HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2019

STUDENT NUMBER: ____________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- NESA-approved calculators and drawing templates may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 8 – 18

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 i^{2019} simplifies to

(A) 1

(B) -1

(C) i

(D) $-i$

2. The hyperbola $\frac{x^2}{\lambda-3}-\frac{y^2}{\lambda+2}=1$ has an asymptotic equation $y=\frac{3}{2}x$

The value of λ for this equation is:

(A) -1

(B) 2

(C) 6

(D) 7

3 The locus defined by $z\bar{z}+3(z+\bar{z})<0$ is the region inside the circle

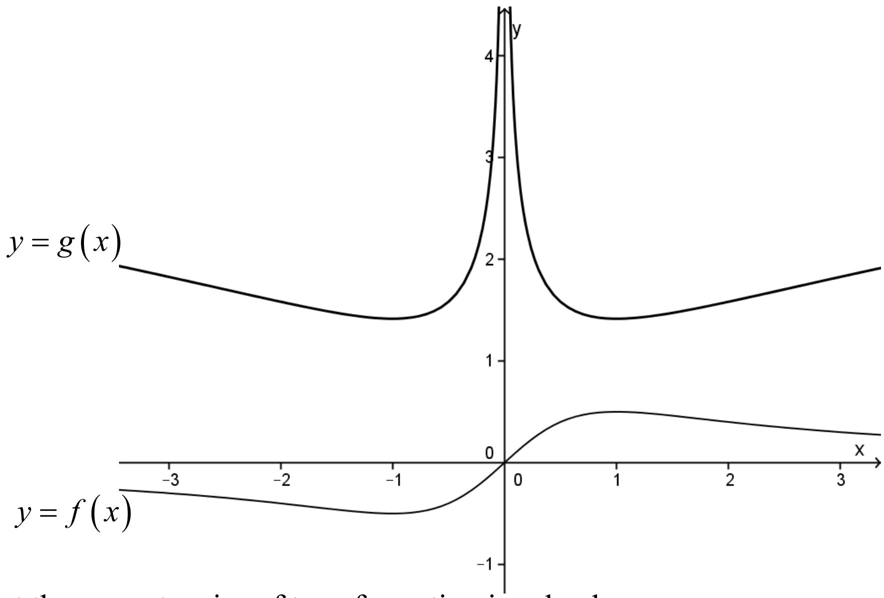
(A) $(x-3)^2+y^2=9$

(B) $(x+3)^2+y^2=9$

(C) $x^2+(y-3)^2=9$

(D) $x^2+(y+3)^2=9$

4. The graphs below are functions $y = f(x)$ and $y = g(x)$ where $y = g(x)$ is the outcome of the original function $y = f(x)$ undergoing a series of transformation.



Select the correct series of transformation involved.

(A) $g(x) = |x| + \dfrac{1}{|f(x)|}$

(B) $g(x) = \dfrac{1}{\left|\sqrt{f(x)}\right|}$

(C) $g(x) = \dfrac{1}{\sqrt{f(|x|)}}$

(D) $g(x) = |x| + \dfrac{1}{f(|x|)}$

5 Find the value of k when $P(x) = x^3 - kx^2 - 10kx + 24$ has a factor of $(x+2)$.

(A) 1

(B) -1

(C) $\dfrac{1}{2}$

(D) $-\dfrac{1}{2}$

6. The possible roots of $P(x) = a_0x^0 + a_1x^1 + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n = 0$ could be:

(A) $\pm 1,\ \pm a_n,\ \pm \frac{1}{a_n}, ...$

(B) $\pm 1,\ \pm a_n,\ \pm \frac{a_0}{a_n}, ...$

(C) $\pm 1,\ \pm a_0,\ \pm \frac{a_n}{a_0}, ...$

(D) $\pm 1,\ \pm a_0,\ \pm \frac{1}{a_n}, ...$

7 Consider the function $f(x) = \frac{e^x - 1}{e^x + 1}$. Which of the following is correct?

(A) $f(x)$ is even and increasing

(B) $f(x)$ is odd and increasing

(C) $f(x)$ is even and decreasing

(D) $f(x)$ is odd and decreasing

8. If $\int_{-a}^{a} f(x)\,dx = 0$ and $\int_{0}^{a} f(a-x)\,dx = \int_{0}^{a} f(x)\,dx$, which of the following functions below possess both of these properties for $a = \pi$?

(A) $f(x) = x\sin^2 x$

(B) $f(x) = x^2 \cos x$

(C) $f(x) = e^x \cos^2 x$

(D) $f(x) = \frac{e^x}{1+e^x}\cos x$

9 If a car with mass M , moving with velocity v is opposed by wind resistance αv^2 and road frictional force β , where α and β are constants, then

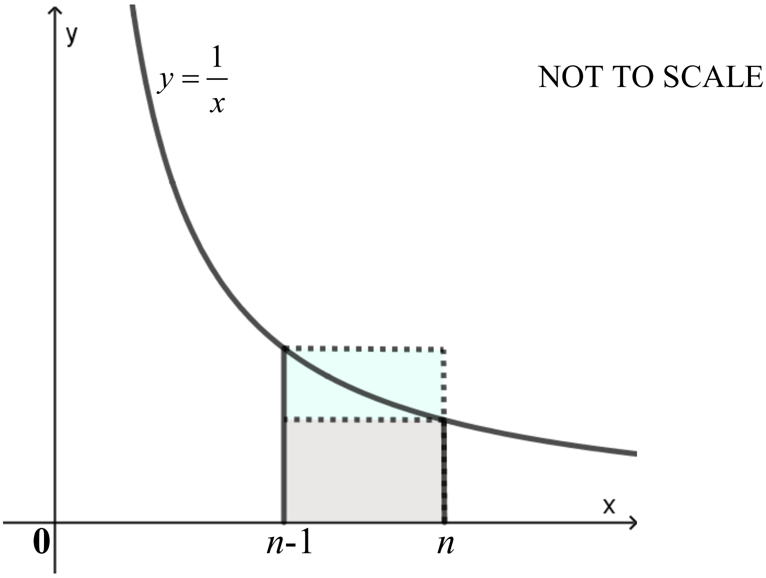
(A) $\frac{dv}{dx} = -\frac{1}{M}(\alpha v + \frac{\beta}{v})$

(B) $\frac{dv}{dx} = \frac{1}{M}(\alpha v + \frac{\beta}{v})$

(C) $\frac{dv}{dx} = -\frac{1}{M}(\alpha v + \beta)$

(D) $\frac{dv}{dx} = \frac{1}{M}(\alpha v + \beta)$

10. Let n be a positive integer greater than 1. Which of the statements below best describe the area of the region under the curve $y = \frac{1}{x}$, $x > 0$ from $x = n-1$ to $x = n$.



(A) $\frac{1}{n} < \ln x < \frac{1}{n+1}$

(B) $\frac{1}{n} \le \ln x \le \frac{1}{n+1}$

(C) $e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$

(D) $e^{-\frac{n}{n-1}} \le \left(1 - \frac{1}{n}\right)^n \le e^{-1}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) Let $z = 5 + 3i$ and $w = -3 + 2i$, find in the form $a + ib$ where a and b are real

(i) $z\overline{w}$ **1**

(ii) $\dfrac{2}{iw}$ **2**

(b) Find the equations of the asymptotes and vertices of the hyperbola **2**

$$\frac{y^2}{12} - \frac{x^2}{4} = 1$$

(c) Sketch the region on the Argand Diagram such that **2**

$$2 < |z| < 3 \text{ and } \frac{\pi}{6} < \arg z < \frac{\pi}{2}$$

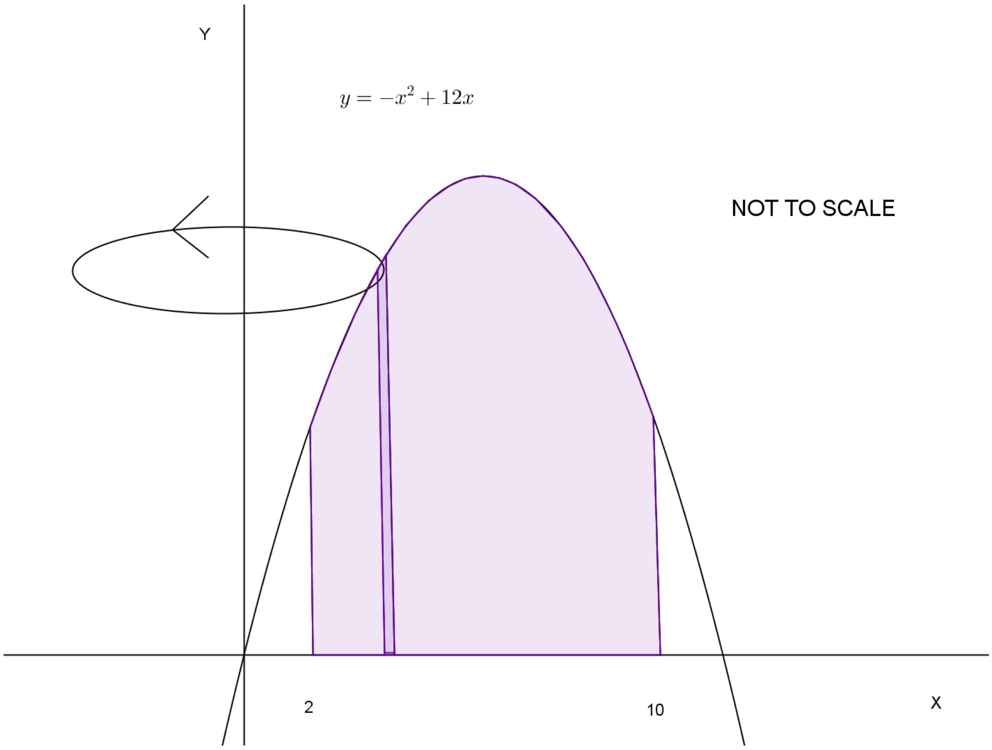
(d) Use the substitution $t = \tan\dfrac{x}{2}$ to evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{13 + 5\sin x + 12\cos x}$$

Question 11 continues on page 9

Question 11 (continued)

(e) The region on the diagram below, between the curve $y = 12x - x^2$, the x axis, $x = 2$ and $x = 10$ is rotated about the y axis. Use the method of cylindrical shells to find the volume of the solid. **3**

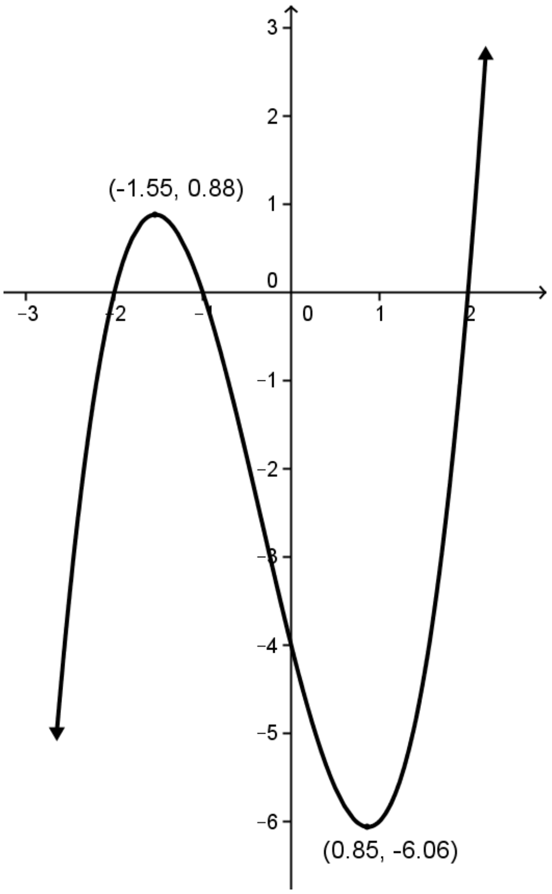


(f) Evaluate $\displaystyle\int_{-3}^{5} \frac{x+7}{\sqrt{x+4}}\,dx$ **2**

End of Question 11.

Question 12 (15 marks) Start a new writing booklet

(a) The graph of $y = f(x)$ is shown.



Using the templates provided construct the following transformations of $f(x)$.

(i) $y = f(|x|)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \tan^{-1} f(x)$ **2**

(b) (i) Express $\sqrt{8-6i}$ in the form of $a+ib$ where a and b are real and $a > 0$. **2**

(ii) Hence solve the quadratic equation $2z^2 + (1-3i)z - 2 = 0$, expressing the answers in the form $c + id$, where c and d are real. **1**

Question 12 continues on page 11

(c) Find $\int_0^{\pi} e^{2x} \sin x \; dx$ **3**

(d) Let $P(x)$ be a polynomial.

(i) Given that $P(x)$ has a root α of multiplicity 3, show that $P(\alpha) = P''(\alpha) = 0$. **2**

(ii) Given that $P(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$ has the factor $(x-2)^3$. Find the other root. **2**

End of Question 12.

Question 13 (15 marks) Start a new writing booklet

(a) Show that if $x \geq 0, y \geq 0$ then

(i) $x^2 + y^2 \geq 2xy$ **1**

(ii) $x^3 + y^3 \geq xy(x+y)$ **2**

(iii) hence $2(x^3 + y^3 + z^3) \geq xy(x+y) + yz(y+z) + xz(x+z)$ **2**

(b) A particle of Unit mass is projected vertically upwards against gravitational force mg and resistance $\frac{mv}{k}$ where v is the velocity of the particle and k is a constant.

Thus the motion in the upward direction is given by

$$m\ddot{x} = -mg - \frac{mv}{k}, \quad \text{where } x \text{ is the displacement.}$$

(DO NOT PROVE THIS RESULT)

Initially, the particle has zero displacement and velocity $v_0 = k(h-g)$.

(i) Show that the time (t) of the motion is given by **2**

$$t = k\ln\left(\frac{kh}{kg+v}\right)$$

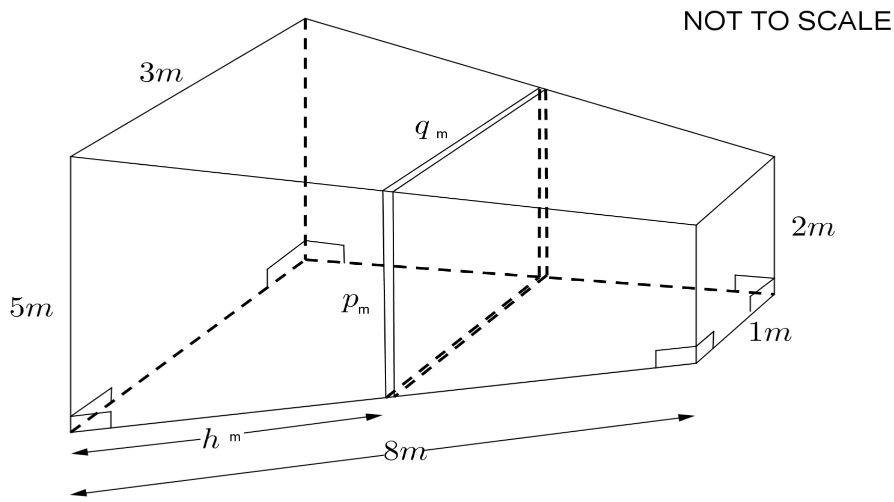
(ii) Show the maximum height (H) of the particle is **3**

$$H = k\left[k(h-g) + kg\ln\left(\frac{g}{h}\right)\right]$$

Question 13 continues on page 13

Question 13 (continued)

(c) A wooden beam of length 8 metres has plane sides with cross-sections parallel to the rectangular ends with dimensions as shown in the diagram below.



(i) Show $p = 5 - \frac{3h}{8}$ and $q = 3 - \frac{h}{4}$ **2**

(ii) Calculate the area of the cross-section in terms of h **1**

(iii) Calculate the Volume of the beam **2**

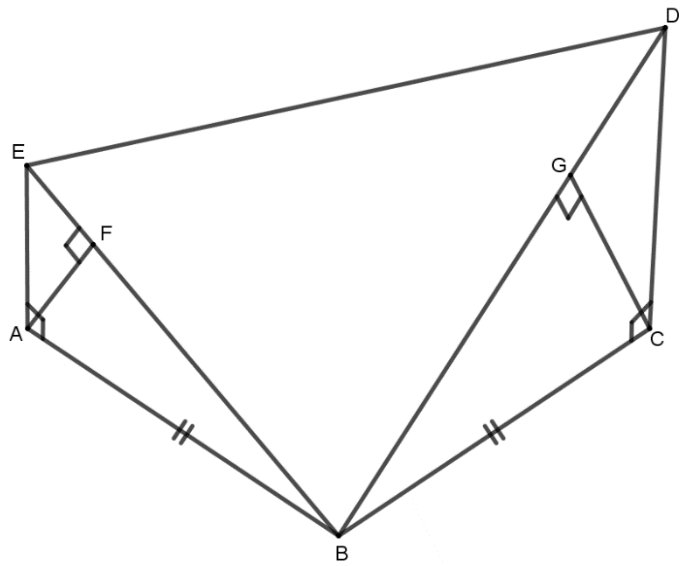
End of Question 13

Question 14 (15 marks)

(a) (i) It is given that $\dfrac{x}{x^3-8} \equiv \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+4}$. Find the values of A, B and C. **1**

(ii) Hence, or otherwise find $\displaystyle\int \frac{x}{x^3-8}\,dx$. **3**

(b) $ABCDE$ is a two dimensional convex polygon such that $AB = BC$, $\angle BCD = \angle EAB = 90^\circ$, $AF \perp EB$, $BD \perp CG$.



NOT TO SCALE

(i) By using similar triangles prove that $AB^2 = BF.BE$. **1**

(ii) Hence, assuming that $BC^2 = BG.BD$, prove that $\Delta BEG \,|||\, \Delta BDF$. **2**

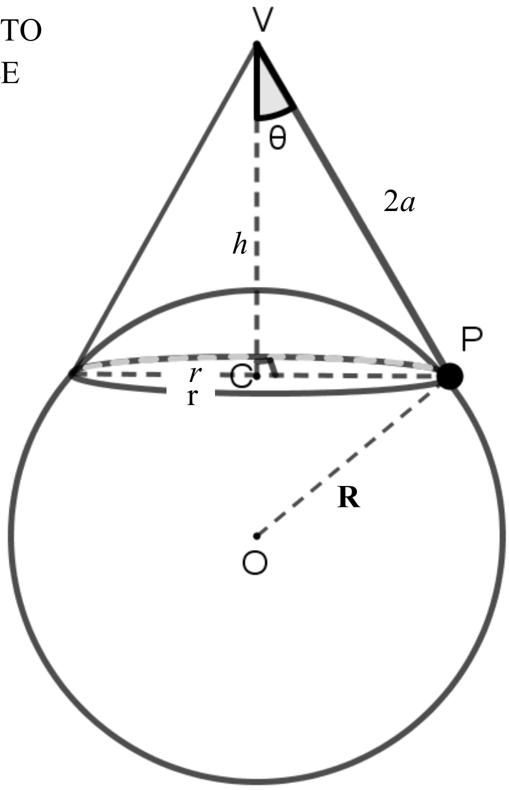
(iii) Show that $DEFG$ is concyclic. **2**

Question 14 continues on page 15

Question 14 (continued)

(c) A particle P, of mass $2m$ kg, at the end of a light inextensible string of length $2a$ metres, is held h metres at V, vertically above point C, the centre of the circular path of the particle which rests on a smooth sphere of radius R metres.

NOT TO SCALE



The string forms a semi vertical angle θ with the vertical. The particle follows a radius r metres on the surface of the sphere with a uniform angular speed of ω radians/second on the outside of the sphere and in contact with it, as shown on the diagram.

(i) Show that the tension (T) in the string, in Newtons is **2**

$$T = 2m\left(g\cos\theta + 2a\omega^2\sin^2\theta\right).$$

(ii) Show the normal force (N) on P, in Newtons is **2**

$$N = 2m\left(g\sin\theta - 2a\omega^2\cos\theta\sin\theta\right).$$

(iii) Show that, for the particle to remain in uniform circular motion on the surface of the surface of the sphere, then $\omega < \left(\dfrac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$, where g is acceleration due to gravity. **2**

End of Question 14

Question 15 (15 marks) Start a new writing booklet

(a) Given $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$ for $n = 1,\ 2,\ 3.....$

(i) Show that $I_n = \frac{2n}{2n+3} I_{n-1}$ **3**

(ii) Hence Evaluate $\int_0^1 x^3 \sqrt{1-x}\,dx$ **2**

(b) Consider the curve $x^2 + y^2 + xy = 3$

(i) Show that $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. **1**

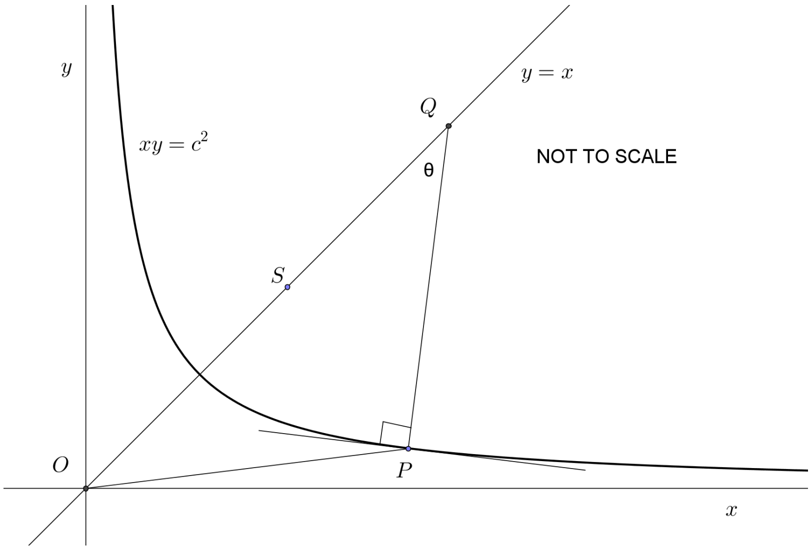
(ii) Deduce the curve has vertical tangents at $(-2,1)$ and $(2,-1)$ and horizontal tangents at $(-1,2)$ and $(1,-2)$. **2**

(iii) Sketch the curve showing these tangents. **2**

Question 15 continues on page 17

Question 15 (continued)

(c) In the following diagram P is the point $P(ct, \frac{c}{t})$ on the rectangular hyperbola $xy = c^2$, where $t > 0$.



The normal to the hyperbola at P meets the line $y = x$ at Q.
The acute angle between PQ and the line $y = x$ is θ.
S is the focus of the hyperbola nearest to P.

(i) Show $\tan\theta = \left|\frac{t^2 - 1}{1 + t^2}\right|$. **1**

(ii) Show PQ and PO are equally inclined to $y = x$. **2**

(iii) If PS is perpendicular to $y = x$, show that $\tan\theta = \frac{1}{\sqrt{2}}$ (Hint: consider $\tan^2\theta$) **2**

End of Question 15

Question 16 (15 marks)

(a) (i) By considering the expansion of $(\cos\theta + i\sin\theta)^5$ and by using De Moivre's Theorem show that **2**

. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(ii) Hence find all the four roots of the equation **2**

$$16x^4 - 20x^2 + 5 = 0.$$

(iii) Hence, or otherwise, show that **3**

$$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}.$$

(iv) Find the exact value of **2**

$$\sin\frac{3\pi}{5}\sin\frac{6\pi}{5}.$$

(b) (i) Show that $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$. **1**

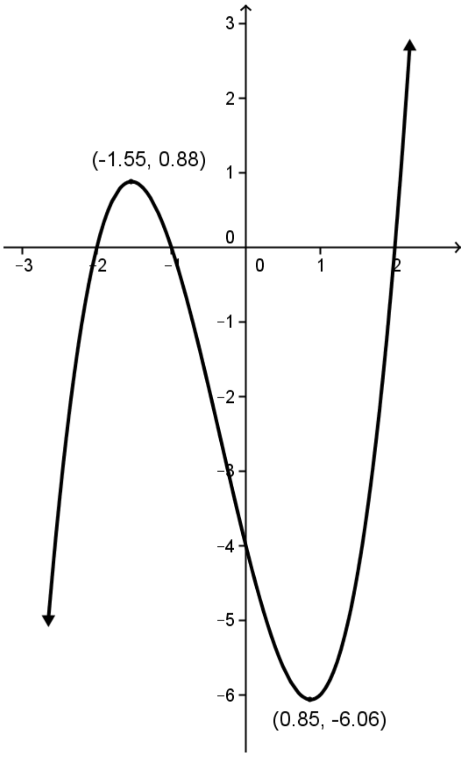
(ii) Hence, or otherwise, find $\int \cos nx \cos mx \ dx, \quad n > m > 0$ **2**

(iii) Find the exact value of $\sum_{r=1}^{r=9} \int_0^{\frac{\pi}{2}} \sin rx \sin x \ dx$. **3**

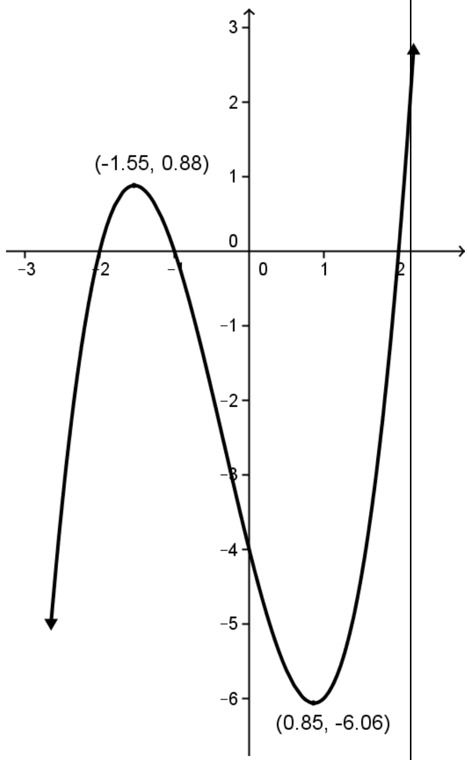
End of Examination

Template for Question 12(a)

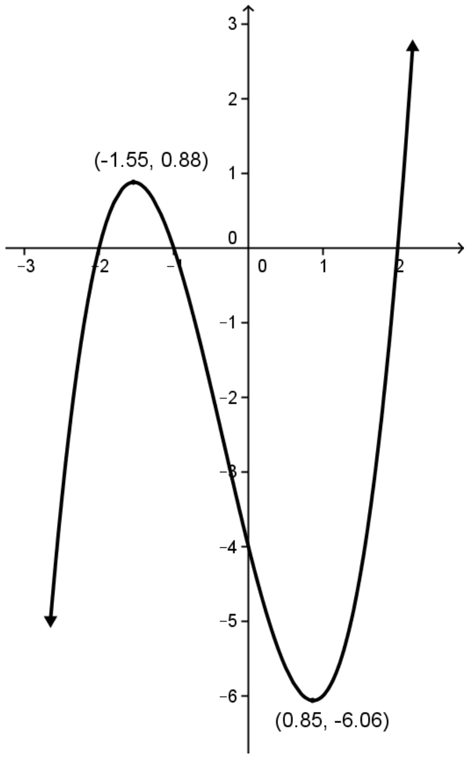
(i) $y = f\left(|x|\right)$



(ii) $y = \dfrac{1}{f\left(x\right)}$



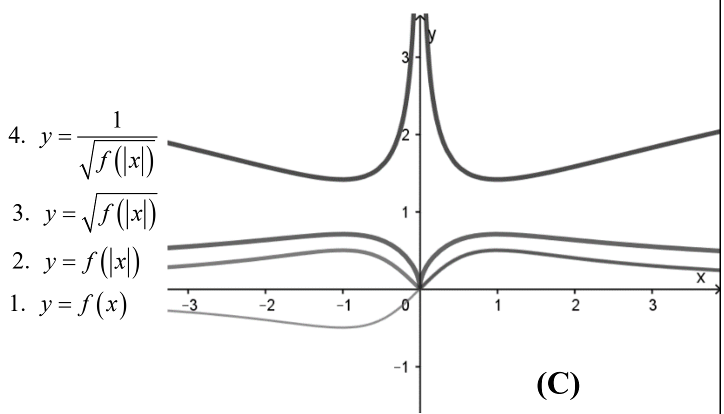
(ii) $y = \tan^{-1} f\left(x\right)$



(iii)

Year 12 Mathematics Extension 2 Trial Term 3 2019 Solutions

MULTIPLE CHOICE

Solution	Comment
1. $\frac{2019}{4} = 504\,r3$ $i^{2019} = i^3$ $= -i$ **(D)**	
2. $\frac{x^2}{\lambda-3} - \frac{y^2}{\lambda+2} = 1$ $\begin{cases} a^2 = \lambda - 3 \\ b^2 = \lambda + 2 \end{cases}$ $\frac{b}{a} = \frac{3}{2}$ $\left(\frac{b}{a}\right)^2 = \left(\frac{3}{2}\right)^2$ $\frac{\lambda+2}{\lambda-3} = \frac{9}{4}$ $4(\lambda+2) = 9(\lambda-3)$ $4\lambda + 8 = 9\lambda - 27$ $35 = 5\lambda$ $\therefore \lambda = 7$ **(D)**	
3. $z\bar{z} + 3(z+\bar{z}) < 0$ $(x+iy)(x-iy) + 3(x+iy+x-iy) < 0$ $x^2 + y^2 + 6x < 0$ $x^2 + 6x + 9 + y^2 < 9$ $(x+3)^2 + y^2 < 9$ **(B)**	
4.  **(C)**	
5. $P(-2) = -8 - 4k + 20k + 24 = 0$ $16k + 16 = 0$ $k = -1$ **(B)**	

Solution	**Comment**
6. $P(x)=a_0x^0+a_1x^1+a_2x^2+...+a_{n-1}x^{n-1}+a_nx^n=0$ Possible roots must be of the form $\dfrac{\pm \textit{the factors of } a_0\ (i.e\ \pm 1, \pm a_0)}{\pm \textit{the factors of } a_n\ (i.e\ \pm 1, \pm a_n)}$ **(D)**	
7. $f(x)=\dfrac{e^x-1}{e^x+1}$ $f(-x)=\dfrac{e^{-x}-1}{e^{-x}+1}$ $=\dfrac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}$ $=\dfrac{\frac{1-e^x}{e^x}}{\frac{1+e^x}{e^x}}$ $=\dfrac{1-e^x}{1+e^x}$ $f(-x)=-f(x)$ odd $f'(x)=\dfrac{e^x(e^x+1)-e^x(e^x-1)}{(e^x+1)^2}$ $=\dfrac{2e^x}{(e^x+1)^2}$ >0 for all x (increasing) **(B)**	
8. $\int_{-a}^{a} f(x)\,dx=0$ $\rightarrow$ odd function i.e. only A $\int_0^a f(a-x)\,dx=\int_0^a f(x)\,dx \rightarrow$ A, B only. **(A)**	

9.  $M\ddot{x} = -\alpha v^2 - \beta$ $\ddot{x} = -\frac{1}{M}\left(\alpha v^2 + \beta\right)$ $v\frac{dv}{dx} = -\frac{1}{M}\left(\alpha v^2 + \beta\right)$ $\frac{dv}{dx} = -\frac{1}{M}\left(\alpha v + \frac{\beta}{v}\right)$ **(A)**	
10. Area of small rectangle $A_1 = \frac{1}{n}\ u^2$ Area of large rectangle $A_2 = \frac{1}{n-1}\ u^2$ $\int_{n-1}^{n} \frac{1}{x}\,dx = [\ln x]_{n-1}^{n}$ $= \ln(n) - \ln(n-1)$ $= \ln\left(\frac{n}{n-1}\right)$ $A_1 < \int_{n-1}^{n} \frac{1}{x}\,dx < A_2$ $\frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$ $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right)$ or $\ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$ $\frac{1}{n} < -\ln\frac{n-1}{n}$ $\quad -\ln\left(\frac{n-1}{n}\right) < \frac{1}{n-1}$ $-\frac{1}{n} > \ln\left(1-\frac{1}{n}\right)$ $\quad \ln\left(1-\frac{1}{n}\right) > -\frac{1}{n-1}$ $e^{-\frac{1}{n}} > \left(1-\frac{1}{n}\right)$ $\quad \left(1-\frac{1}{n}\right) > e^{-\frac{1}{n-1}}$ $\therefore e^{-1} > \left(1-\frac{1}{n}\right)^n$ $\quad \therefore \left(1-\frac{1}{n}\right)^n > e^{-\frac{n}{n-1}}$ **(C)**	

Question 11 Solutions

(a)(i)

$$z\,\overline{w} = (5+3i)(-3-2i)$$
$$= -15 - 9i - 10i - 6i^2$$
$$= -15 - 9i - 10i + 6$$
$$= -9 - 19i$$

(ii)

$$\frac{2}{iw} = \frac{2}{i(-3+2i)}$$
$$= \frac{2}{-3i-2} \times \frac{(-2+3i)}{(-2+3i)}$$
$$= \frac{2(-2+3i)}{4+9}$$
$$= \frac{-4+6i}{13}$$
$$= -\frac{4}{13} + \frac{6i}{13}$$

(b)

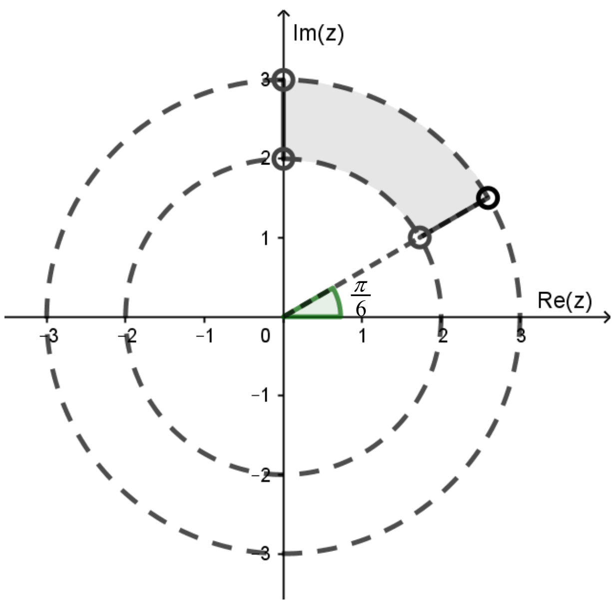
$$\frac{y^2}{12} - \frac{x^2}{4} = 1 \quad \text{where } a = 2\sqrt{3}$$
$$b = 2$$

Equation of asymptotes $x = \pm \frac{b}{a} y$

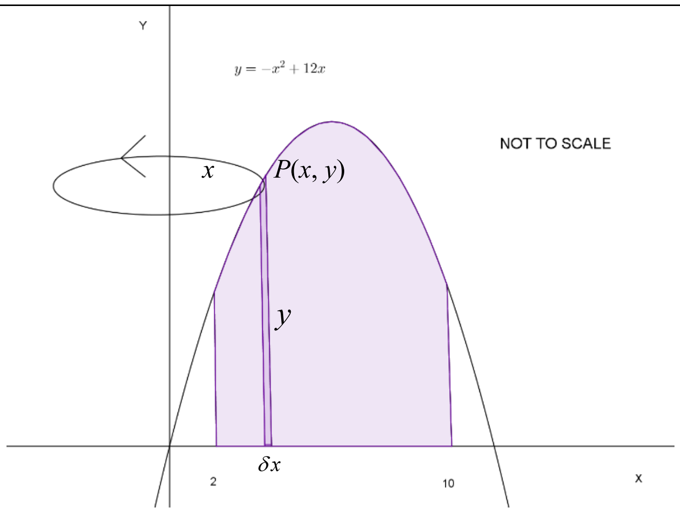
$$x = \pm \frac{2}{2\sqrt{3}} y$$
$$x = \pm \frac{1}{\sqrt{3}} y$$
$$\therefore y = \pm \sqrt{3} x$$

$\therefore$ Vertices $\left(0,\ \pm 2\sqrt{3}\right)$

(c)



Question 11 Solutions	
(d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{13+5\sin x+12\cos x}$ Let $t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$ $dx=\frac{2}{1+t^2}\,dt$ When $x=0,\ t=0$ $x=\frac{\pi}{2},\ t=1$ $\displaystyle=\int_0^1 \frac{\frac{2}{1+t^2}\,dt}{13+5\left(\frac{2t}{1+t^2}\right)+12\left(\frac{1-t^2}{1+t^2}\right)}$ $\displaystyle=\int_0^1 \frac{\frac{2}{1+t^2}\,dt}{13+\frac{10t}{1+t^2}+\frac{12-12t^2}{1+t^2}}$ $\displaystyle=\int_0^1 \frac{2\,dt}{13\left(1+t^2\right)+10t+12-12t^2}$ $\displaystyle=\int_0^1 \frac{2\,dt}{t^2+10t+25}$ $\displaystyle=\int_0^1 \frac{2\,dt}{(t+5)^2}$ $\displaystyle=\int_0^1 2(t+5)^{-2}\,dt$ $=-\left[2(t+5)^{-1}\right]_0^1$ $=\left[\frac{2}{t+5}\right]_1^0$ $=\frac{2}{(0)+5}-\frac{2}{(1)+5}$ $=\frac{2}{5}-\frac{1}{3}$ $=\frac{1}{15}$	

Question 11 Solutions	
(e)  S.A. of hollow cylinder $= 2\pi xy$ $\delta V = 2\pi xy\,\delta x$ where $y = 12x - x^2$ $\delta V = 2\pi x\left(12x - x^2\right)\delta x$ Volume of solid $V = \lim_{\delta x \to 0} \sum_{x=2}^{10} 2\pi x\left(12x - x^2\right)\delta x$ $= 2\pi \int_2^{10} 12x^2 - x^3 \; dx$ $= 2\pi\left[4x^3 - \frac{x^4}{4}\right]_2^{10}$ $= 2\pi\left[\left(4000 - \frac{10\,000}{4}\right) - \left(32 - \frac{16}{4}\right)\right]$ $= 2944\pi\ u^3$	
(f) $\int_{-3}^{5} \frac{x+7}{\sqrt{x+4}}\,dx$ $= \int_{-3}^{5} \frac{x+4+3}{\sqrt{x+4}}\,dx$ $= \int_{-3}^{5} \frac{x+4}{\sqrt{x+4}} + \frac{3}{\sqrt{x+4}}\,dx$ $= \int_{-3}^{5} (x+4)^{\frac{1}{2}} + 3(x+4)^{-\frac{1}{2}}\,dx$ $= \left[\frac{2(x+4)^{\frac{3}{2}}}{3} + 6(x+4)^{\frac{1}{2}}\right]_{-3}^{5}$	

Question 11 Solutions	
(f)cont.	

$$=\left[\frac{2(5+4)^{\frac{3}{2}}}{3}+6(5+4)^{\frac{1}{2}}\right]-\left[\frac{2(-3+4)^{\frac{3}{2}}}{3}+6(-3+4)^{\frac{1}{2}}\right]$$

$$=\left[\frac{2(27)}{3}+6(3)\right]-\left[\frac{2(1)}{3}+6(1)\right]$$

$$=36-6\frac{2}{3}$$

$$=29\frac{1}{3}$$

OR $\int_{-3}^{5}\frac{x+7}{\sqrt{x+4}}\,dx$ Let $u^2=x+4$

$$x=u^2-4$$
$$dx=2u\,du$$

When $x=5,\ u=3$

$x=-3,\ u=1$

$$=\int_{1}^{3}\frac{u^2+3}{u}2u\,du$$

$$=2\int_{1}^{3}u^2+3\,du$$

$$=2\left[\frac{u^3}{3}+3u\right]_1^3$$

$$=2\left[\left(\frac{(3)^3}{3}+3(3)\right)-\left(\frac{(1)^3}{3}+3(1)\right)\right]$$

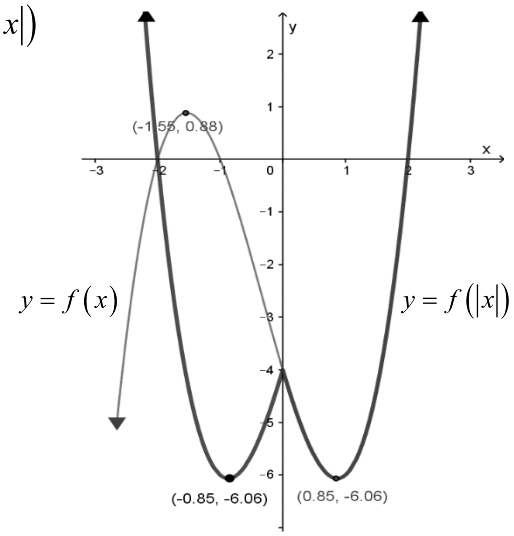
$$=2\left[\left(\frac{27}{3}+9\right)-\left(\frac{1}{3}+3\right)\right]$$

$$=2\left[14\frac{2}{3}\right]$$

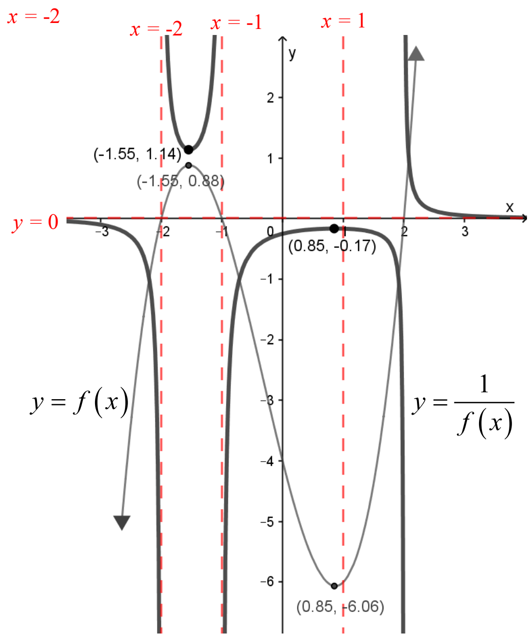
$$=29\frac{1}{3}$$

Question 12 Solutions

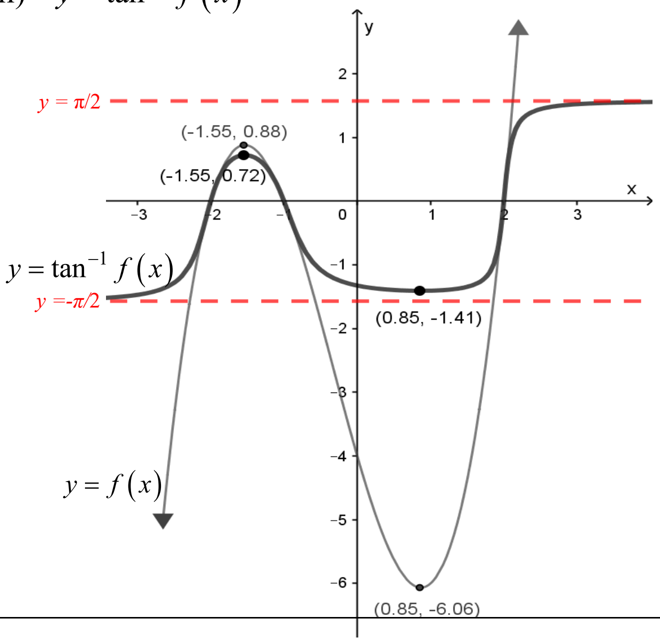
(a)(i) $y = f(|x|)$



(ii) $y = \dfrac{1}{f(x)}$



(iii) $y = \tan^{-1} f(x)$



Question 12 Solutions	
(b)(i) Let $x+iy=\sqrt{8-6i}$ $(x+iy)^2=8-6i$ $x^2+2xyi+y^2i^2=8-6i$ $x^2-y^2+2xyi=8-6i$ Equating like terms $x^2-y^2=8$ — (1) $2xy=-6$ $y=-\frac{3}{x}$ — (2) Sub (2) into (1), $x^2-\left(-\frac{3}{x}\right)^2=8$ $x^2-\frac{9}{x^2}=8$ $x^4-9=8x^2$ $x^4-8x^2-9=0$ $(x^2-9)(x^2+1)=0$ Since x is the real component of the complex number, $\therefore x^2=9$ $\therefore x=\pm 3$ Sub $x=\pm 3$ into (2), $\therefore y=\mp 1$ $\therefore \sqrt{8-6i}=3-i$ or $-3+i$ (ii) $2z^2+(1-3i)z-2=0$ $a=2$ $b=1\text{-}3i$ $c=\text{-}2$ $z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $=\frac{-(1-3i)\pm\sqrt{(1-3i)^2-4(2)(-2)}}{2(2)}$ $=\frac{-1+3i\pm\sqrt{1-6i-9+16}}{4}$ $=\frac{-1+3i\pm\sqrt{8-6i}}{4}$ $z=\frac{-1+3i+(3-i)}{4}$ or $=\frac{-1+3i+(-3+i)}{4}$ $z=\frac{2+2i}{4}$ or $=\frac{-4+4i}{4}$ $\therefore z=\frac{1}{2}+\frac{1}{2}i$ or $-1+i$	

Question 12 Solutions	
(c) Let $I = \int_0^{\pi} e^{2x} \sin x \ dx$ $= [uv]_0^{\pi} - \int_0^{\pi} u'v \ dx$ where $u = e^{2x}$ and $v' = \sin x$ $u' = 2e^{2x}$ $v = -\cos x$ $= \left[-e^{2x}\cos x\right]_0^{\pi} - (-)2\int_0^{\pi} e^{2x}\cos x \ dx$ $= \left[-e^{2(\pi)}\cos(\pi) - \left(-e^{2(0)}\cos(0)\right)\right] + 2\int_0^{\pi} e^{2x}\cos x \ dx$ $= \left[-e^{2\pi}(-1) + (1\times 1)\right] + 2\int_0^{\pi} e^{2x}\cos x \ dx$ $= \left(e^{2\pi}+1\right) + 2\int_0^{\pi} e^{2x}\cos x \ dx$ $= \left(e^{2\pi}+1\right) + 2\left[[uv]_0^{\pi} - \int_0^{\pi} u'v \ dx\right]$ where $u = e^{2x}$ and $v' = \cos x$ $u' = 2e^{2x}$ $v = \sin x$ $= \left(e^{2\pi}+1\right) + 2\left[\left[e^{2x}\sin x\right]_0^{\pi} - 2\int_0^{\pi} e^{2x}\sin x \ dx\right]$ $= \left(e^{2\pi}+1\right) + 2\left[\left[e^{2(\pi)}\sin(\pi) - e^{2(0)}\sin(0)\right] - 2I\right]$ $= \left(e^{2\pi}+1\right) + 2\left[[0-0] - 2I\right]$ $I = \left(e^{2\pi}+1\right) - 4I$ $5I = \left(e^{2\pi}+1\right)$ $\therefore I = \frac{e^{2\pi}+1}{5}$	
(d) Let $P(x) = (x-\alpha)^3 Q(x)$ with α being the root of multiplicity of 3 $P(\alpha) = (\alpha-\alpha)^3 Q(\alpha)$ $P(\alpha) = (0)^3 Q(\alpha)$ $\therefore P(\alpha) = 0$	

Question 12 Solutions	
(d)(i)cont. $P(x)=(x-\alpha)^3 Q(x)$ $P'(x)=u'v+v'u$ where $u=(x-\alpha)^3$ $u'=3(x-\alpha)^2$ $v=Q(x)$ $v'=Q'(x)$ $P'(x)=3(x-\alpha)^2 Q(x)+(x-\alpha)^3 Q'(x)$ $=(x-\alpha)^2\left[3Q(x)+(x-\alpha)Q'(x)\right]$ and let $M(x)=3Q(x)+(x-\alpha)Q'(x)$ $\therefore P'(x)=(x-\alpha)^2 M(x)$ $P''(x)=u'v+v'u$ where $u=(x-\alpha)^2$ $u'=2(x-\alpha)$ $v=M(x)$ $v'=M'(x)$ $P''(x)=2(x-\alpha)M(x)+(x-\alpha)^2 M'(x)$ $=(x-\alpha)\left[2M(x)+(x-\alpha)M'(x)\right]$ $\therefore P''(\alpha)=(\alpha-\alpha)\left[2M(\alpha)+(\alpha-\alpha)M'(\alpha)\right]$ $\therefore P''(\alpha)=(0)\left[2M(\alpha)+(0)M'(\alpha)\right]$ $\therefore P''(\alpha)=0$ $\therefore P(\alpha)=P''(\alpha)=0$ (as required) (ii) $P(x)=x^4-5x^3+6x^2+4x-8$ divided by factor $(x-2)^3$. Method 1: $(x-2)^3=x^3-3x^2(2)+3x(2)^2-8$ $=x^3-6x^2+12x-8$ $x+1$ $x^3-6x^2+12x-8\overline{)x^4-5x^3+6x^2+4x-8}$ $-(x^4-6x^3+12x^2-8x)\downarrow$ $x^3-6x^2+12x-8$ $-(x^3-6x^2+12x-8)$ 0 $\therefore$ the remainder root is -1. Method 2: Let the roots be $2, 2, 2, \alpha$ $2(2)(2)\alpha=\frac{e}{a}$	

Question 12 Solutions

(d)(ii)cont.

$$8\alpha = \frac{(-8)}{(1)}$$

$$8\alpha = -8$$

$$\therefore \alpha = -1 \text{ i.e. the remainder root}$$

$$8\alpha = -8$$

$$\therefore \alpha = -1 \text{ i.e. the remainder root}$$

Question 13 Solutions

(a)(i)

$$(x-y)^2 \geq 0$$

$$x^2 - 2xy + y^2 \geq 0$$

$$x^2 + y^2 \geq 2xy \quad \text{(as required)}$$

(ii)

$$x^2 + y^2 \geq 2xy$$

$$(x^2 + y^2)(x+y) \geq 2xy(x+y).$$

$$x^3 + xy^2 + x^2y + y^3 \geq 2xy(x+y)$$

$$x^3 + y^3 + xy(x+y) \geq 2xy(x+y)$$

$$\therefore x^3 + y^3 \geq xy(x+y) \quad \text{(as required)}$$

(iii) Since $x^3 + y^3 \geq xy(x+y)$

Similarly $x^3 + z^3 \geq xz(x+z)$

$$z^3 + y^3 \geq zy(z+y)$$

$$\therefore 2(x^3 + y^3 + z^3) \geq xy(x+y) + xz(x+z) + zy(z+y)$$

(as required)

(b)(i)

$$m\ddot{x} = -mg - \frac{mv}{k}$$

$$m\ddot{x} = -m\left(g + \frac{v}{k}\right)$$

$$\ddot{x} = -\left(g + \frac{v}{k}\right)$$

$$\frac{dv}{dt} = -\left(g + \frac{v}{k}\right)$$

$$\frac{dv}{dt} = -\left(\frac{gk+v}{k}\right)$$

$$\frac{dt}{dv} = -\left(\frac{k}{gk+v}\right)$$

$$-dt = \frac{k\,dv}{gk+v}$$

Question 13 Solutions	
(b)(i)cont. $\int_0^t dt = -\int_{v_0}^{v} \frac{k}{gk+v}\,dv$ $t = -k\left[\ln(gk+v)\right]_{v_0}^{v}$ $= -k\left[\ln(gk+v) - \ln(gk+v_0)\right]$ $= -k\left[\ln\left(\frac{gk+v}{gk+v_0}\right)\right]$ $= k\ln\left(\frac{gk+v_0}{gk+v}\right)$ since $v_0 = k(h-g)$ $\therefore t = k\ln\left(\frac{kh}{gk+v}\right)$ (as required) (ii) Time to reach max height is when $v = 0$, $t = k\ln\left(\frac{gh}{gk+(0)}\right)$. $\therefore t = k\ln\left(\frac{h}{g}\right)$ Max height $\ddot{x} = -\left(g + \frac{v}{k}\right)$ $v\frac{dv}{dx} = -\left(\frac{gk+v}{k}\right)$ $\frac{dv}{dx} = -\left(\frac{gk+v}{kv}\right)$ $\frac{dx}{dv} = -\left(\frac{kv}{gk+v}\right)$ $\frac{dx}{k} = -\left(\frac{v}{gk+v}\right)dv$ $\int_0^H \frac{dx}{k} = -\int_{v_0}^{0}\left(\frac{v}{gk+v}\right)dv$ $\left[\frac{x}{k}\right]_0^H = \int_0^{v_0}\left(\frac{gk+v-gk}{gk+v}\right)dv$ $\frac{H}{k} = \int_0^{v_0}\left(1 - \frac{gk}{gk+v}\right)dv$ $\frac{H}{k} = \left[v - gk\ln(gk+v)\right]_0^{v_0}$ $H = k\left[v_0 - gk\ln(gk+v_0) - \left(-gk\ln(gk)\right)\right]$	$H = k\left[v_0 - gk\ln\left(\frac{gk+v_0}{gk}\right)\right]$

Question 13 Solutions

(b)(ii)cont. $H = k\left[v_0 - gk\ln\left(gk + v_0\right) + gk\ln\left(gk\right)\right]$

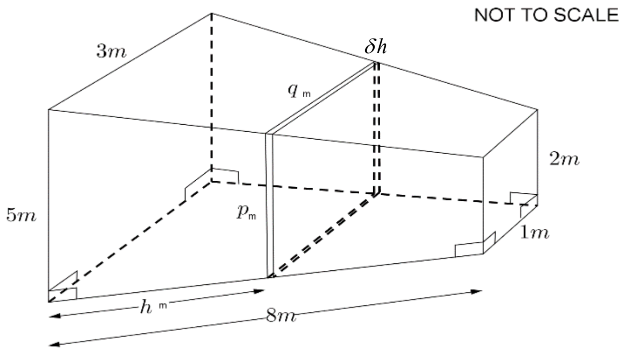
$$H = k\left[v_0 + gk\ln\left(\frac{gk}{gk + v_0}\right)\right]$$

Since $v_0 = kh - kg$

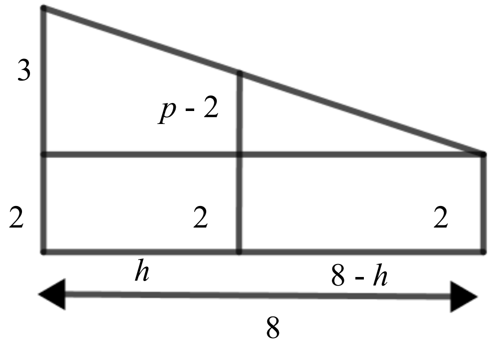
$$= k\left[\left(kh - kg\right) + gk\ln\left(\frac{gk}{gk + kh - kg}\right)\right]$$

$$\therefore H = k\left[\left(kh - kg\right) + gk\ln\left(\frac{g}{h}\right)\right] \quad \text{(as required)}$$

(c)



(i)

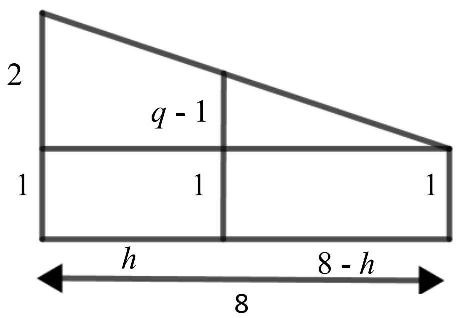


$$\frac{p-2}{3} = \frac{8-h}{8}$$

$$8p - 16 = 24 - 3h$$

$$8p = 40 - 3h$$

$$\therefore p = 5 - \frac{3h}{8} \quad \text{(as required)}$$



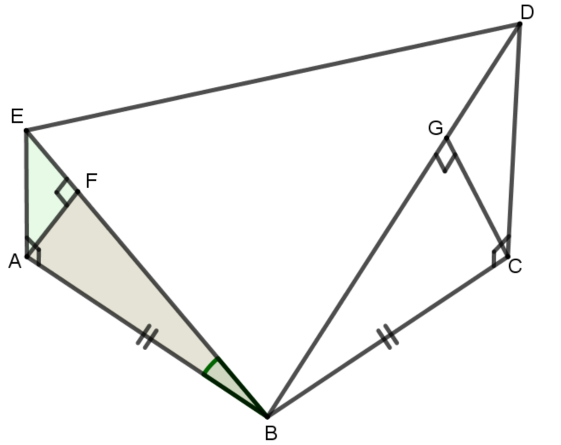
Question 13 Solutions	
(c)(i)cont. $\frac{q-1}{2}=\frac{8-h}{8}$ $8q-8=16-2h$ $8q=24-2h$ $\therefore q=3-\frac{h}{4}$ (as required) (ii) $A_{pq}=pq$ $=\left(5-\frac{3h}{8}\right)\left(3-\frac{h}{4}\right)$ $=15-\frac{5h}{4}-\frac{9h}{8}+\frac{3h^2}{32}$ $=15-\frac{19h}{8}+\frac{3h^2}{32}$ (iii) $\delta V=A_{pq}\delta h$ $V=\lim_{\delta h\to 0}\sum_{h=0}^{8}\left(15-\frac{19h}{8}+\frac{3h^2}{32}\right)\delta h$ $=\int_0^8 15-\frac{19h}{8}+\frac{3h^2}{32}\,dh$ $=\left[15h-\frac{19h^2}{16}+\frac{h^3}{32}\right]_0^8$ $=\left[15(8)-\frac{19(8)^2}{16}+\frac{(8)^3}{32}\right]-0$ $=120-76+16$ $=60\ m^3$	

Question 14 Solutions	
(a)(i) $\frac{x}{x^3-8}\equiv\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$ $x\equiv A\left(x^2+2x+4\right)+\left(Bx+C\right)\left(x-2\right)$ $x\equiv Ax^2+2Ax+4A+\left(Bx^2-2Bx+Cx-2C\right)$ $x\equiv Ax^2+2Ax+4A$ $+Bx^2-2Bx-2C$ $+Cx$ $x\equiv\left(A+B\right)x^2+\left(2A+C-2B\right)x+\left(4A-2C\right)$ Equating like terms: $A+B=0$ $B=-A$ — (1) $2A+C-2B=1$ — (2)	

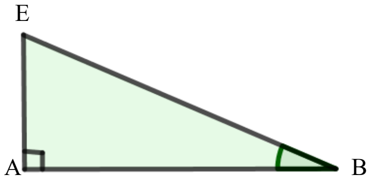
Question 14 Solutions	
(a)(ii)cont. $4A-2C=0$ $2C=4A$ $C=2A \quad — (3)$ Sub (1) and (3) into (2) $2A+(2A)-2(-A)=1$ $6A=1$ $A=\frac{1}{6} \quad (4)$ Sub (4) into (1) $\therefore B=-\frac{1}{6}$ Sub (4) into (3) $\therefore C=\frac{2}{6}$ $\therefore A=\frac{1}{6},\ B=-\frac{1}{6}$ and $C=\frac{1}{3}$ (ii) $\int \frac{x}{x^3-8}\ dx$ $=\frac{1}{6}\int\left(\frac{1}{x-2}+\frac{-x+2}{x^2+2x+4}\right)\ dx$ $=\frac{1}{6}\int\left(\frac{1}{x-2}-\frac{x-2}{x^2+2x+4}\right)\ dx$ $=\frac{1}{6}\int\left(\frac{1}{x-2}-\frac{(x+1)-2-1}{x^2+2x+4}\right)\ dx$ $=\frac{1}{6}\int\left(\frac{1}{x-2}-\frac{(x+1)-3}{x^2+2x+4}\right)\ dx$ $=\frac{1}{6}\left[\int\left(\frac{1}{x-2}-\frac{(x+1)}{x^2+2x+4}+\frac{3}{x^2+2x+4}\right)\ dx\right]$ $=\frac{1}{6}\left[\int\left(\frac{1}{x-2}-\frac{1}{2}.\frac{2(x+1)}{x^2+2x+4}+\frac{3}{(x^2+2x+1)+3}\right)\ dx\right]$ $=\frac{1}{6}\left[\int\left(\frac{1}{x-2}-\frac{1}{2}.\frac{2x+2}{x^2+2x+4}+\frac{3}{(x+1)^2+3}\right)\ dx\right]$ $=\frac{1}{6}\left[\ln\|x-2\|-\frac{1}{2}\ln(x^2+2x+4)+\frac{3}{\sqrt{3}}\int\left(\frac{\sqrt{3}}{(x+1)^2+3}\right)\ dx\right]$ $=\frac{1}{6}\left[\ln\|x-2\|-\frac{1}{2}\ln(x^2+2x+4)+\sqrt{3}\int\left(\frac{\sqrt{3}}{(x+1)^2+3}\right)\ dx\right]$ $=\frac{\ln\|x-2\|}{6}-\frac{\ln(x^2+2x+4)}{12}+\frac{\sqrt{3}}{6}\tan^{-1}\frac{x+1}{\sqrt{3}}+C$	

Question 14 Solutions

(b)(i)



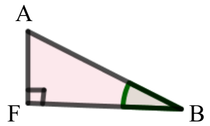
In ΔBAE and ΔBFA,



$\angle BAE = \angle BFA = 90^\circ$ (given)

$\angle ABE = \angle FBA$ (common)

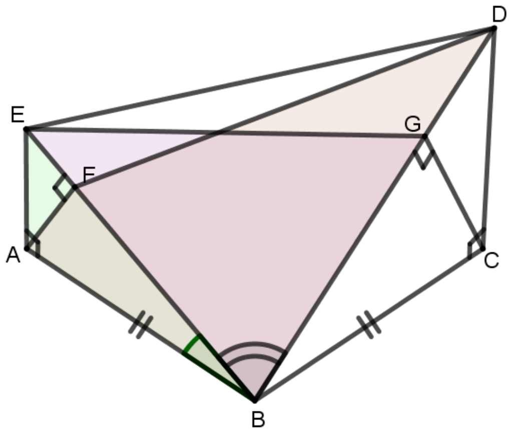
$\therefore \Delta BAE \, ||| \, \Delta BFA$ (equiangular)



$\frac{BE}{BA} = \frac{BA}{BF}$ (corresponding sides of similar triangles in same ratio)

$\therefore BE.BF = BA^2$

(ii) Assume $BC^2 = BG.BD$,



$AB = CB$ (given)

$AB^2 = CB^2$

$BF.BE = BG.BD$ [from (a) and given]

$\frac{BG}{BF} = \frac{BE}{BD}$

In ΔBEG and ΔBDF,

$\angle EBG = \angle DBF$ (common)

$\frac{BG}{BF} = \frac{BE}{BD}$ (proven above)

Question 14 Solutions

(b)(ii)cont. $\therefore \Delta BEG \parallel\!| \Delta BDF$ (two sides in same ratio and an equal included angle)

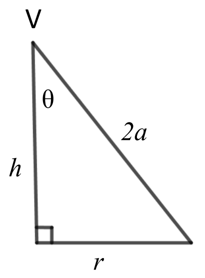
(iii) In ΔBEG and ΔBDF,

$\angle BEG = \angle BDF$ (corresponding angles of similar triangles)

$\therefore D,\ E,\ F$ and G are concyclic points (angles in the same segment)

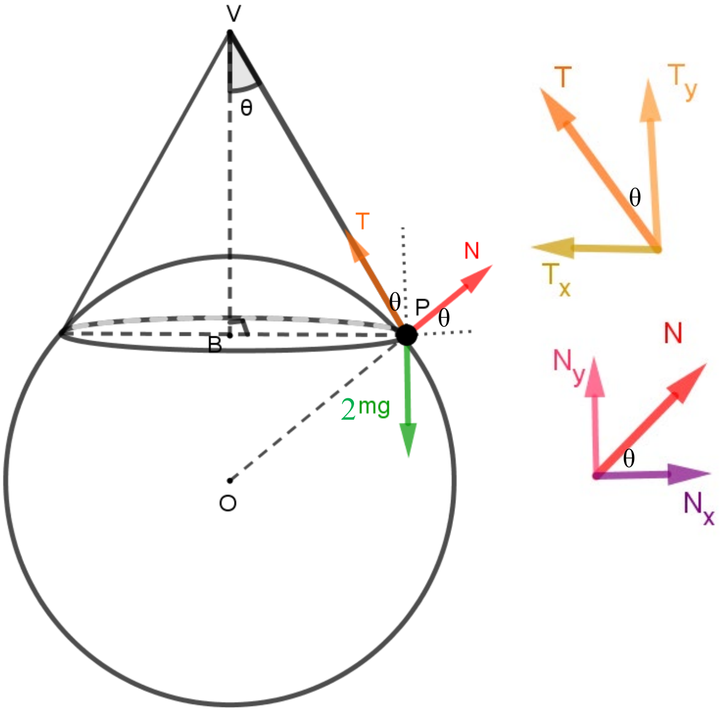
$\therefore DEFG$ is a cyclic quadrilateral.

(c)(i) In ΔVBP,



$r = 2a\sin\theta$

$h = 2a\cos\theta$



Vertically: $T\cos\theta + N\sin\theta = 2mg$ (1)

Horizontally: $T\sin\theta - N\cos\theta = 2mr\omega^2$ (2)

(1) $\times\cos\theta$ $T\cos^2\theta + N\sin\theta\cos\theta = 2mg\cos\theta$ (1)'

(2) $\times\sin\theta$ $T\sin^2\theta - N\sin\theta\cos\theta = 2mr\omega^2\sin\theta$ (2)'

(1)'+(2)' $T\cos^2\theta + T\sin^2\theta = 2mg\cos\theta + 2mr\omega^2\sin\theta$

$$T\left(\cos^2\theta + \sin^2\theta\right) = 2m\left(g\cos\theta + r\omega^2\sin\theta\right)$$

$$\therefore T = 2m\left(g\cos\theta + r\omega^2\sin\theta\right)$$

$$= 2m\left(g\cos\theta + \left(2a\sin\theta\right)\omega^2\sin\theta\right)$$

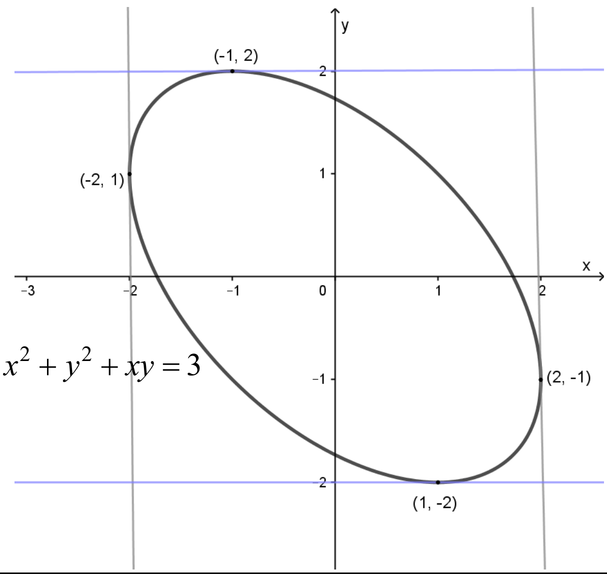
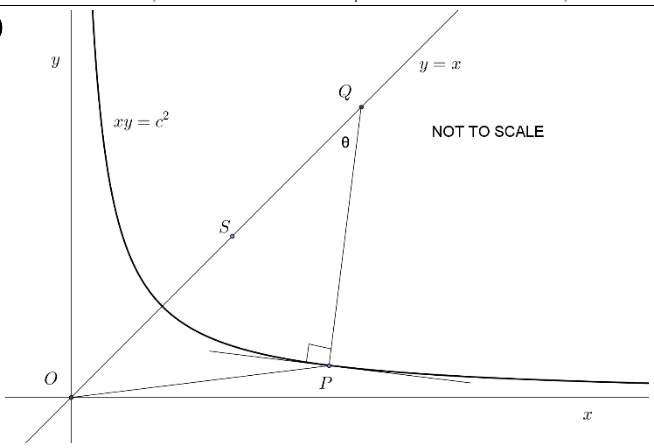
$$\therefore T = 2m\left(g\cos\theta + 2a\omega^2\sin^2\theta\right) \text{ Newtons}$$

Question 14 Solutions	
(c)(ii) (1) $\times \sin\theta$ $T\cos\theta\sin\theta + N\sin^2\theta = 2mg\sin\theta$ (1)" (2) $\times\cos\theta$ $T\sin\theta\cos\theta - N\cos^2\theta = 2mr\omega^2\cos\theta$ (2)" (1)"-(2)" $N\cos^2\theta + N\sin^2\theta = 2mg\sin\theta - 2mr\omega^2\cos\theta$ $N\left(\cos^2\theta+\sin^2\theta\right) = 2m\left(g\sin\theta - r\omega^2\cos\theta\right)$ $\therefore N = 2m\left(g\sin\theta - r\omega^2\cos\theta\right)$ $\therefore N = 2m\left(g\sin\theta - 2a\omega^2\cos\theta\sin\theta\right)$ Newton (iii) For the particle to remain in contact with the surface of the sphere, then $T > 0$ and $N > 0$ for all the values of ω. Since T is always positive, thus need to consider N. Hence $N > 0$ $g\sin\theta - 2a\omega^2\sin\theta\cos\theta > 0$ $\sin\theta\left(g - 2a\omega^2\cos\theta\right) > 0$ and since $0^\circ < \theta < 90^\circ$ $\therefore \sin\theta > 0$ i.e. $g - 2a\omega^2\cos\theta > 0$ $g > 2a\omega^2\cos\theta$ $\dfrac{g}{2a\cos\theta} > \omega^2$ $\therefore \omega^2 < \dfrac{g}{2a\cos\theta}$ $\therefore \omega < \left(\dfrac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$	

Question 15 Solutions	
(a)(i) $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x}\,dx$ for $n = 0, 1, 2, \ldots$ i.e. $I_{n-1} = \displaystyle\int_0^1 x^{n-1}\sqrt{1-x}\,dx$ $= \left[uv\right]_0^1 - \displaystyle\int_0^1 u'v\,dx$ where $u = x^n$ and $v' = \sqrt{1-x}$ $u' = nx^{n-1}$ $v = \dfrac{2(1-x)^{\frac{3}{2}}}{-3}$ $= \left[-\dfrac{2(1-x)^{\frac{3}{2}}x^n}{3}\right]_0^1 - n\displaystyle\int_0^1 x^{n-1}\left(-\dfrac{2(1-x)^{\frac{3}{2}}}{3}\right)dx$	

Question 15 Solutions	
(a)(i)cont. $=\left[\dfrac{2(1-x)^{\frac{3}{2}}x^n}{3}\right]_1^0+\dfrac{2n}{3}\displaystyle\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\,dx$ $=\left[\dfrac{2(1-0)^{\frac{3}{2}}(0)^n}{3}-\dfrac{2(1-1)^{\frac{3}{2}}(1)^n}{3}\right]+\dfrac{2n}{3}\displaystyle\int_0^1\left(x^{n-1}-x^n\right)\sqrt{1-x}\,dx$ $=0+\dfrac{2n}{3}\displaystyle\int_0^1\left(x^{n-1}\sqrt{1-x}-x^n\sqrt{1-x}\right)dx$ $=\dfrac{2n}{3}\displaystyle\int_0^1 x^{n-1}\sqrt{1-x}\,dx-\dfrac{2n}{3}\int_0^1 x^n\sqrt{1-x}\,dx$ $I_n=\dfrac{2n}{3}I_{n-1}-\dfrac{2n}{3}I_n$ $I_n+\dfrac{2n}{3}I_n=\dfrac{2n}{3}I_{n-1}$ $\dfrac{3+2n}{3}I_n=\dfrac{2n}{3}I_{n-1}$ $I_n=\dfrac{2n}{3}\left(\dfrac{3}{3+2n}\right)I_{n-1}$ $\therefore I_n=\dfrac{2n}{3+2n}I_{n-1}$ (as required) (ii) Let $I_3=\displaystyle\int_0^1 x^3\sqrt{1-x}\,dx$ $=\dfrac{2(3)}{3+2(3)}I_2$ $=\dfrac{6}{9}I_2$ $=\dfrac{2}{3}I_2$ $I_2=\dfrac{2(2)}{3+2(2)}I_1$ $=\dfrac{4}{7}I_1$ $I_1=\dfrac{2(1)}{3+2(1)}I_0$ $=\dfrac{2}{5}I_0$ $I_0=\displaystyle\int_0^1 x^0\sqrt{1-x}\,dx$	

Question 15 Solutions	
(a)(ii)cont. $I_0 = \int_0^1 (1-x)^{\frac{1}{2}}\,dx$ $= \left[-\frac{2(1-x)^{\frac{3}{2}}}{3} \right]_0^1$ $= \frac{2}{3}\left[(1-x)^{\frac{3}{2}} \right]_1^0$ $= \frac{2}{3}\left[(1-0)^{\frac{3}{2}} - (1-1)^{\frac{3}{2}} \right]$ $= \frac{2}{3}[1-0]$ $\therefore I_0 = \frac{2}{3}$ $\therefore I_1 = \frac{2}{5}\left(\frac{2}{3}\right)$ $= \frac{4}{15}$ $\therefore I_2 = \frac{4}{7}\left(\frac{4}{15}\right)$ $= \frac{16}{105}$ $\therefore I_3 = \frac{2}{3}\left(\frac{16}{105}\right)$ $\therefore I_3 = \frac{32}{315}$	
(b)(i) $x^2 + y^2 + xy = 3$ $2x + 2y\frac{dy}{dx} + u'v + v'u = 0$ where $u = x$ and $v = y$ $u' = 1 \qquad v' = \frac{dy}{dx}$ $2x + 2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$ $2x + y + 2y\frac{dy}{dx} + x\frac{dy}{dx} = 0$ $2x + y + (2y + x)\frac{dy}{dx} = 0$ $(2y + x)\frac{dy}{dx} = -(2x + y)$ $\frac{dy}{dx} = -\frac{(2x + y)}{x + 2y}$ (as required)	

Question 15 Solutions	
(b)(ii) For vertical tangents when $x+2y=0$ $x=-2y$ $(-2y)^2+y^2+(-2y)y=3$ $4y^2+y^2-2y^2=3$ $3y^2=3$ $y^2=1$ $\therefore y=\pm 1 \quad \Rightarrow x=\mp 2$ $\therefore$Vertical tangents at (2, -1) and (-2, 1). For horizontal tangents when $2x+y=0$ $y=-2x$ $x^2+(-2x)^2+x(-2x)=3$ $x^2+4x^2-2x^2=3$ $3x^2=3$ $x^2=1$ $\therefore x=\pm 1 \quad \Rightarrow y=\mp 2$ $\therefore$Horizontal tangents at (1, -2) and (-1, 2). (iii) 	
(c)(i) 	

Question 15 Solutions	
(c)(i)cont. $x = ct$ and $y = \frac{c}{t}$ $\frac{dx}{dt} = c$ $\frac{dy}{dt} = -\frac{c}{t^2}$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $= -\frac{c}{t^2} \times \frac{1}{c}$ $= -\frac{1}{t^2}$ Since tangent at $P \perp PQ$, $\therefore m_{PQ} = t^2$ $\tan\theta = \left\| \frac{m_{PQ} - m_{OQ}}{1 + m_{PQ} \times m_{OQ}} \right\|$ $\therefore \tan\theta = \left\| \frac{t^2 - 1}{1 + t^2} \right\|$ (as required) (ii) $m_{PO} = \frac{\frac{c}{t} - 0}{ct - 0}$ $= \frac{\left(\frac{c}{t}\right)}{ct}$ $= \frac{1}{t^2}$ $\tan\angle POQ = \left\| \frac{1 - \frac{1}{t^2}}{1 + \frac{1}{t^2}} \right\|$ $= \left\| \frac{\frac{t^2 - 1}{t^2}}{\frac{t^2 + 1}{t^2}} \right\|$ $= \left\| \frac{t^2 - 1}{t^2 + 1} \right\|$ $\therefore \tan\angle POQ = \tan\theta$ (as required) (iii) $S\left(c\sqrt{2},\ c\sqrt{2}\right)$ If $PS \perp OQ$, $\therefore m_{PS} = -1$ Hence $\left\| \frac{\frac{c}{t} - c\sqrt{2}}{ct - c\sqrt{2}} \right\| = -1$	

Question 15 Solutions	
(c)(iii)cont. $\frac{1}{t}-\sqrt{2}=-t+\sqrt{2}$ $t+\frac{1}{t}=2\sqrt{2}$ $\frac{t^2+1}{t}=2\sqrt{2}$ Now $\tan\theta=\tan\angle POQ$ from (ii), $\tan\theta=\frac{t^2-1}{t^2+1}$ $=\frac{\frac{t^2-1}{t}}{\frac{t^2+1}{t}}$ $=\frac{t-\frac{1}{t}}{t+\frac{1}{t}}$ $\therefore \tan^2\theta=\left(\frac{t-\frac{1}{t}}{t+\frac{1}{t}}\right)^2$ $=\frac{t^2-2+\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2}$ $=\frac{\left(t^2+2+\frac{1}{t^2}\right)-2-2}{\left(t+\frac{1}{t}\right)^2}$ $=\frac{\left(t+\frac{1}{t}\right)^2-4}{\left(t+\frac{1}{t}\right)^2}$ $=\frac{\left(2\sqrt{2}\right)^2-4}{\left(2\sqrt{2}\right)^2}$ $=\frac{8-4}{8}=\frac{1}{2}$ $\therefore \tan\theta=\frac{1}{\sqrt{2}}$ $(\theta\geq 0)$ (as required)	

Question 16 Solutions	
(a)(i) $(\cos\theta+i\sin\theta)^5$ $=\cos^5\theta+5\cos^4\theta(i\sin\theta)+10\cos^3\theta(i\sin\theta)^2$ $+10\cos^2\theta(i\sin\theta)^3+5\cos\theta(i\sin\theta)^4+(i\sin\theta)^5$ $=\cos^5\theta+5i\cos^4\theta\sin\theta+10\cos^3\theta(i^2)\sin^2\theta$ $+10\cos^2\theta(i^3)\sin^3\theta+5\cos\theta(i^4)\sin^4\theta+(i^5)\sin^5\theta$ $=\cos^5\theta+5i\cos^4\theta\sin\theta+10\cos^3\theta(-1)\sin^2\theta$ $+10\cos^2\theta(-i)\sin^3\theta+5\cos\theta(1)\sin^4\theta+(i)\sin^5\theta$ $=\cos^5\theta+5i\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta$ $-10i\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+i\sin^5\theta$ And by De Moivre's Theorem, $(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$ Equating the real terms: $\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ $=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta(\sin^2\theta)^2$ $=\cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-\cos^2\theta)^2$ $=\cos^5\theta-10\cos^3\theta+10\cos^5\theta$ $+5\cos\theta(1-2\cos^2\theta+\cos^4\theta)$ $=\cos^5\theta-10\cos^3\theta+10\cos^5\theta+5\cos\theta-10\cos^3\theta$ $+5\cos^5\theta$ $\therefore \cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$ (as required) (ii) Let $\cos 5\theta=0$ $16\cos^5\theta-20\cos^3\theta+5\cos\theta=0$ and let $x=\cos\theta$ $16x^5-20x^3+5x=0$ $x(16x^4-20x^2+5)=0$ $x=0$ or $16x^4-20x^2+5=0$ The roots of x can be obtained from $\cos 5\theta=0$; $5\theta=2k\pi+\frac{\pi}{2}$ where $k=0, 1, 2, 3, 4$ $5\theta=\frac{4k\pi+\pi}{2}$ $\therefore \theta=\frac{4k\pi+\pi}{10}$ When $k=0,\quad \theta=\frac{\pi}{10},\quad x_1=\cos\frac{\pi}{10}$ $k=1,\quad \theta=\frac{\pi}{2},\quad x_2=0$ $k=2,\quad \theta=\frac{9\pi}{10},\quad x_3=\cos\frac{9\pi}{10}=-\cos\frac{\pi}{10}$	

Question 16 Solutions	
(a)(ii)cont. $k=3,\quad \theta=\frac{13\pi}{10},\quad x_4=\cos\frac{13\pi}{10}=-\cos\frac{3\pi}{10}$ $k=4,\quad \theta=\frac{17\pi}{10},\quad x_5=\cos\frac{17\pi}{10}=\cos\frac{3\pi}{10}$ $\therefore 16x^4-20x^2+5=0$ has 4 non – zero roots are $\cos\frac{\pi}{10}, -\cos\frac{\pi}{10}, \cos\frac{3\pi}{10}$ and $-\cos\frac{3\pi}{10}$. (iii) $16x^4-20x^2+5=0$ $x_1x_3x_4x_5=\frac{e}{a}$ where $e=5$ and $a=16$ $x_1x_3x_4x_5=\frac{5}{16}$ $\cos\frac{\pi}{10}\left(-\cos\frac{\pi}{10}\right)\cos\frac{3\pi}{10}\left(-\cos\frac{3\pi}{10}\right)=\frac{5}{16}$ $\left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)^2=\frac{5}{16}$ $\therefore \cos\frac{\pi}{10}\cos\frac{3\pi}{10}=\frac{\sqrt{5}}{4}$ since $\cos\frac{\pi}{10}>0$ and $\cos\frac{3\pi}{10}>0$ (iv) $\sin\frac{3\pi}{5}\sin\frac{6\pi}{5}=\cos\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)\cos\left(\frac{\pi}{2}-\frac{6\pi}{5}\right)$ $=\cos\left(\frac{5\pi-6\pi}{10}\right)\cos\left(\frac{5\pi-12\pi}{10}\right)$ $=\cos\left(\frac{-\pi}{10}\right)\cos\left(\frac{7\pi}{10}\right)$ $=\cos\frac{\pi}{10}\left(-\cos\frac{3\pi}{10}\right)$ $=-\left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)$ $\therefore \sin\frac{3\pi}{5}\sin\frac{6\pi}{5}=-\frac{\sqrt{5}}{4}$	
(b)(i) To prove: $\cos(\alpha+\beta)+\cos(\alpha-\beta)=2\cos\alpha\cos\beta$ Proof: LHS $=\cos(\alpha+\beta)+\cos(\alpha-\beta)$ $=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ $+\cos\alpha\cos\beta+\sin\alpha\sin\beta$ $=2\cos\alpha\cos\beta$ $\therefore$LHS = RHS (QED)	

Question 16 Solutions	Comment
(b)(ii) $\int \cos nx \cos mx \; dx$ $= \frac{1}{2}\int \cos(n+m)x + \cos(n-m)x \; dx$ $= \frac{1}{2}\left[\frac{\sin(n+m)x}{(n+m)} + \frac{\sin(n-m)x}{(n-m)}\right] + C$ $= \frac{\sin(n+m)x}{2(n+m)} + \frac{\sin(n-m)x}{2(n-m)} + C$ (iii) If $\alpha > \beta > 0$ $-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$ $\therefore 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$ $\therefore \sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$ $\sum_{r=1}^{r=9} \int_0^{\frac{\pi}{2}} \sin r x \sin x \; dx$ $= \int_0^{\frac{\pi}{2}} \sin x \sin x \; dx + \int_0^{\frac{\pi}{2}} \sin 2x \sin x \; dx + \int_0^{\frac{\pi}{2}} \sin 3x \sin x \; dx$ $+ \ldots + \int_0^{\frac{\pi}{2}} \sin 8x \sin x \, dx + \int_0^{\frac{\pi}{2}} \sin 9x \sin x \, dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 0 - \cos 2x) \; dx + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos x - \cos 3x) \; dx$ $+ \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 2x - \cos 4x) \; dx + \ldots + \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 7x - \cos 9x) \; dx$ $+ \frac{1}{2}\int_0^{\frac{\pi}{2}} (\cos 8x - \cos 10x) \; dx$ $= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \cos 0 - \cos 2x + \cos x - \cos 3x + \cos 2x - \cos 4x\right.$ $\left. + \ldots + \cos 8x - \cos 9x - \cos 10x \; dx\right]$	

Question 16 Solutions	Comment
(b)(iii)cont. $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\cos 0 + \cos x - \cos 9x - \cos 10x\right)\ dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 + \cos x - \cos 9x - \cos 10x\right)\ dx$ $= \frac{1}{2}\left[x + \sin x - \frac{\sin 9x}{9} - \frac{\sin 10x}{10}\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{2}\left[\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) - \frac{\sin\left(\frac{9\pi}{2}\right)}{9} - \frac{\sin\left(\frac{10\pi}{2}\right)}{10} - 0\right]$ $= \frac{1}{2}\left[\frac{\pi}{2} + 1 - \frac{(1)}{9} - \frac{\sin 5\pi}{10}\right]$ $= \frac{1}{2}\left[\frac{\pi}{2} + 1 - \frac{1}{9} + 0\right]$ $= \frac{1}{2}\left[\frac{\pi}{2} + \frac{8}{9}\right]$ $= \frac{1}{2}\left(\frac{9\pi - 16}{18}\right)$ $= \frac{9\pi - 16}{36}$ ☺ **THE END** ☺	