HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12

MATHEMATICS EXTENSION 2

2005

HSC COURSE

TRIAL HSC ~ASSESSMENT TASK 4

EXAMINERS ~ J. DILLON AND G. RAWSON

GENERAL INSTRUCTIONS

- READING TIME – 5 MINUTES.
- WORKING TIME – THREE HOURS.
- ATTEMPT ALL QUESTIONS.
- QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING SHOULD BE SHOWN IN EVERY QUESTION.
- THIS PAPER CONTAINS EIGHT (8) QUESTIONS.
- TOTAL MARKS – 120 MARKS
- MARKS MAY NOT BE AWARDED FOR CARELESS OR BADLY ARRANGED WORK.
- BOARD APPROVED CALCULATORS MAY BE USED.
- A TABLE OF INTEGRALS IS SUPPLIED.
- EACH QUESTION IS TO BE STARTED IN A NEW EXAMINATION BOOKLET.
- THIS ASSESSMENT TASK MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

STUDENT NAME / NUMBER: ____________________

TEACHER: ____________________

QUESTION 1: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) Find:

(i) $\int \frac{1}{x \ln x} dx$ **2**

(ii) $\int \frac{x}{x^2 + 2x + 5} dx$ **3**

(b) Evaluate, using integration by parts, $\int_0^{\frac{\pi}{2}} x \cos x \, dx$. **2**

(c) Evaluate, using partial fractions, $\int_1^3 \frac{dx}{x^2 + 2x}$. **3**

(d) The integral I_n is defined by $I_n = \int_0^1 x^n e^{-x} \, dx$.

(i) Show that $I_n = nI_{n-1} - e^{-1}$. **2**

(ii) Hence, show that $I_3 = 6 - 16e^{-1}$. **3**

QUESTION 2: **(USE A SEPARATE ANSWER BOOKLET)**

MARKS

(a) Given $z = \sqrt{6} - \sqrt{2}\,i$, find:

(i) $\text{Re}(z^2)$; 1

(ii) $|z|$; 1

(iii) $\arg z$; 2

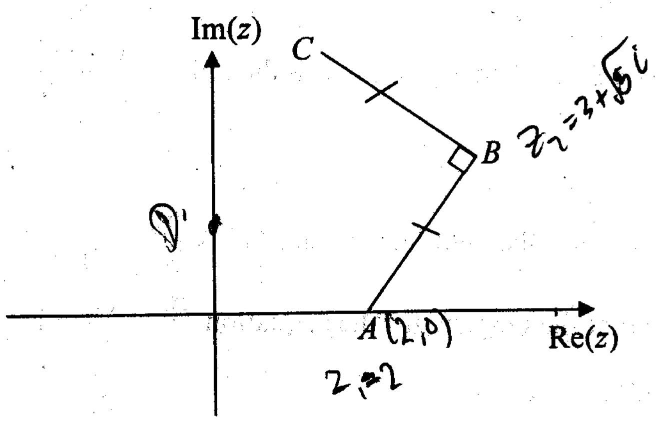
(iv) z^4 in the form $x + iy$. 2

(b) The equations $|z - 8 - 6i| = 2\sqrt{10}$ and $\arg z = \tan^{-1} 2$ both represent loci on the Argand plane.

(i) Write down the Cartesian equations of the loci, and hence show that the points of intersection of the loci are $2 + 4i$ and $6 + 12i$. 3

(ii) Sketch both loci on the same diagram, showing their points of intersection. (You need not show the intercepts with the axes.) 2

(c)



The diagram above shows the fixed points A, B and C in the Argand plane, where $AB = BC$, $\angle ABC = \dfrac{\pi}{2}$, and A, B and C are in anticlockwise order. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + \sqrt{5}\,i$.

(i) Find the complex number z_3 represented by the point C. 2

(ii) D is the point on the Argand plane such that $ABCD$ is a square. Find the complex number z_4 represented by D. 2

QUESTION 3: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **2**

(ii) Show that the equation of the normal at P is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. **2**

(iii) The line through P parallel to the y – axis meets the asymptote $y = \frac{b}{a}x$ at Q.
The tangent at P meets the same asymptote at R. The normal at P meets the x – axis at G. Prove that $\angle RQG$ is a right angle. **3**

(iv) What sort of quadrilateral is $RQPG$?

(b) The tangents at two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at $T(x_0, y_0)$.

(i) Show that the equation of the chord of contact PQ is $\frac{xx_0}{16} + \frac{yy_0}{9} = 1$. **2**

(You may assume that the tangent at P has equation $\frac{xx_1}{16} + \frac{yy_1}{9} = 1$, and similarly for Q)

(ii) If the chord PQ touches the circle $x^2 + y^2 = 9$, then by considering the distance of the chord from the origin, or otherwise, show that the point $T(x_0, y_0)$ satisfies $\frac{9x_0^2}{256} + \frac{y_0^2}{9} = 1$. **4**

(iii) Give a geometrical description of the locus of T. **1**

Question 4: (Use a SEPARATE answer booklet)

Marks

(a) (i) Let $P(x)$ be a polynomial of degree 4 with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence, or otherwise, find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. ***You do not need to give the coordinates of turning points or points of inflection.*** **1**

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**

$$\theta = (2n+1)\frac{\pi}{5},\ n = 0, \pm 1, \pm 2, \ldots .$$

Hence, solve the equation $\cos 5\theta = -1$, for $0 \le \theta \le 2\pi$.

(ii) Use De Moirve's Theorem to show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

(iii) Find the exact trigonometric roots of the equation **2**

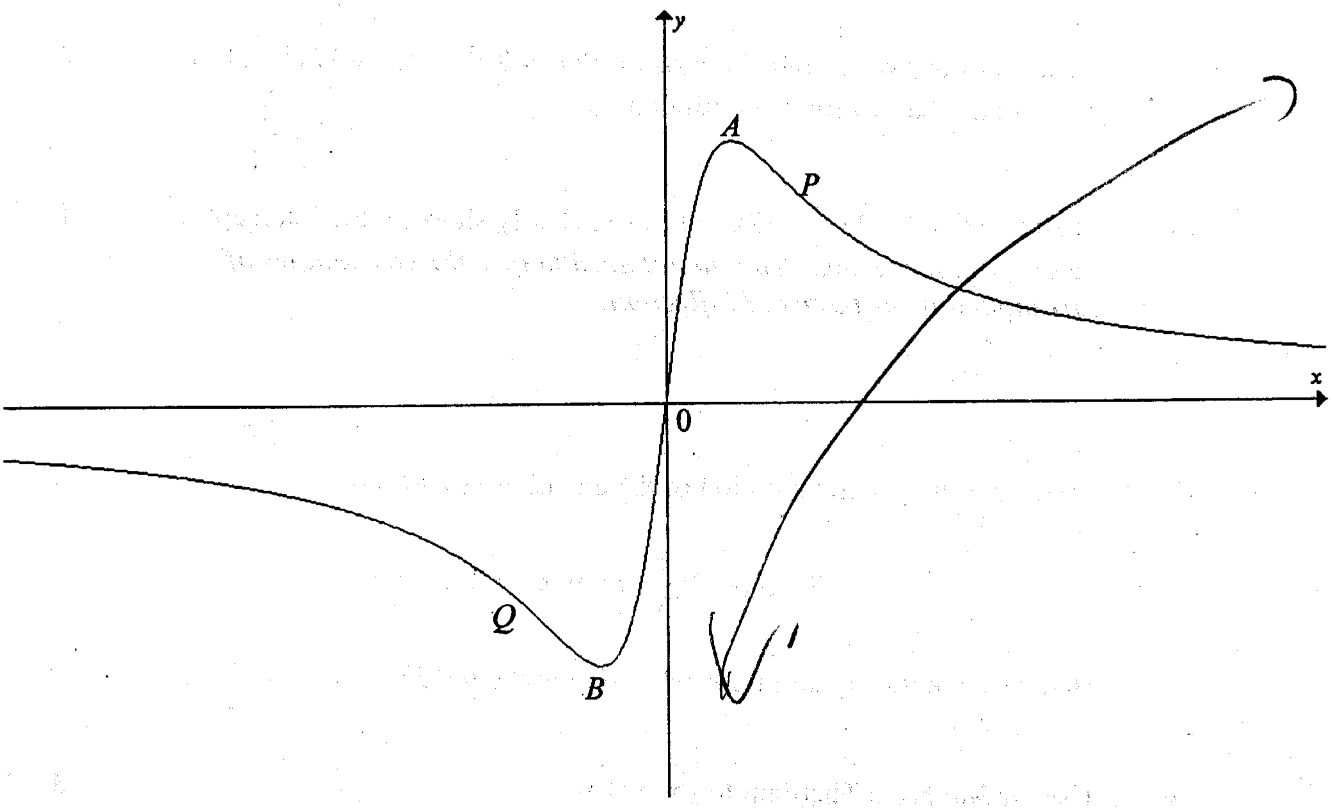
$$16x^5 - 20x^3 + 5x + 1 = 0$$

(iv) Hence, find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} \times \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1 = 0$ into irreducible factors over the rational numbers. **3**

QUESTION 5: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) In the diagram below, the curve $y = \dfrac{2x}{1+x^2}$ is sketched.



(i) Find the coordinates of the turning points A and B. **2**

(ii) Find the coordinates of the inflection points X and R. **3**

(b) Draw separate sketches of:

(i) $y = \dfrac{|2x|}{1+x^2}$ **1**

(ii) $y = \dfrac{1+x^2}{2x}$ **2**

(iii) $y^2 = \dfrac{2x}{1+x^2}$ **2**

(iv) $y = \log_e\left(\dfrac{2x}{1+x^2}\right)$ **2**

QUESTION 5 CONTINUES ON THE NEXT PAGE ....

QUESTION 4: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Let $P(x)$ be a polynomial of degree 4 with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. **2**

(ii) Hence, or otherwise, find all zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. **2**

(iii) Sketch $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, clearly showing the intercepts on the coordinate axes. ***You do not need to give the coordinates of turning points or points of inflection.*** **1**

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**

$$\theta = (2n+1)\frac{\pi}{5}, n = 0, \pm 1, \pm 2, \ldots .$$

Hence, solve the equation $\cos 5\theta = -1$, for $0 \le \theta \le 2\pi$.

(ii) Use De Moirve's Theorem to show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

(iii) Find the exact trigonometric roots of the equation **2**

$$16x^5 - 20x^3 + 5x + 1 = 0$$

(iv) Hence, find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} \times \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1 = 0$ into irreducible factors over the rational numbers. **3**

(c) (i) Show that the equation $kx^3 + (k-2)x = 0$ can be written in the form $\frac{2x}{1+x^2} = kx$. **1**

(ii) Using a graphical approach based on the curve $y = \frac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3 + (k-2)x = 0$ has exactly one real root. **2**

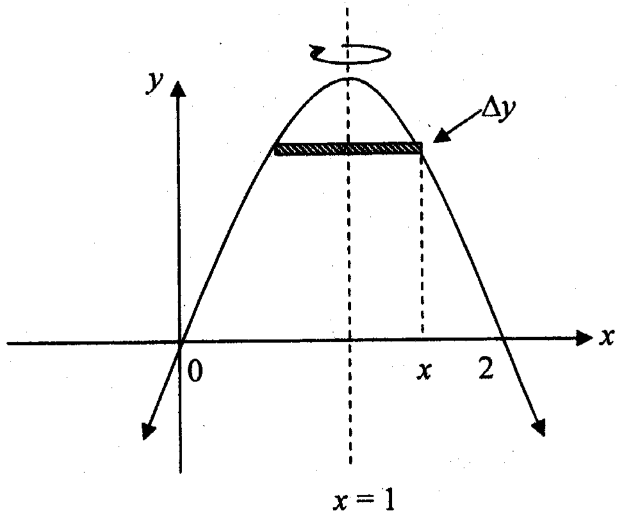
QUESTION 6: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) A Mini-League football has a volume the same as the volume generated by rotating the region inside the ellipse $\frac{x^2}{20}+\frac{y^2}{15}=1$ about the x-axis. **3**

Find the volume of this football.

(b) The area bounded by the curve $y=2x-x^2$ and the x-axis is rotated through 180° about the line $x=1$.



(i) Show that the volume, ΔV, of a representative horizontal slice of width Δy is given by **2**

$$\Delta V=\pi(x-1)^2\Delta y$$

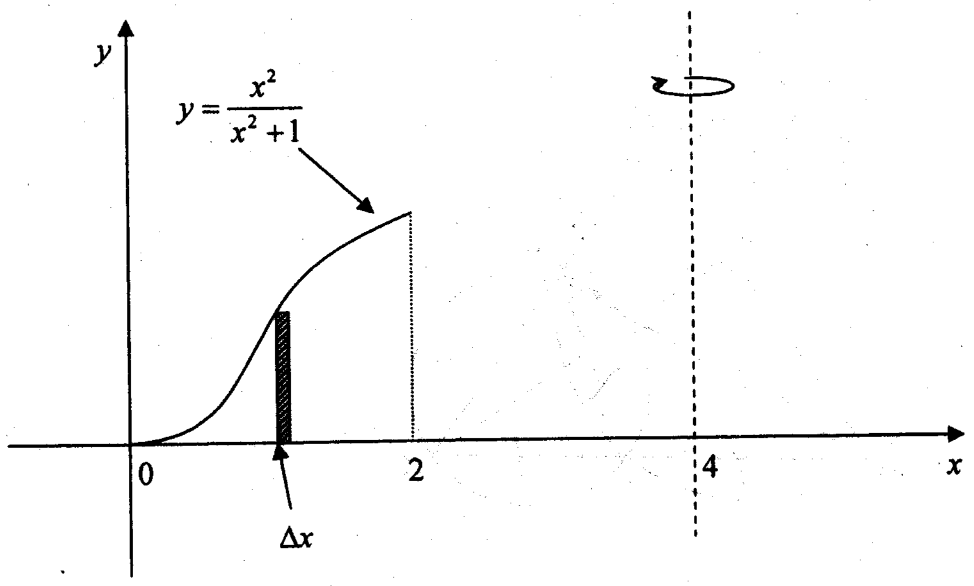
(ii) Hence show that the volume of the solid of revolution is given by **2**

$$V=\lim_{\Delta y\to 0}\sum_{y=0}^{1}\pi(1-y)\Delta y$$

(iii) Hence, find the volume of the solid of revolution **2**

QUESTION 6 CONTINUES ON THE NEXT PAGE ....

(c) The region shown below, bounded by the curve $y = \dfrac{x^2}{x^2+1}$, the x-axis and the line $x = 2$, is rotated about the line $x = 4$.



(i) Using the method of cylindrical shells, show that the volume ΔV of a Shell distant x from the origin and with thickness Δx is given by **3**

$$\Delta V = 2\pi(4-x)\left(1-\frac{1}{1+x^2}\right)\Delta x$$

(ii) Hence, find the volume of the solid **3**

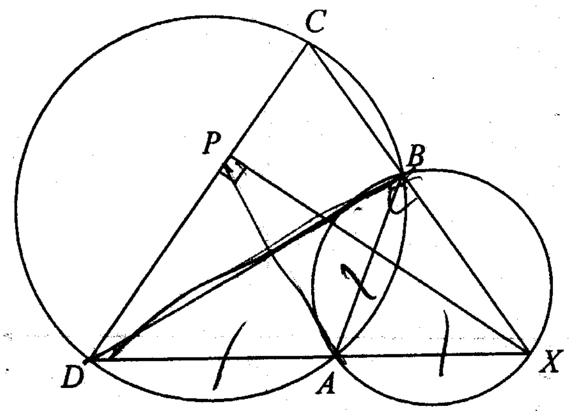
$$V = 2\pi\int (4-x)\left(1-\frac{1}{1+x^2}\right)dx$$

$$= 2\pi\int \left(4 - \frac{4}{1+x^2} - x + \frac{1}{1+x^2}\right)dx$$

QUESTION 7: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$.

(i) Prove that $\angle DBX = 90°$ **2**

(ii) Hence, or otherwise, prove that $AB = AP$. **3**

(b) (i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. **1**

(ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. **2**

(iii) Hence, or otherwise, prove that **2**

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a + b + c} > abc,$$

where a, b and c are distinct positive real numbers.

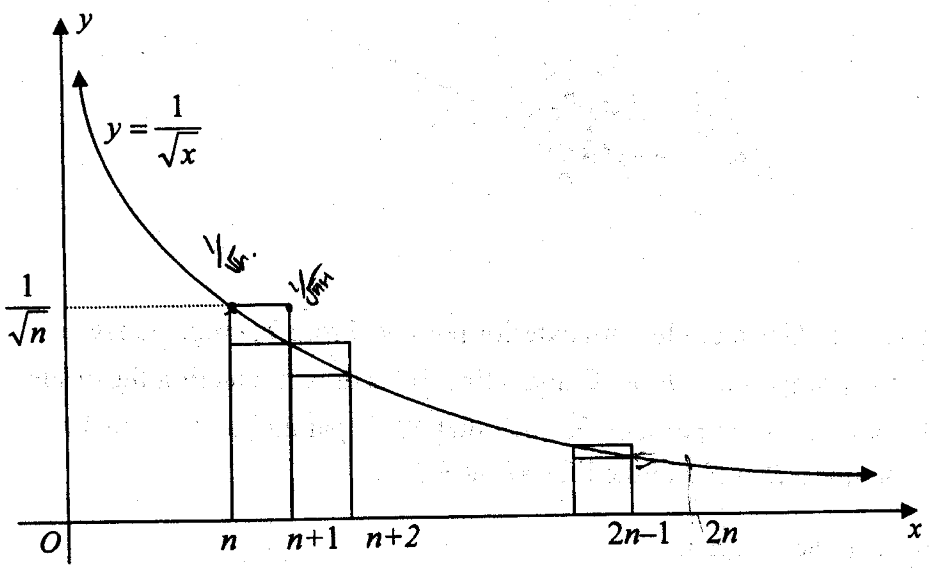
(c) A sequence, T_n, is such that $T_1 = 3$, $T_2 = 5$ and $T_{n+2} = 4T_{n+1} - 3T_n$. **5**
Prove by mathematical induction that $T_n = 3^{n-1} + 2$.

QUESTION 8: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Show that $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = 2\sqrt{n}\left(\sqrt{2}-1\right)$. **2**

(ii)



In the diagram above, the graph of $y=\dfrac{1}{\sqrt{x}}$ has been drawn, and n upper and lower rectangles have been constructed between $x=n$ and $x=2n$, each of width 1 unit.

Let $S_n = \dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \dfrac{1}{\sqrt{n+3}} + \cdots + \dfrac{1}{\sqrt{2n}}$.

(α) By considering the sums of areas of upper and lower rectangles, show that: **4**

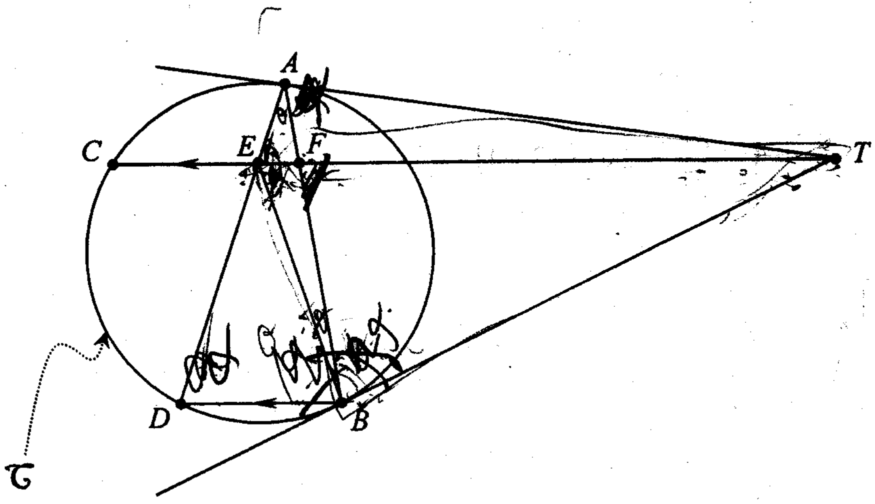
$$2\sqrt{n}\left(\sqrt{2}-1\right)+\frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}\left(\sqrt{2}-1\right)$$

(β) Hence find, correct to four decimal places, **1**

$$\frac{1}{\sqrt{10^8+1}}+\frac{1}{\sqrt{10^8+2}}+\frac{1}{\sqrt{10^8+3}}+\cdots+\frac{1}{\sqrt{2\times10^8}}$$

QUESTION 8 CONTINUES ON THE NEXT PAGE ....

(b)



In the diagram, $\mathcal{C}$ is a circle with exterior point T. From T, tangents are drawn to the points A and B on $\mathcal{C}$ and a line TC is drawn, meeting the circle at C. The point D is the point on $\mathcal{C}$ such that BD is parallel to TC. The line TC cuts the line AB at F and the line AD at E.

Copy or trace the diagram.

(i) Prove that ΔTFA is similar to ΔTAE. **3**

(ii) Deduce that $TE.TF = TB^2$. **2**

(iii) Show that ΔEBT is similar to ΔBFT. **2**

(iv) Prove that ΔDEB is isosceles. **1**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x = \log_e x, \quad x > 0$

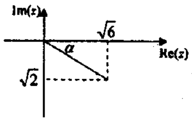
Year 12 Mathematics Extension 2		Trial HSC Examination 2005
Question No. 1	Solutions and Marking Guidelines	
	Outcomes Addressed in this Question	

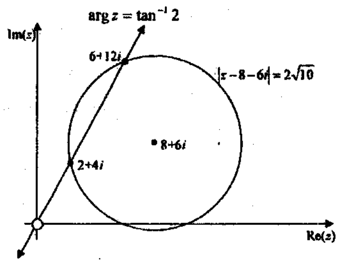
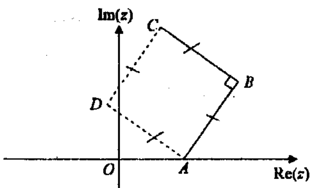
E8 Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

Outcome	Solutions	Marking Guidelines
E8 (a) (i)	$\int \frac{1}{x\ln x}dx$ Let $u=\ln x$ Then $\frac{du}{dx}=\frac{1}{x}$ $\therefore dx = x du$ Hence, $\int \frac{1}{x\ln x}dx = \int \frac{du}{u} = \ln u + c = \ln(\ln x)+c$	Correct solution ....2 Appropriate substitution done correctly ....1 OR Correct modified primitive or equivalent merit but fails to get the correct solution 1
E8 (a) (ii)	$\int \frac{xdx}{x^2+2x+5}$ $=\int \frac{\left[\frac{1}{2}(2x+2)-1\right]}{x^2+2x+5}dx$ $=\frac{1}{2}\int \frac{2x+2}{x^2+2x+5}dx - \int \frac{dx}{x^2+2x+5}$ $=\frac{1}{2}\ln(x^2+2x+5) - \int \frac{dx}{(x+1)^2+4}$ $=\frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$	Correct solution .... 3 Decomposes correctly into two known integrals, but fails to obtain correct answer .... 2 Partially decomposes into one or two known integrals and fails to obtain correct answer 1
E8 (b)	$\int_0^{\frac{\pi}{2}} x\cos x\,dx$ Let $u=x$ $\frac{dv}{dx}=\cos x$ $\frac{du}{dx}=1$ $v=\sin x$ Hence, $\int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx$ $=\frac{\pi}{2}-[\cos x]_0^{\frac{\pi}{2}}$ $=\frac{\pi}{2}-1$	Correct solution .... 2 Reasonable attempt to use the method of IBP 1
E8 (c)	$\frac{1}{x(x+2)}=\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)$ $\int_1^3 \frac{1}{x(x+2)}dx = \frac{1}{2}\left(\int_1^3\left(\frac{1}{x}-\frac{1}{x+2}\right)dx\right)$ $=\frac{1}{2}[\ln x - \ln(x+2)]_1^3$ $=\frac{1}{2}\left[\ln\left(\frac{3}{5}\right)-\ln\left(\frac{1}{3}\right)\right]$ $=\frac{1}{2}\ln\left(\frac{9}{5}\right)$	Correct solution .... 3 Applies method of partial fractions correctly, but fails to get the correct answer ...... 2 Reasonable attempt to use the method of partial fractions 1
E8 (d) (i)	$I_n=\int x^n e^{-x}dx$ Let $u=x^n$ $\frac{dv}{dx}=e^{-x}$ $\frac{du}{dx}=nx^{n-1}$ $v=-e^{-x}$ Hence, $I_n=\left[-e^{-x}x^n\right]_0^1 - \int_0^1 -e^{-x}nx^{n-1}dx$ $=-e^{-1}+nI_{n-1}$ $=nI_{n-1}-e^{-1}$ QED	Correct solution .... 2 Reasonable attempt to use the method of IBP 1
E8 (d) (ii)	Using the recurrence relation, $I_3=3I_2-e^{-1}$ $I_2=2I_1-e^{-1}$ $I_1=I_0-e^{-1}$ Also, $I_0=\int_0^1 e^{-x}dx=\left[-e^{-x}\right]_0^1=-e^{-1}+1$ Hence, $I_1=-e^{-1}+1-e^{-1}=1-2e^{-1}$ $I_2=2(1-2e^{-1})-e^{-1}=2-5e^{-1}$ $I_3=3(2-5e^{-1})-e^{-1}=6-16e^{-1}$ QED	Correct solution .... 3 Applies recurrence relation correctly, but fails to get the correct answer .... 2 Reasonable attempt to use the recurrence relation 1

Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 2	Solutions and Marking Guidelines	
	Outcomes Addressed in this Question	

E3 uses the relationship between algebraic and geometric representations of complex numbers

Outcome	Solutions	Marking Guidelines
E3 (a)(i)	$\text{Re}(z^2)=\text{Re}(\sqrt{6}-\sqrt{2}i)^2$ $=\text{Re}(6-4\sqrt{3}i-2)$ $=4$	**1 mark** : correct answer
E3 (a)(ii)	$\|z\|=\sqrt{\sqrt{6}^2+\sqrt{2}^2}$ $=2\sqrt{2}$	**1 mark** : correct answer
E3 (a)(iii)	$\tan\alpha=\frac{\sqrt{2}}{\sqrt{6}}=\frac{1}{\sqrt{3}}$ $\alpha=\frac{\pi}{6}$ so $\arg z=-\frac{\pi}{6}$  	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (a)(iv)	From (i), $z^2=4-4\sqrt{3}i$. So $z^4=[4(1-\sqrt{3}i)]^2$ $=16(1-3-2\sqrt{3}i)$ $=-32(1+\sqrt{3}i)$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (b)(i)	$\arg z=\tan^{-1}2$ becomes $y=2x,\ (x\neq 0)$ and $\|z-8-6i\|=2\sqrt{10}$ becomes $(x-8)^2+(y-6)^2=40$. Points of intersection occur when $(x-8)^2+(2x-6)^2=40$ $x^2-16x+64+4x^2-24x+36=40$ $5x^2-40x+60=0$ $x^2-8x+12=0$ $(x-6)(x-2)=0$ $x=6$ or 2 $\therefore$ Points of intersection are $z=6+12i$ and $z=2+4i$	**3 marks** : correct solution (NB: must solve simultaneously to show these are the only points of intersection) **2 marks** : substantially correct **1 mark** : demonstrating some knowledge of the cartesian equations

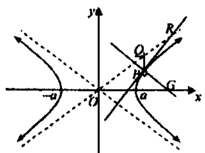
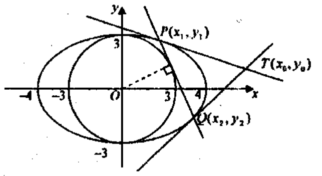
Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 2	Solutions and Marking Guidelines	
Outcome	Solutions	
E3 (b)(ii)	 	**2 marks** : correct solution (NB : must show points of intersection, and open circle on $y=2x$ at origin) **1 mark** : substantially correct
E3 (c)(i)	 $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}$ $=2+(1+\sqrt{5}i)+i(1+\sqrt{5}i)$ $\therefore z_3=3-\sqrt{5}+(1+\sqrt{5})i$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (c)(ii)	$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$ $=\overrightarrow{OA}+\overrightarrow{BC}$ $=2+i(1+\sqrt{5}i)$ $\therefore z_4=2-\sqrt{5}+i$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)

Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 3	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

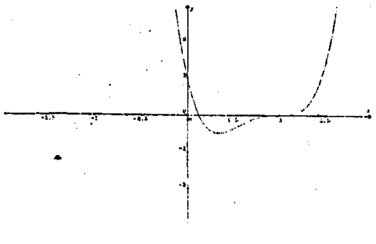
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Outcome	Solutions	Marking Guidelines
E4 (a)(i)	$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$ $=\frac{b\sec^2\theta}{a\sec\theta\tan\theta}$ $=\frac{b\sec\theta}{a\tan\theta}$  $\therefore$ Equation of tangent at $P(a\sec\theta, b\tan\theta)$ is $y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$ $\frac{y\tan\theta}{b}-\tan^2\theta=\frac{x\sec\theta}{a}-\sec^2\theta$ $1=\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}$	**2 marks** : correct solution **1 mark** : substantially correct, including correct expression for $\frac{dy}{dx}$.
E4 (a)(ii)	Equation of normal is $y-b\tan\theta=-\frac{a\tan\theta}{b\sec\theta}(x-a\sec\theta)$ $\times$ by $\frac{b}{\tan\theta}$ $\frac{by}{\tan\theta}-b^2=-\frac{ax}{\sec\theta}+a^2$ $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$	**2 marks** : correct solution **1 mark** : substantially correct, including correct expressions subbed into point-gradient formula.
E4 (a)(iii)	Equation of asymptote given is $y=\frac{b}{a}x\,(a,b>0)$ ie $m_{OR}=\frac{b}{a}$ At Q, $x=a\sec\theta$ so $y=\frac{b}{a}x=\frac{b}{a}a\sec\theta=b\sec\theta$ ie Q is $(a\sec\theta, b\sec\theta)$ Normal crosses x-axis at $y=0$, so $\frac{ax}{\sec\theta}+\frac{b\times 0}{\tan\theta}=a^2+b^2$ $x=\frac{(a^2+b^2)\sec\theta}{a}$ ie G is $\left(\frac{(a^2+b^2)\sec\theta}{a},0\right)$	**3 marks** : correct solution **2 marks** : susbstantially correct, including correct expression for point G, and correctly finding one gradient. **1 mark** : partially correct, including correct expression for point G, or correctly finding one gradient.
Question 3 continued	so, $m_{QG}=\frac{b\sec\theta-0}{a\sec\theta-\frac{(a^2+b^2)\sec\theta}{a}}$ $=\frac{ab}{a^2-(a^2+b^2)}=-\frac{a}{b}$ now, $m_{OR}\times m_{QG}=\frac{b}{a}\times-\frac{a}{b}=-1$ $\therefore \angle PQR=90°$	
E3 (a)(iv)	$RQPG$ is a cyclic quadrilateral with RG as diameter, since RG subtends right angles at P and Q.	**1 mark** : for stating cyclic quadrilateral.
E4 (b)(i)	 Equation of tangent at P is $\frac{xx_1}{16}+\frac{yy_1}{9}=1$. Similarly, equation of tangent at Q is $\frac{xx_2}{16}+\frac{yy_2}{9}=1$. Now, $T(x_0,y_0)$ satisfies both these equations. $\therefore \frac{x_0x_1}{16}+\frac{y_0y_1}{9}=1$ and $\frac{x_0x_2}{16}+\frac{y_0y_2}{9}=1$. Ie both P and Q satisfy $\frac{xx_0}{16}+\frac{yy_0}{9}=1$, and so this is the equation of the chord of contact.	**2 marks** : correct solution **1 mark** : substantially correct
E4 (b)(ii)	If PQ touches $x^2+y^2=9$, then the distance of PQ from $(0, 0)$ is 3. $\therefore$ by perpendicular distance formula, $\frac{\left\lvert\frac{x_0}{16}\times 0+\frac{y_0}{9}\times 0-1\right\rvert}{\sqrt{\left(\frac{x_0}{16}\right)^2+\left(\frac{y_0}{9}\right)^2}}=3$ $\frac{1}{\sqrt{\frac{x_0^2}{256}+\frac{y_0^2}{81}}}=3$ ie $\frac{x_0^2}{256}+\frac{y_0^2}{81}=\frac{1}{9}$ $\frac{9x_0^2}{256}+\frac{y_0^2}{9}=1$	**4 marks** : correct solution **3 marks** : substantially correct – must include correct radius of circle, and correctly using perpendicular distance formula. **2 marks** : partially correct – must include correct radius of circle, or correctly using perpendicular distance formula. **1 mark** : basic method is correct
E3 (b)(iii)	The locus of T is an ellipse, centred at the origin.	**1 mark** : stating ellipse

Year 12 Mathematics Extension 2		Trial HSC Examination 2005
Question No. 4	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

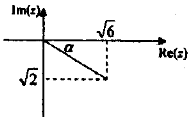
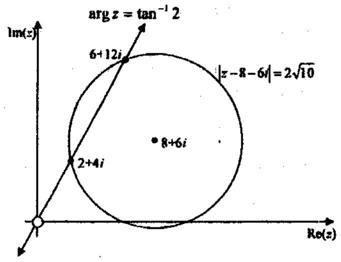
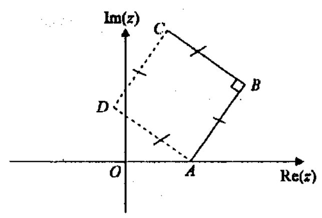
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Outcome	Solutions	Marking Guidelines
E4 (a) (i)	For $P(x)$ to have a zero with multiplicity of 3, we can write $P(x)$ as $P(x)=(x-\alpha)^3Q(x)$ where $Q(\alpha)\neq 0$. Differentiating $P(x)$ gives $P'(x)=(x-\alpha)^3Q'(x)+3(x-\alpha)^2Q(x)$ $=(x-\alpha)^2[(x-\alpha)Q'(x)+3Q(x)]$ $=(x-\alpha)^2R(x)$ where $R(\alpha)\neq 0$. So $P(x)$ has a zero of multiplicity 2.	Correct solution .... 2 Substantially correct solution 1
E4 (a) (ii)	Let $P(x)=8x^4-25x^3+27x^2-11x+1$ and let $x=\alpha$ be the zero of multiplicity 3. Differentiating $P'(x)=32x^3-75x^2+54x-11$ $P''(x)=96x^2-150x+54$ $=6(16x^2-25x+9)$ $=6(x-1)(16x-9)$ So the zeros of $P''(x)$ are $x=1$ and $x=\frac{9}{16}$. Testing $x=1$, $P(1)=0$ and $P'(0)=0$, so $P(x)=(x-1)^3Q(x)$. Let $x=\beta$ be the other zero. $\alpha+\alpha+\alpha+\beta=\frac{25}{8}$ $\beta=\frac{25}{8}-3\alpha=\frac{25}{8}-3=\frac{1}{8}$ So, the zeros of $P(x)=8x^4-25x^3+27x^2-11x+1$ are $x=1,1,1,\frac{1}{8}$.	Correct solution .... 2 Substantially correct solution 1
E8(a) (iii)		Correct solution 1
E4 (b) (i)	$\cos 5\theta=-1$ $\therefore 5\theta=2n\pi+\pi=(2n+1)\pi$ for $n=0,\pm1,\pm2,....$ $\therefore \theta=(2n+1)\frac{\pi}{5}$ for $n=0,\pm1,\pm2,....$ For $0\le\theta\le 2\pi$, $\theta=\frac{\pi}{5},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{9\pi}{5}$	Correct solution .... 2 Determination of general solution only .... 1 OR Particular solution only 1
E4 (b) (ii)	Using the binomial expansion, $\mathrm{Re}\{(\cos\theta+i\sin\theta)^5\}$ $=\cos^5\theta+10\cos^3\theta(i\sin\theta)^2+5\cos\theta(i\sin\theta)^4$ $=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ $=\cos^5\theta-10\cos^3\theta(1-\cos^2\theta)+5\cos\theta(1-\cos^2\theta)^2$ $=16\cos^5\theta-20\cos^3\theta+5\cos\theta$ Using De Moivre's Theorem $(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$ Hence, $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$	Correct solution .... 3 Determination of $\mathrm{Re}\{(\cos\theta+i\sin\theta)^5\}$ in terms of $\cos\theta$ only .... 2 Determination of $\mathrm{Re}\{(\cos\theta+i\sin\theta)^5\}$ in terms of $\cos\theta$ and $\sin\theta$ .... 1 OR Correct use of De Moivre's Theorem 1
E4(b) (iii)	In the given equation let $x=\cos\theta$. The equation becomes $16\cos^5\theta-20\cos^3\theta+5\cos\theta+1=0$. This can be written as $\cos 5\theta+1=0$, $\therefore \cos 5\theta=-1$ $\therefore \theta=\frac{\pi}{5},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{9\pi}{5}$ $\therefore x=\cos\frac{\pi}{5},\cos\frac{3\pi}{5},\cos\pi,\cos\frac{7\pi}{5},\cos\frac{9\pi}{5}$ $\therefore x=\cos\frac{\pi}{5},\cos\frac{3\pi}{5},-1,\cos\left(-\frac{3\pi}{5}\right),\cos\left(-\frac{\pi}{5}\right)$ $\therefore x=\cos\frac{\pi}{5},\cos\frac{\pi}{5},\cos\frac{3\pi}{5},\cos\frac{3\pi}{5},-1$	Correct solution .... 2 Substantially correct solution 1
E4(b) (iv)	Sum of roots = 0 $2\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)-1=0$ $\therefore \cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$	Correct solution .... 3 Correct values for $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}$ only .... 2 OR Reasonable attempt at factorisation based upon values of $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}$ 2

Year 12 Mathematics Extension 2		Trial HSC Examination 2005
Question No. 1	**Solutions and Marking Guidelines**	
	Outcomes Addressed in this Question	
E8	Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems	

Outcome	Solutions	Marking Guidelines
E8 (a) (i)	$\int \frac{1}{x\ln x}\,dx$ Let $u=\ln x$ Then $\frac{du}{dx}=\frac{1}{x}$ $\therefore dx = x\,du$ Hence, $\int \frac{1}{x\ln x}\,dx = \int \frac{du}{u} = \ln u + c = \ln(\ln x)+c$	Correct solution ……………2 Appropriate substitution done correctly ……………………1 OR Correct modified primitive or equivalent merit but fails to get the correct solution ………1
E8 (a) (ii)	$\int \frac{x\,dx}{x^2+2x+5}$ $=\int \frac{\left[\frac{1}{2}(2x+2)-1\right]}{x^2+2x+5}dx$ $=\frac{1}{2}\int \frac{2x+2}{x^2+2x+5}dx - \int \frac{dx}{x^2+2x+5}$ $=\frac{1}{2}\ln(x^2+2x+5) - \int \frac{dx}{(x+1)^2+4}$ $=\frac{1}{2}\ln(x^2+2x+5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$	Correct solution …………… 3 Decomposes correctly into two known integrals, but fails to obtain correct answer …….. 2 Partially decomposes into one or two known integrals and fails to obtain correct answer ……. 1
E8 (b)	$\int_0^{\frac{\pi}{2}} x\cos x\,dx$ Let $u=x$ $\frac{dv}{dx}=\cos x$ $\frac{du}{dx}=1$ $v=\sin x$ Hence, $\int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$ $=\frac{\pi}{2} - [\cos x]_0^{\frac{\pi}{2}}$ $=\frac{\pi}{2}-1$	Correct solution ………… 2 Reasonable attempt to use the method of IBP …………… 1
E8 (c)	$\frac{1}{x(x+2)} = \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x+2}\right)$ $\int_1^3 \frac{1}{x(x+2)}dx = \frac{1}{2}\left(\int_1^3\left(\frac{1}{x}-\frac{1}{x+2}\right)dx\right)$ $=\frac{1}{2}\left[\ln x - \ln(x+2)\right]_1^3$ $=\frac{1}{2}\left[\ln\left(\frac{3}{5}\right) - \ln\left(\frac{1}{3}\right)\right]$ $=\frac{1}{2}\ln\left(\frac{9}{5}\right)$	Correct solution ………… 3 Applies method of partial fractions correctly, but fails to get the correct answer …… 2 Reasonable attempt to use the method of partial fractions …… 1
E8 (d) (i)	$I_n = \int x^n e^{-x}dx$ Let $u=x^n$ $\frac{dv}{dx}=e^{-x}$ $\frac{du}{dx}=nx^{n-1}$ $v=-e^{-x}$ Hence, $I_n = \left[-e^{-x}x^n\right]_0^1 - \int_0^1 -e^{-x}nx^{n-1}dx$ $=-e^{-1}+nI_{n-1}$ $=nI_{n-1}-e^{-1}$ QED	Correct solution ………… 2 Reasonable attempt to use the method of IBP ……………1
E8 (d) (ii)	Using the recurrence relation, $I_3 = 3I_2 - e^{-1}$ $I_2 = 2I_1 - e^{-1}$ $I_1 = I_0 - e^{-1}$ Also, $I_0 = \int_0^1 e^{-x}dx = \left[-e^{-x}\right]_0^1 = -e^{-1}+1$ Hence, $I_1 = -e^{-1}+1-e^{-1} = 1-2e^{-1}$ $I_2 = 2(1-2e^{-1})-e^{-1} = 2-5e^{-1}$ $I_3 = 3(2-5e^{-1})-e^{-1} = 6-16e^{-1}$ QED	Correct solution ………… 3 Applies recurrence relation correctly, but fails to get the correct answer ………….. 2 Reasonable attempt to use the recurrence relation ……… 1

Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 2	Solutions and Marking Guidelines	
	Outcomes Addressed in this Question	
E3	uses the relationship between algebraic and geometric representations of complex numbers	

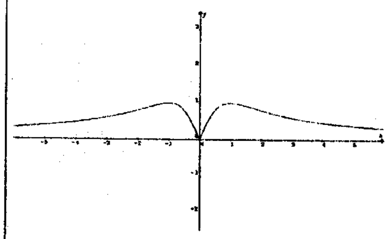
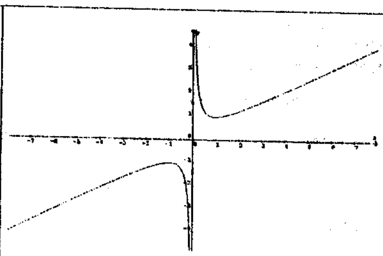
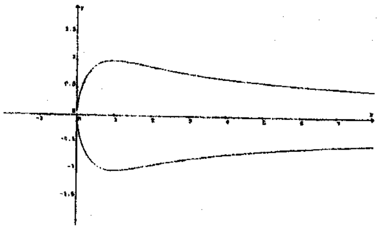
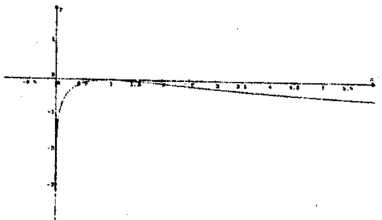
Outcome	Solutions	Marking Guidelines
E3 (a)(i)	$\mathrm{Re}(z^2) = \mathrm{Re}(\sqrt{6}-\sqrt{2}i)^2$ $=\mathrm{Re}(6-4\sqrt{3}i-2)$ $=4$	**1 mark** : correct answer
E3 (a)(ii)	$\lvert z\rvert = \sqrt{\sqrt{6}^2+\sqrt{2}^2}$ $=2\sqrt{2}$	**1 mark** : correct answer
E3 (a)(iii)	$\tan\alpha = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$ $\alpha = \frac{\pi}{6}$ so $\arg z = -\frac{\pi}{6}$ 	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (a)(iv)	From (i), $z^2 = 4-4\sqrt{3}i$. So $z^4 = \left[4(1-\sqrt{3}i)\right]^2$ $=16(1-3-2\sqrt{3}i)$ $=-32(1+\sqrt{3}i)$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (b)(i)	$\arg z = \tan^{-1}2$ becomes $y=2x$, $(x\neq 0)$ and $\lvert z-8-6i\rvert = 2\sqrt{10}$ becomes $(x-8)^2+(y-6)^2=40$. Points of intersection occur when $(x-8)^2+(2x-6)^2=40$ $x^2-16x+64+4x^2-24x+36=40$ $5x^2-40x+60=0$ $x^2-8x+12=0$ $(x-6)(x-2)=0$ $x=6$ or 2 $\therefore$ Points of intersection are $z=6+12i$ and $z=2+4i$	**3 marks** : correct solution (NB: must solve simultaneously to show these are the only points of intersection) **2 marks** : substantially correct **1 mark** : demonstrating some knowledge of the cartesian equations
E3 (b)(ii)		**2 marks** : correct solution (NB : must show points of intersection, and open circle on $y=2x$ at origin) **1 mark** : substantially correct
E3 (c)(i)	 $\overrightarrow{OC} = \overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}$ $=2+(1+\sqrt{5}i)+i(1+\sqrt{5}i)$ $\therefore z_3 = 3-\sqrt{5}+(1+\sqrt{5})i$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)
E3 (c)(ii)	$\overrightarrow{OD} = \overrightarrow{OA}+\overrightarrow{AD}$ $=\overrightarrow{OA}+\overrightarrow{BC}$ $=2+i(1+\sqrt{5}i)$ $\therefore z_4 = 2-\sqrt{5}+i$	**2 marks** : correct answer **1 mark** : substantially correct (basically, correct method with incorrect values)

	Solutions	Marking Guidelines
	Product of roots = $-\frac{1}{16}$ $-\left(\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}\right)^2=-\frac{1}{16}$ $\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}=\pm\frac{1}{4}$ Since $\cos\frac{\pi}{5}>0$ and $\cos\frac{3\pi}{5}<0$, $\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}=-\frac{1}{4}$ This means that $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are the roots of $x^2+\left(-\frac{1}{2}\right)x+\left(-\frac{1}{4}\right)=0$ $4x^2-2x-1=0$ Hence, $16x^5-20x^3+5x+1=(x+1)(4x^2-2x-1)^2$	Correct value of $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}$ or $\cos\frac{\pi}{5}\times\cos\frac{3\pi}{5}$ only ..... 1 OR Attempt at factorisation of given polynomial 1

Question No. 5 **Solutions and Marking Guidelines**

Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

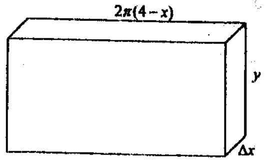
Outcome	Solutions	Marking Guidelines
E6 (a) (i)	By the quotient rule, $y'=\frac{(1+x^2).2-2x.2x}{(1+x^2)^2}=\frac{2-2x^2}{(1+x^2)^2}$ Turning points are where $y'=0$, $y'=0$ when $x=\pm1$ Turning points are $A(1, 1)$ and $B(-1,-1)$	Correct solution .... 2 Correctly applies the quotient rule, but does not determine coordinates of turning points 1
E6 (a) (ii)	By the quotient rule, $y''=\frac{(1+x^2)^2.(-4x)-2(1-x^2).4x(1+x^2)}{(1+x^2)^4}$ $=\frac{(1+x^2)^2(-4x-4x^3-8x+8x^3)}{(1+x^2)^4}$ $=\frac{4x(x^2-3)}{(1+x^2)^3}$ Inflexion points are where $y''=0$, $y''=0$ when $x=0$ or $\pm\sqrt{3}$ Points of inflexion are $P\left(\sqrt{3},\frac{\sqrt{3}}{2}\right)$ and $Q\left(-\sqrt{3},-\frac{\sqrt{3}}{2}\right)$	Correct solution .... 3 Correctly applies the quotient rule, but does not determine coordinates of the points of inflection .... 2 OR Reasonable attempt to use quotient rule and then find the points of inflection .... 2 Attempts to use the quotient rule to determine points of inflection .... 1 OR Attempts to find the points of inflection by some method 1
E6 (b) (i)		Correct graph 1
E6 (b) (ii)		Correct graph showing asymptote .... 2 Correct graph, without asymptote 1
E6(b) (iii)		Correct graph showing symmetry .... 2 Graph, without obvious symmetry 1
E6(b) (iv)		Correct graph showing correct concavity as $x\to\infty$......... 2 Graph, without obvious concavity as $x\to\infty$......... 1
E6 (c) (i)	$kx^3+(k-2)x=0$ $kx^3+kx-2x=0$ $2x=kx+kx^3=kx(1+x^2)$ $kx=\frac{2x}{1+x^2}$	Correct solution 1
E6 (c) (ii)	The curve $y=\frac{2x}{1+x^2}$ has gradient 2 at (0, 0). So $y=kx$ and the curve will intersect exactly once for $k\geq2$ or $k\leq0$.	Correct solution .... 2 Substantially correct solution 1

Question No. 6 **Solutions and Marking Guidelines**

Outcomes Addressed in this Question

E7 uses the techniques of slicing and cylindrical shells to determine volumes

Outcome	Solutions	Marking Guidelines
E7 (a)	$\frac{x^2}{20}+\frac{y^2}{15}=1$ $y=\sqrt{15\left(1-\frac{x^2}{20}\right)}$ $V=\pi\int_{-\sqrt{20}}^{\sqrt{20}}y^2dx$ $V=2\pi\int_0^{\sqrt{20}}15\left(1-\frac{x^2}{20}\right)dx$ $V=30\pi\left[x-\frac{x^3}{60}\right]_0^{\sqrt{20}}=40\pi\sqrt{5}$ unit^3	Correct solution .... 3 Solution indicating an expression to be evaluated in order to determine the volume and correctly determining this volume .... 2 Solution indicating an expression to be evaluated in order to determine the volume 1
E7 (b) (i)	Radius of strip is $(x-1)$, width of strip is Δy $\therefore A=\pi(x-1)^2$ and hence $\Delta V=A\times\Delta y=\pi(x-1)^2\Delta y$	Correct solution .... 2 Solution without sufficient justification 1
E7 (b) (ii)	$y=2x-x^2$ $x^2-2x=-y$ $(x-1)^2=1-y$ $\therefore\Delta V=\pi(1-y)\Delta y$ The volume of the solid is the sum of all strips from $y=0$ to $y=1$ $\therefore V=\lim_{\Delta y\to0}\sum_{y=0}^{1}\pi(1-y)\Delta y$	Correct solution .... 2 Solution without sufficient justification 1
E7(b) (iii)	$V=\pi\int_0^1(1-y)dy$ $=\pi\left[y-\frac{y^2}{2}\right]_0^1$ $=\frac{\pi}{2}$ unit^3	Correct solution .... 2 Correct statement for volume, with volume incorrectly determined 1

E7 (c) (i)	Radius of shell is $(4-x)$ and height is y A typical shell when cut open looks like  $A=2\pi(4-x)y$ $=2\pi(4-x)\left(\frac{x^2}{x^2+1}\right)$ $=2\pi(4-x)\left(\frac{x^2+1-1}{x^2+1}\right)$ $=2\pi(4-x)\left(1-\frac{1}{x^2+1}\right)$ Hence, $\Delta V=A\times\Delta x$ $=2\pi(4-x)\left(\frac{x^2+1-1}{x^2+1}\right)\Delta x$ $=2\pi(4-x)\left(1-\frac{1}{x^2+1}\right)\Delta x$	Correct solution .... 3 Solution which attempts to find the elemental volume ~ correctly stating radius and height ~ but failing to derive the desired result .... 2 Solution which attempts to find the elemental volume ~ with only radius or height correct 1
E7 (c) (ii)	$V=\lim_{\Delta x\to 0}\sum_{x=0}^{2}2\pi(4-x)\left(1-\frac{1}{x^2+1}\right)\Delta x$ $=2\pi\int_0^2(4-x)\left(1-\frac{1}{x^2+1}\right)dx$ $=2\pi\int_0^2\left\{4-\frac{4}{x^2+1}-x+\frac{x}{x^2+1}\right\}dx$ $=2\pi\left[4x-4\tan^{-1}x-\frac{x^2}{2}+\frac{1}{2}\ln\left(1+x^2\right)\right]_0^2$ $=\pi(12-8\tan^{-1}2+\ln 5)$ unit^3	Correct solution .... 3 Solution which correctly finds the volume ~ but does not indicate the limit .... 2 OR Solution which attempts to find the volume ~ showing all steps ~ but incorrectly calculates the volume .... 2 Solution which only states the limit in correct form .... 1 OR Solution which correctly evaluates the volume based upon incorrect assumptions 1

Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 7	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

Outcome	Solutions	Marking Guidelines
PE3 (a)(i)	The circle through D, B and X has centre A, since $AD=AB=AX$. Hence DAX is a diameter. Hence $\angle DBX=90°$ (angle in a semicircle).	**2 marks** : Writes correct argument. **1 mark** : Recognises that D, B and X lie on a circle centred at A.
PE3(a)(ii)	By the converse of the angle in a semicircle, since $\angle DPX$ is a right angle, the circle with diameter DAX also passes through P. Hence $AP=AB$ (radii).	**3 marks** : writes correct argument. **2 marks** : argues that the circle with diameter DAX also passes through P without giving reasons. **1 mark** : Attempts to determine AP
PE3 (b)(i)	Since $a\neq b$, $a-b\neq 0$, so $(a-b)^2>0$ Hence, $a^2-2ab+b^2>0$ $a^2+b^2>2ab$	**1 mark** : Gives appropriate explanation.
PE3(b)(ii)	From (i), $a^2+b^2>2ab$ $b^2+c^2>2bc$ and $a^2+c^2>2ac$. Adding, $2\left(a^2+b^2+c^2\right)>2(ab+ac+bc)$ $a^2+b^2+c^2>ab+ac+bc$	**2 marks** : Correctly derives the inequation **1 mark** : Provides an unfinished derivation of the inequation, showing $b^2+c^2>2bc$
PE3(b)(iii)	Let $A=ab$, $B=bc$ and $C=ac$ Then A, B and C are distinct positive numbers, and from (ii), $A^2+B^2+C^2>AB+AC+BC$ Substituting, $a^2b^2+b^2c^2+a^2c^2>(ab)(bc)+(ab)(ac)+(bc)(ac)$. Now $(ab)(bc)+(ab)(ac)+(bc)(ac)=abc(a+b+c)$ Hence, $a^2b^2+b^2c^2+a^2c^2>abc(a+b+c)$ $\frac{a^2b^2+b^2c^2+a^2c^2}{a+b+c}>abc$	**2 marks** : Establishes result **1 mark** : Obtains correct inequality for $a^2b^2+b^2c^2+a^2c^2$

	Question 7 cont'd	
E2 (c)	*Step 1* : Show true for $n=1,2$ $T_1=3^{1-1}+2=3$ $T_2=3^{2-1}+2=5$ ✓ *Step 2* : Assume true for $n=k$, and $n=k-1$ ie $T_k=3^{k-1}+2$ $T_{k-1}=3^k+2$ *Step 3* : Prove true for $n=k+1$ ie prove $T_{k+1}=3^k+2$ is true $T_{k+1}=4T_k-3T_{k-1}$ $=4.3^{k-1}+8-3.3^{k-2}-6$ $=4.3.3^{k-2}-3.3^{k-2}+2$ $=3^{k-2}(12-3)+2$ $=3^{k-2}.3^2+2$ $=3^k+2$ *Step 4* ∴ if true for $n=k-1$, k then true for $n=k+1$ but true for $n=1,2$ ∴ true for $n=2+1=3$, $n=3+1=4$, etc ∴ true for all integer values of n.	**5 marks** : complete solution **4 marks** : substantially correct solution **3 marks** : significant progress towards correct solution **2 marks** : attempts to follow the process of mathematical induction but fails to make significant progress **1 mark** : Shows true for numerical value of n only

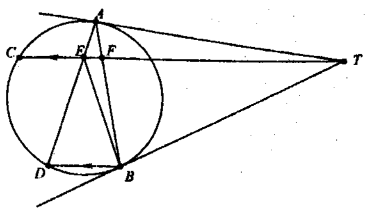
Year 12	Mathematics Extension 2	TRIAL Examination 2005
Question No. 8	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

Outcome	Solutions	Marking Guidelines
(a)(i) PE3	$\int_n^{2n}\frac{dx}{\sqrt{x}}=\left[2\sqrt{x}\right]_n^{2n}$ $=2\sqrt{2n}-2\sqrt{n}$ $=2\sqrt{n}\left(\sqrt{2}-1\right)$	**2 marks** : correct soltion **1 mark** : substantially correct.
(a)(ii)(α) PE3	The upper rectangles have area greater than the integral, so $\int_n^{2n}\frac{dx}{\sqrt{x}}<\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+\ldots+\frac{1}{\sqrt{2n-1}}$ Adding $\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{n}}$ to both sides, and using (i) gives $2\sqrt{n}\left(\sqrt{2}-1\right)+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{n}}<\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{2n}}$ ...① The lower rectangles have area less than the integral, so $\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{2n}}<\int_n^{2n}\frac{dx}{\sqrt{x}}$ ie $\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\ldots+\frac{1}{\sqrt{2n}}<2\sqrt{n}\left(\sqrt{2}-1\right)$ ...② From ① and ② we get $2\sqrt{n}\left(\sqrt{2}-1\right)+\frac{1}{\sqrt{2n}}-\frac{1}{\sqrt{n}}<S_n<2\sqrt{n}\left(\sqrt{2}-1\right)$ ie $2\sqrt{n}\left(\sqrt{2}-1\right)+\frac{1-\sqrt{2}}{\sqrt{2n}}<S_n<2\sqrt{n}\left(\sqrt{2}-1\right)$	**4 marks** : Establishes result **3 marks** : Substantially correct - Derives inequalities ① and ② (or equivalent) without final inequality **2 marks** : Partially correct – eg, establishing $S_n<\int<2\sqrt{n}\left(\sqrt{2}-1\right)$ **1 mark** : establishing one inequality correctly.
(a)(ii)(β) PE3	When $n=10^8$, $S_n\approx 8284\cdot 2712$ on substitution into result from (ii)(α)	**1 mark** : correct result

Question 8 continued



(b)(i) PE3	$\angle AET = \angle ADB$ (corresponding, $CT \parallel DB$) $\angle ADB = \angle FAT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment) $\therefore \angle AET = \angle FAT$ (both = $\angle ADB$) Also, $\angle ATF = \angle ETA$ (common) So, $\Delta TFA \parallel \Delta TAE$ (equiangular)	**3 marks** : writes correct argument. **2 marks** : incomplete, but relevent argument. **1 mark** : recognises relevent data without constructing an argument
(b)(ii) PE3	$\frac{TA}{TE} = \frac{TF}{TA}$ [scale of ratios of similar $\Delta's$ in (i)] $\therefore TE.TF = TA^2$ But $TA = TB$ (tangents from an external point are equal) $\therefore TE.TF = TB^2$	**2 marks** : writes correct argument. **1 mark** : incomplete, but relevent argument.
(b)(iii) PE3	$\angle BTE = \angle FTB$ (common) $\frac{TE}{TB} = \frac{TB}{TF}$ (from (ii)) $\therefore \Delta EBT \parallel \Delta BFT$ (2 pairs of corresponding sides are in proportion and their included angles are equal)	**2 marks** : writes correct argument. **1 mark** : incomplete, but relevent argument.
(b)(iv) PE3	$\angle EDB = \angle FBT$ ($\angle$ between chord and tangent equals $\angle$ in alternate segment) $\angle FBT = \angle BET$ (corresponding, $\Delta EBT \parallel \Delta BFT$) $\angle BET = \angle EBD$ (alternate, $CT \parallel DB$) $\therefore \angle EDB = \angle EBD$ $\therefore \Delta DEB$ is isosceles (base angles equal)	**1 marks** : writes correct argument.