku^2) - \ln g\right]$$

$$= \frac{1}{2k}\left[\ln\frac{g+ku^2}{g}\right]$$

$$= \frac{1}{2k}\left[\ln\left(1+\frac{ku^2}{g}\right)\right]$$

$$\therefore H = h + \frac{1}{2k}\ln\left(1+\frac{ku^2}{g}\right)$$

as required

(d) 1 package: 3 people together; rest: $n-3$ people $= n-3$ packages

$\therefore$ 1 package $+$ $n-3$ packages

$= n-2$ packages of people

no. of ways 3 particular people sit together.

$$= \frac{3! \times (n-2)!}{(n-2)}$$

$\leftarrow$ divide since circle.

$$= 3!\,(n-3)!$$

$$= 6(n-3)!$$

(e) RTS $^{n+2}C_r = {}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$

$$\text{LHS} = {}^{n+2}C_r$$

$$= \frac{(n+2)!}{r!(n-r+2)!}$$

$$\text{RHS} = {}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$$

$$= \frac{n!}{r!(n-r)!} + \frac{2\,n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r-2)!(n-r+2)!}$$

$$= \frac{n!(n-r+1)(n-r+2) + 2n!(n-r+2)r + n!r(r\ldots}{r!(n-r+2)!}$$

$$= \frac{n![n^2-nr+2n-nr+r^2-2r+2-r+n+2rn-2r^2+4r+r^2-r\ldots}{r!(n-r+2)!}$$

$$= \frac{n![n^2+3n+2]}{r!(n-r+2)!}$$

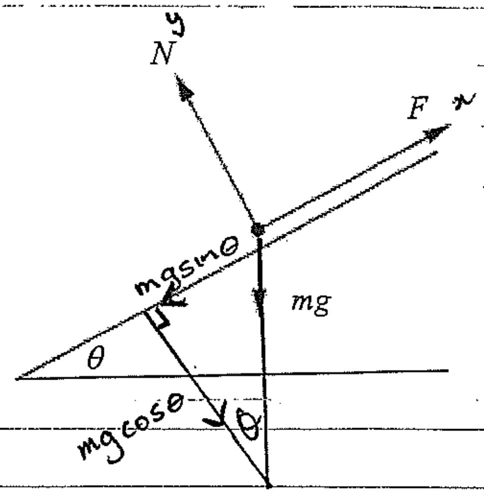
$$= \frac{n!(n+2)(n+1)}{r!(n-r+2)!}$$

$$= \frac{(n+2)!}{r!(n-r+2)!}$$

$= \text{LHS}$ as required.

Q7

(a)



$F_x = F - mg\sin\theta \quad \ldots ①$ parallel

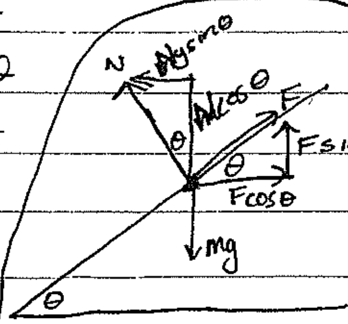
$F_y = N - mg\cos\theta \quad \ldots ②$ perpend

Since not moving $F_x = 0$, $F_y = 0$

$\therefore F = mg\sin\theta$

$N = mg\cos\theta$

$\frac{F}{N} = \tan\theta$



Alternatively

Horizontally

$F\cos\theta - N\sin\theta = 0$ — ①

Vertically

$N\cos\theta + F\sin\theta - mg = 0$ — ②

from ① $F\cos\theta = N\sin\theta$

$\frac{F}{N} = \frac{\sin\theta}{\cos\theta}$

$= \tan\theta$

(b) $P(x) = x^4 - 4x^3 + 3x^2 - 14x + 10$

(i) Roots $a+ib$, $a-ib$, $a-2ib$, $a+2ib$.

sum of roots

$4a = \frac{-(-4)}{1}$

$= 4$

$a = 1$.

Product of Roots:

$(1+ib)(1-ib)(1-2ib)(1+2ib) = 10$

$(1+b^2)(1+4b^2) = 10$

$1 + 4b^2 + b^2 + 4b^4 = 10$

$4b^4 + 5b^2 - 9 = 0$

$4b^4 + 9b^2 - 4b^2 - 9 = 0$

$b^2(4b^2+9) - (4b^2+9) = 0$

$(b^2-1)(4b^2+9) = 0$

$b^2 = 1$ [b real]

$b = \pm 1$.

$\left.\begin{array}{l}\alpha = 1+i\\ \bar{\alpha} = 1-i\end{array}\right\}$ $\alpha + \bar{\alpha} = 2$, $\alpha\bar{\alpha} = 1^2 + i^2$ $= 2$

$\left.\begin{array}{l}\beta = 1+2i\\ \bar{\beta} = 1-2i\end{array}\right\}$ $\beta + \bar{\beta} = 2$, $\beta\bar{\beta} = 1^2 + 2^2$ $= 5$

(ii) $(z-\alpha)(z-\bar{\alpha}) = z^2 - z(\alpha+\bar{\alpha}) + \alpha\bar{\alpha}$

$(z-\beta)(z-\bar{\beta}) = z^2 - z(\beta+\bar{\beta}) + \beta\bar{\beta}$

$\therefore (z^2-2z+2)(z^2-2z+5)$ is the factored form of $P(x)$ over the rational field.

Question 7

(c) (i) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x \, dx$

let

$u = \cos^{n-1} x \qquad v' = \cos x$

$u' = (n-1)\cos^{n-2} x . -\sin x \qquad v = \sin x$

$\therefore \quad I_n = \left[\sin x \cos^{n-1} x\right]_0^{\pi/2}$

$- \int_0^{\pi/2} \sin x (n-1)\cos^{n-2} x . (-\sin x) dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x . \sin^2 x \, dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x (1-\cos^2 x) dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$= (n-1) I_{n-2} - (n-1) I_n$

$\therefore \quad n I_n = (n-1) I_{n-2}$

$\therefore \quad I_n = \frac{n-1}{n} I_{n-2}$

as required.

(ii) $\therefore \int_0^{\frac{\pi}{2}} \cos^5 x \, dx = I_5$

$= \frac{4}{5} I_3$

$= \frac{4}{5}\left(\frac{2}{3} I_1\right)$

$= \frac{4}{5} \times \frac{2}{3} \times 1$

$= \frac{8}{15}$

$I_1 = \int_0^{\pi/2} \cos x \, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= 1$

(d) RTP $1\times 1! + 2\times 2! + 3\times 3! + \dots + n\times n! = (n+1)! - 1$

for $n \geq 1$

Step 1 prove for $n=1$

$LHS = 1\times 1! \qquad RHS = (1+1)! - 1$

$= 1\times 1 \qquad = 2! - 1$

$= 1 \qquad = 1$

$\therefore LHS = RHS \quad \therefore$ true for $n=1$

Step 2 assume true for $n=k$

ie. $1\times 1! + 2\times 2! + \dots + n\times n! = (k+1)! - 1$

Step 3 prove for $n=k+1$

ie. $1\times 1! + 2\times 2! + \dots + (k+1)\times(k+1)! = [(k+1)+1]! - 1$

$LHS = 1\times 1! + 2\times 2! + \dots + k\times k! + (k+1)\times(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)! + (k+1)!(k+1) - 1$

$= (k+1)!\left[1 + (k+1)\right] - 1$

$= (k+1)!(k+2) - 1$

$= (k+2)! - 1$

$= \left[(k+1)+1\right]! - 1$

$= RHS.$

$\therefore$ result is true for $n=k+1$

Step 4 Since result is true for $n=1$, it is also true for $n=2$. Since it is true for $n=2$ it is also true for $n=3$ and so on.

$\therefore$ Statement is true $\forall$ positive integral values of n:

Question 8

(a) $f(x) = \sqrt{3 - \sqrt{x}}$

(i)

Domain $3 - \sqrt{x} \geqslant 0$ and $\sqrt{x} \geqslant 0$

$-\sqrt{x} \geqslant -3 \qquad x \geqslant 0$

$\sqrt{x} \leq 3$

$x \leq 9$

$\therefore \quad 0 \leq x \leq 9$

(ii) $f(x) = \left(3 - x^{1/2}\right)^{1/2}$

$\therefore f'(x) = \frac{1}{2}\left(3 - x^{1/2}\right)^{-1/2} \times -\frac{1}{2}x^{-1/2}$

$= \dfrac{-1}{4\sqrt{x}\sqrt{3-\sqrt{x}}}$

< 0 for $0 < x < 9$

and grad. undefined for

$x = 0$ and $x = 9$

(ie. vertical)

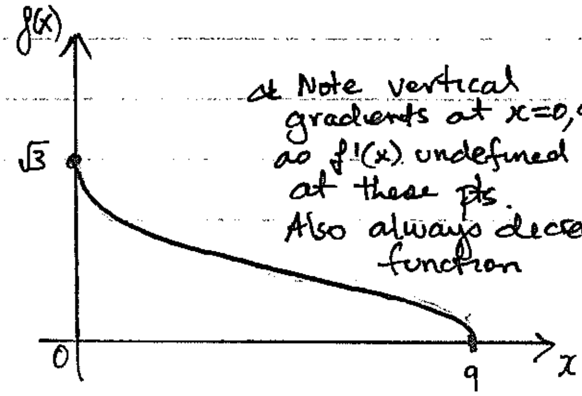
$\therefore f(x)$ is a decreasing function.

at $x = 0$ $f(0) = \sqrt{3}$

at $x = 9$ $f(9) = 0$

$\therefore \quad 0 \leq f(x) \leq \sqrt{3}$

(iii)



Note vertical gradients at $x = 0, 9$ as $f'(x)$ undefined at these pts.
Also always decreasing function

(iv) $\displaystyle\int_0^9 \sqrt{3 - \sqrt{x}}\, dx = \int_3^0 \sqrt{u}\,(-2)(3-u)\,du$

let $u = 3 - \sqrt{x}$

$\sqrt{x} = 3 - u$

$x = (3-u)^2$

$dx = -2(3-u)\,du$

$x = 0 \quad u = 3$

$x = 9 \quad u = 0$

$\displaystyle = 2\int_0^3 \left(3u^{1/2} - u^{3/2}\right) du$

$= 2\left[2u^{3/2} - \dfrac{2u^{5/2}}{5}\right]_0^3$

$= 4\left[3\sqrt{3} - \dfrac{9\sqrt{3}}{5}\right]$

$= 4\sqrt{3}\left[3 - \dfrac{9}{5}\right]$

$= 4\sqrt{3} \times \dfrac{6}{5}$

$= \dfrac{24\sqrt{3}}{5}$ as required.

Question 8

b) RTS: $\frac{d}{du}(\sec u+\tan u)=\sec u(\sec u+\tan u)$

$$LHS = \frac{d}{du}(\sec u+\tan u)$$

$$= \frac{d}{du}\left((\cos u)^{-1}+\tan u\right)$$

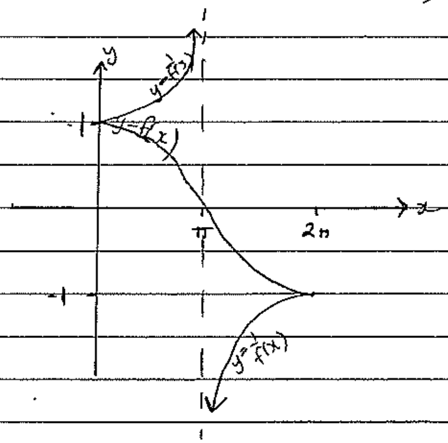
$$= -(\cos u)^{-2}\times -\sin u+\sec^2 u$$

$$= \frac{\sin u}{\cos u}\cdot\frac{1}{\cos u}+\sec^2 u$$

$$= \sec u\cdot\tan u+\sec^2 u$$

$$= \sec u(\sec u+\tan u)$$

c) (i)



(ii) $A=\int_{\pi/3}^{\pi/2}\sec\frac{x}{2}\,dx$

Let $u=\frac{x}{2}$

$\frac{du}{dx}=\frac{1}{2}$

$2du=dx$

When $x=\pi/3$, $u=\pi/6$

When $x=\pi/2$, $u=\pi/4$.

$$=2\int_{\pi/6}^{\pi/4}\sec u\,du$$

$$=2\int_{\pi/6}^{\pi/4}\frac{\sec u(\sec u+\tan u)}{\sec u+\tan u}du$$

$$=2\Big[\ln(\sec u+\tan u)\Big]_{\pi/6}^{\pi/4}$$

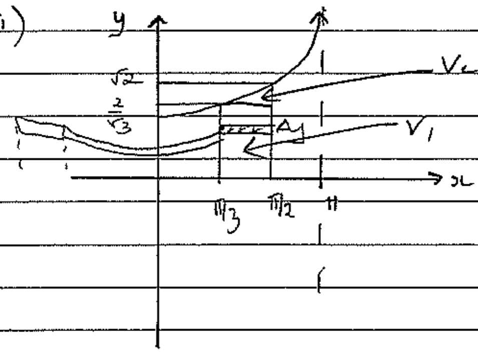
$$=2\left[\ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\ln\left(\sec\pi/6+\tan\frac{\pi}{6}\right)\right]$$

$$=2\left[\ln(\sqrt{2}+1)-\ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)\right]$$

$$=2\left[\ln(\sqrt{2}+1)-\ln\left(\frac{3}{\sqrt{3}}\right)\right]$$

$$=2\ln\left(\frac{\sqrt{2}+1}{\sqrt{3}}\right)\text{ units}^2.$$

(iii)



When $x=\pi/2$

$$y=\frac{1}{\cos\pi/4}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}$$

When $x=\pi/3$

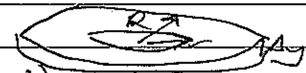
$$y=\frac{1}{\cos\pi/6}=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}$$

$$V_1=\pi\left(\left(\frac{\pi}{2}\right)^2-\left(\frac{\pi}{3}\right)^2\right)\times\frac{2}{\sqrt{3}}$$

$$=\pi\left(\frac{\pi^2}{4}-\frac{\pi^2}{9}\right)\times\frac{2}{\sqrt{3}}$$

$$=\pi\left(\frac{5\pi^2}{36}\right)\times\frac{2}{\sqrt{3}}$$

$$=\frac{10\pi^3}{36\sqrt{3}}\text{ unit}^3$$



$$A_2=\pi\left(\frac{\pi^2}{4}-x^2\right)$$

$$V_2\doteq\sum_{y=\frac{2}{\sqrt{3}}}^{\sqrt{2}}\pi\left(\frac{\pi^2}{4}-x^2\right)\Delta y$$

$$=\pi\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}}\left(\frac{\pi^2}{4}-x^2\right)dy$$

But, $y=\frac{1}{\cos\frac{x}{2}}$

$$\frac{1}{y} = \cos\frac{x}{2}$$

$$\frac{x}{2} = \cos^{-1}\left(\frac{1}{y}\right)$$

$$x = 2\cos^{-1}\left(\frac{1}{y}\right)$$

$$\therefore \; V_2 = \int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \left(\frac{\pi^2}{4} - 4\left[\cos^{-1}\left(\frac{1}{y}\right)\right]^2\right) dy$$

$$\therefore \; V = \frac{10\pi^3}{36\sqrt{3}} + \int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \left(\frac{\pi^2}{4} - 4\left[\cos^{-1}\left(\frac{1}{y}\right)\right]^2\right) dy$$