HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12

MATHEMATICS EXTENSION 2

2008

TRIAL HSC EXAMINATION (ASSESSMENT TASK 4)

EXAMINERS ~ G. RAWSON AND J. DILLON

GENERAL INSTRUCTIONS

- Reading time – 5 minutes.
- Working time – 3 hours.
- Attempt all 8 questions.
- Each question is worth 15 marks.
- Total marks – 120 marks
- All necessary working should be shown in every question.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators and mathematical templates may be used.
- A table of standard integrals is supplied.
- Each question is to be started in a new examination booklet.
- This assessment task must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

Students are advised that this is a trial examination only and cannot in any way guarantee the content or the format of the 2008 HSC Mathematics Extension 2 Examination.

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Total Marks 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet

Marks

(a) (i) Find $\int \frac{x}{\sqrt{9-16x^2}}\, dx$ **2**

(ii) Find $\int \frac{x^2}{x+1}\, dx$ **2**

(iii) Evaluate $\int_0^{\ln 3} xe^x\, dx$ **3**

(b) (i) Find real numbers A, B and C such that $\frac{2}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$ **3**

(ii) Hence, find $\int_0^1 \frac{2}{(t+1)(t^2+1)}\, dt$ **3**

(iii) By using the substitution $t = \tan\left(\frac{x}{2}\right)$ evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x - \cos x}\, dx$ **2**

Question 2 (15 marks) Use a SEPARATE writing booklet

Marks

(a) Let $A = 3 - 4i$ and $B = 2 + i$.

Find in the form $x + iy$

(i) $A - B$ **1**

(ii) $\overline{A}B$ **1**

(iii) $\dfrac{5}{A}$ **2**

(iv) $\dfrac{iB}{\overline{B}}$ **2**

(b) If $P = \sqrt{3} + i$

(i) Express P in modulus-argument form. **2**

(ii) Hence find P^4 in modulus-argument form. **2**

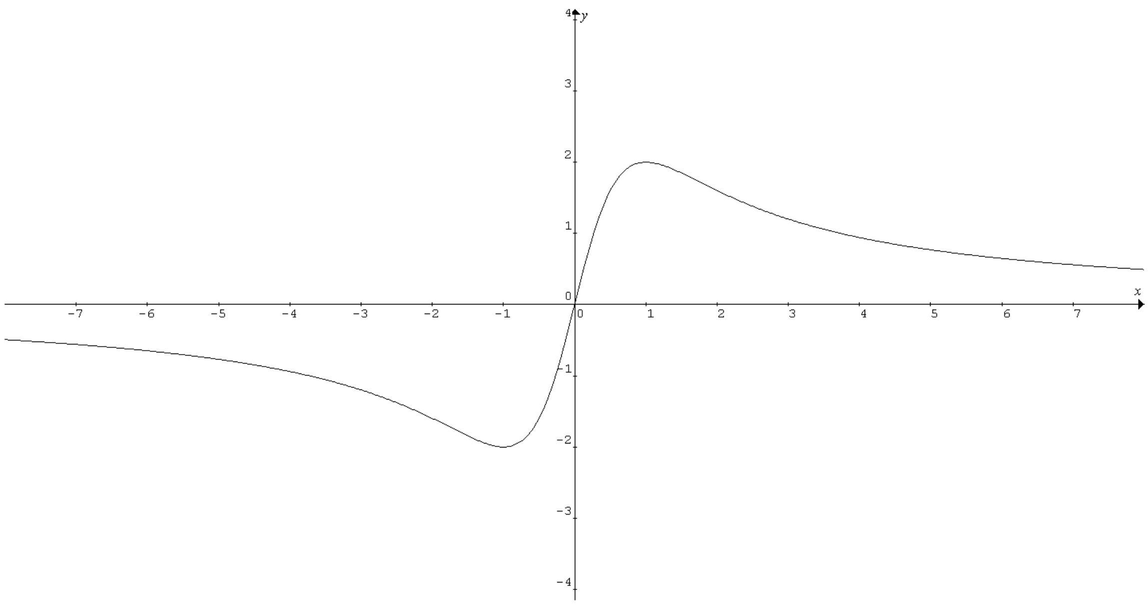
(c) On an Argand diagram, show the region where the inequalities $1 \le |Z| \le 3$ and $\dfrac{\pi}{4} \le \arg Z \le \dfrac{\pi}{2}$ hold simultaneously. **2**

(d) Describe the locus of Z on the Argand diagram if $\arg(Z-1) - \arg(Z+1) = \dfrac{\pi}{3}$, giving its Cartesian equation. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet

Marks

(a) The diagram shows the graph of $y = f(x)$



Draw separate sketches of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \left(f\left(x\right)\right)^2$ **2**

(iii) $y = f'(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(v) $y = x + f(x)$ **2**

(vi) $y^2 = f(x)$ **2**

(b) Find the equation of the tangent to the curve $x^2 + x - xy + y + y^2 = 12$ at the point (0, 3) **3**

Question 4 (15 marks) Use a SEPARATE writing booklet

Marks

(a) If p, q, and r are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, find the equation whose roots are **3**

$$\frac{1}{p}, \frac{1}{q} \text{ and } \frac{1}{r}$$

(b) Find the roots of $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in geometric progression. **4**

(c) (i) Let k be a zero of a polynomial $F(x)$ and also of its derivative $F'(x)$. Prove that k is a zero of multiplicity at least 2. **2**

(ii) Show that $y = 1$ is a double root of the equation **2**

$$y^{2t} - ty^{t+1} = 1 - ty^{t-1}$$

where t is a positive integer.

(d) Consider the polynomial $k(t) = t^4 + at^3 + bt^2 + at + 1$, where a and b are real numbers.

(i) Show that if θ is a zero of $k(t)$ then $\dfrac{1}{\theta}$ is also a zero of $k(t)$ **2**

(ii) Hence, or otherwise, write down all four zeros of $k(t)$, given that $(1 + i)$ is a zero of $k(t)$. (There is no need to calculate a or b.) **2**

Question 5 (15 marks) Use a SEPARATE writing booklet

Marks

(a) Sketch the graph of the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ showing the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **4**

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a>b>0$, has eccentricity e.

(i) Show that the line through the focus $S(ae,\ 0)$ that is perpendicular to the asymptote $y=\frac{b}{a}x$ has equation $ax+by-a^2e=0$. **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. **3**

(c) $P\left(p,\ \frac{1}{p}\right)$ and $Q\left(q,\ \frac{1}{q}\right)$ are two variable points on the rectangular hyperbola $xy=1$ such that the chord PQ passes through the point $A(0,\ 2)$. M is the midpoint of PQ.

(i) Show that PQ has equation $x+pqy-(p+q)=0$. Hence, deduce that $p+q=2pq$. **3**

(ii) Deduce that the tangent drawn from the point A to the rectangular hyperbola touches the curve at the point $(1,\ 1)$. **1**

(iii) Sketch the rectangular hyperbola showing the points P, Q, A and M. Find the equation of the locus of M and state any restrictions on the domain of this locus. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet

Marks

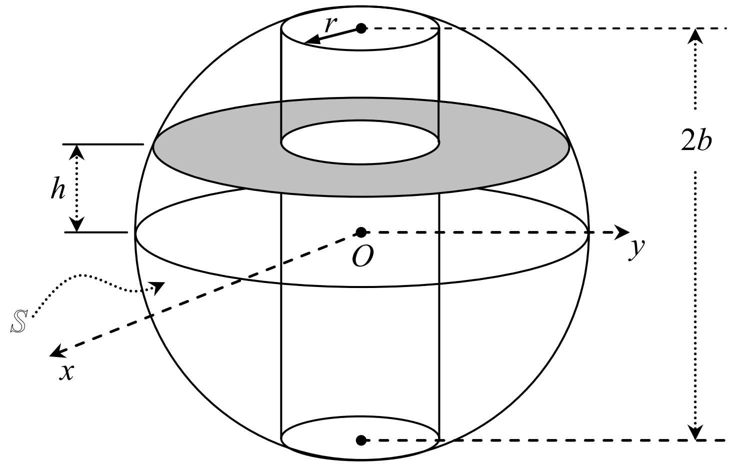
(a) The region bounded by the curves $y = 3 - x^2$, $y = x + x^2$ and $x = -1$ is rotated about the line $x = -1$. The point P is the point of intersection of $y = 3 - x^2$ and $y = x + x^2$ in the first quadrant.

(i) Find the x coordinate of P. **1**

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. **3**

(iii) Evaluate the integral in part (ii). **2**

(b) A cylindrical hole of radius r is bored through a sphere of radius R. The hole is perpendicular to the xy plane and its axis passes through the origin O, which is the centre of the sphere. The resulting solid is denoted by $\mathbb{S}$. The cross section of $\mathbb{S}$ shown in the diagram is a distance h from the xy plane.



(i) Show that the area of the cross-section shown above is $\pi\left(R^2 - h^2 - r^2\right)$. **3**

(ii) Find the volume of $\mathbb{S}$, and express your answer in terms of b alone, where $2b$ is the length of the hole. **2**

(c) A solid has its base as the region bounded by the curves $y = x$ and $y = 9$. **4**
Cross sections parallel to the x-axis are squares, one side of which lies in the base of the solid. Find the volume of the solid.

Question 7 (15 marks) Use a SEPARATE writing booklet

Marks

(a) (i) By considering $f'(x)$ where $f(x) = e^x - x$, show that $e^x > x$ for $x \geq 0$ **2**

(ii) Hence, use Mathematical Induction to show that for $x \geq 0$, $e^x > \dfrac{x^n}{n!}$ for all positive integers $n \geq 1$. **3**

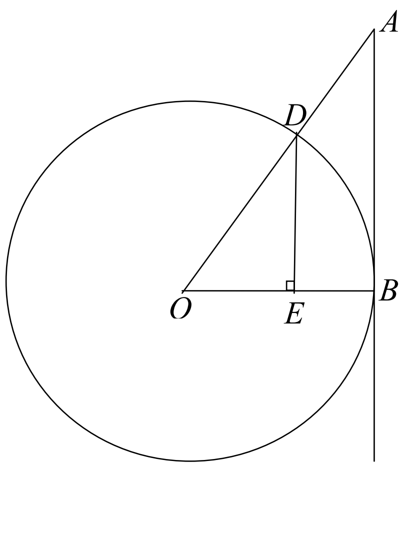
(b) Let $\int_0^1 x\left(1 - x^5\right)^n \, dx$, where $n \geq 0$ is an integer.

(i) Show that $I_n = \dfrac{5n}{5n+2} I_{n-1}$, for $n \geq 1$. **3**

(ii) Show that $I_n = \dfrac{5^n n!}{2 \times 7 \times 12 \times \ldots \times (5n+2)}$, for $n \geq 1$. **2**

(iii) Hence, evaluate I_4. **1**

(c) In the diagram, O is the centre of a circle of radius r units. AB is a tangent to the circle at B and DE is perpendicular to OB at E. $\angle DOE = x$ radians.



Clearly, $DE <$ arc DB $<$ AB.

(i) Show that $\sin x < x < \tan x$ if $0 < x < \dfrac{\pi}{2}$ **2**

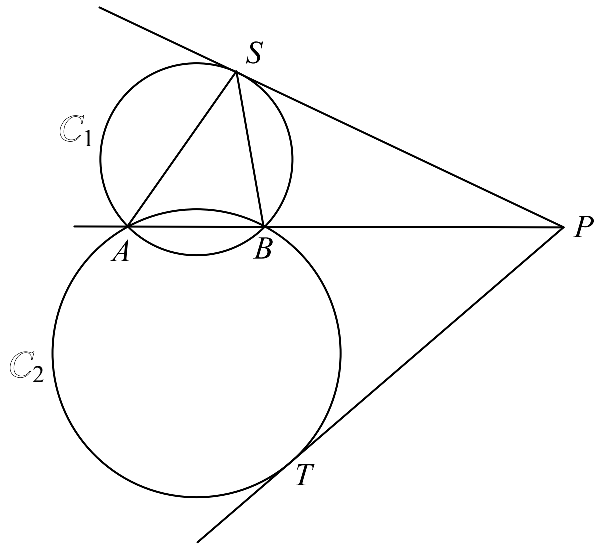
(ii) Hence, prove that **2**

$$\int_0^{\frac{\pi}{6}} x^2 \sin x \, dx < \frac{\pi^4}{2^6 . 3^4} < \int_0^{\frac{\pi}{6}} x^2 \tan x \, dx$$

Question 8 (15 marks) Use a SEPARATE writing booklet

Marks

(a)



Two circles $\mathbb{C}_1$ and $\mathbb{C}_2$ intersect at the points A and B. Let P be a point on AB produced and let PS and PT be tangents to $\mathbb{C}_1$ and $\mathbb{C}_2$ respectively, as shown in the diagram.

(i) Prove that $\Delta ASP \parallel\!\mid \Delta SBP$ **2**

(ii) Hence, prove that $SP^2 = AP \times BP$ and deduce that $PT = PS$. **2**

(iii) The perpendicular to SP drawn from S meets the bisector of $\angle SPT$ at D. Prove that DT passes through the centre of $\mathbb{C}_2$. **3**

(b) At the NSW State elections, a pre-poll survey found that 50% of the electorate were in favour of the sitting member, 40% were opposed and 10% were undecided. If three people are selected at random, find (giving your answers as percentages to the nearest whole number):

(i) the probability that all three are of the same opinion. **2**

(ii) At least two of them are opposed. **3**

(c) Maria and Ferdinando are competing against each other in a competition in which the winner is the first person to score five goals. The outcome is recorded by listing, in order, the initial of the person who scores each goal. For example, one possible outcome could be *MFFMMFMM*.

(i) Explain why there are five different ways in which the outcome could be recorded if Ferdinando scores only one goal in the competition. **1**

(ii) In how many ways could the outcome of this competition be recorded? **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \quad x > 0$