2015
HSC
ASSESSMENT TASK 4

Trial HSC Examination

Mathematics Extension 2

Examiners ~ Mr. J. Dillon, Mr. G. Huxley, Mr. G. Rawson

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen.
- Board approved calculators and mathematical templates and instruments may be used.
- Show all necessary working in Questions 11 – 16.
- This examination booklet consists of 21 pages including a standard integral page and multiple choice answer sheet.

Total marks (100)

Section I

Total marks (**10**)

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided.
- Allow 15 minutes for this section.

Section II

Total marks (**90**)

- Attempt questions 11 – 16
- Answer each question in the Writing Booklets provided.
- Start a new booklet for each question with your name and question number at the top of the page.
- All necessary working should be shown for every question.
- Allow 2 hours 45 minutes for this section.

Name: ____

Teacher: ____

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 – 10.

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1. What is the value of $\dfrac{z_1}{z_2}$ when $z_1 = 4 + i$ and $z_2 = 2 + 3i$?

(A) $\dfrac{-11+10i}{5}$ (B) $\dfrac{5-10i}{5}$

(C) $\dfrac{5-10i}{13}$ (D) $\dfrac{11-10i}{13}$

2. What are the square roots of $z = -5 - 12i$?

(A) $2 - 3i$ and $2 + 3i$ (B) $-2 - 3i$ and $2 + 3i$

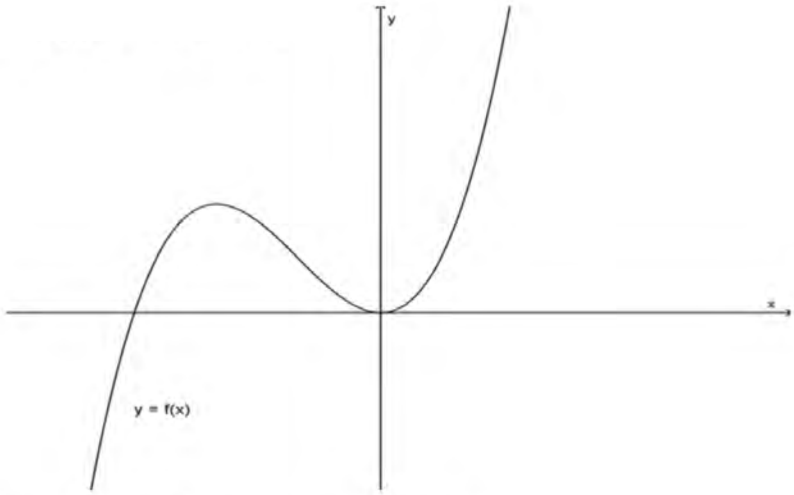
(C) $-2 + 3i$ and $-2 - 3i$ (D) $2 - 3i$ and $-2 + 3i$

3. Which of the following is an expression for $\displaystyle\int \frac{x}{\sqrt{16 - x^2}}\,dx$?

(A) $-\sqrt{16 - x^2} + c$ (B) $-2\sqrt{16 - x^2} + c$

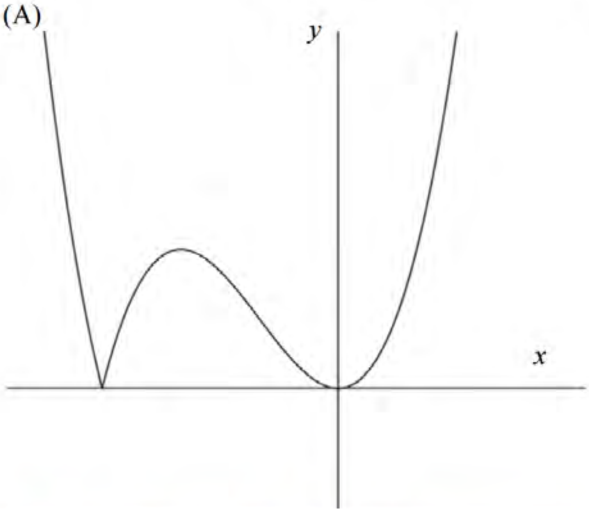
(C) $\dfrac{1}{2}\sqrt{16 - x^2} + c$ (D) $-\dfrac{1}{2}\sqrt{16 - x^2} + c$

4. The graph of $y = f\left(x\right)$ is shown below

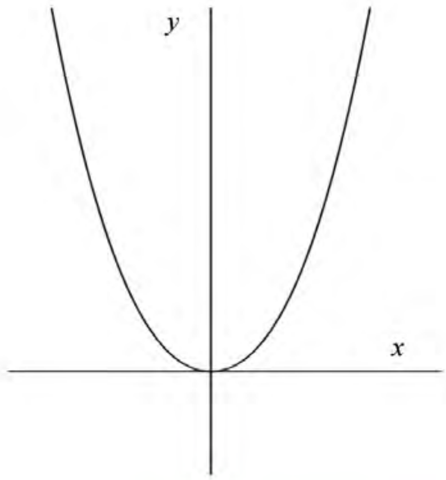


Which of the following graphs is the graph of $y = f\left(|x|\right)$?

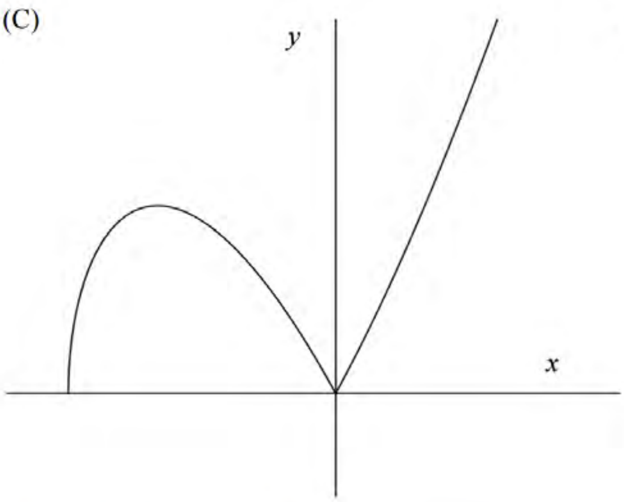
(A)



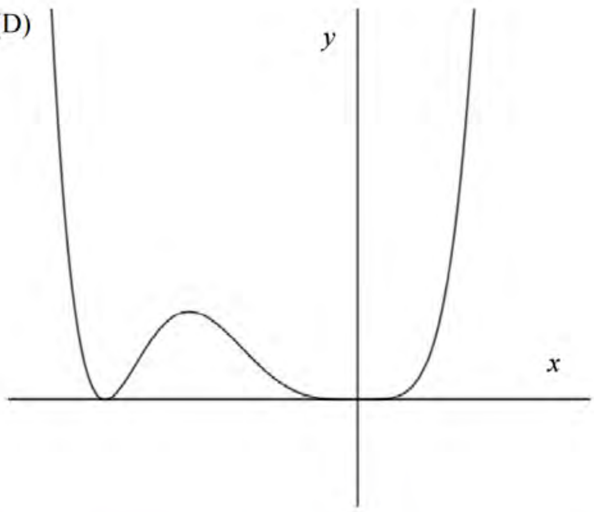
(B)



(C)



(D)



5. The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.
What is the value of $\frac{dy}{dx}$ at the point (1, 2)?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) $\frac{3}{4}$ (D) 1

6. The polynomial $P(z)$ has the equation $P(z) = z^4 - 4z^3 + Az + 20$, where A is real.
Given that $3+i$ is a zero of $P(z)$, which of the following expressions is $P(z)$ as a product of two real quadratic factors?

(A) $(z^2 - 2z + 2)(z^2 - 6z + 10)$ (B) $(z^2 + 2z + 2)(z^2 - 6z + 10)$

(C) $(z^2 - 2z + 2)(z^2 + 6z + 10)$ (D) $(z^2 + 2z + 2)(z^2 + 6z + 10)$

7. The solution to the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ is

$$x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{-3\pi}{16}, \tan\frac{-7\pi}{16}.$$

What is the value of $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16}$?

(A) 4 (B) 16

(C) 28 (D) 32

8. An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has eccentricity e.
What happens to the shape of the ellipse as $e \to 0$?

(A) The ellipse becomes a circle. (B) The ellipse gets flatter and wider.

(C) The ellipse gets taller and narrower (D) The ellipse opens up into a parabola.

9. The length of the transverse axis of the hyperbola $\frac{x^2}{16} - \frac{y^2}{25} = 1$ is:

(A) 4 units

(B) 5 units

(C) 8 units

(D) 10 units

10. Four digit numbers are formed from the digits 1, 2, 3 and 4.
Each digit is used once only.
What is the sum of all the numbers that can be formed?

(A) 6 666

(B) 44 440

(C) 66 660

(D) 266 640

Section II

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

Answer each question in the appropriate writing booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) Start a new booklet **Marks**

(a) Let $z = 3 + 2i$ and $w = 2 - i$. Find, in the form $x + iy$:

(i) $z + 4w$ **1**

(ii) $z^2 w$ **1**

(iii) $\dfrac{2}{\overline{w}}$ **2**

(b) Sketch the locus of z on the Argand diagram where the inequalities

$$|z - 1| \le 3 \text{ and } \operatorname{Im}(z) \ge 3$$

hold simultaneously. **3**

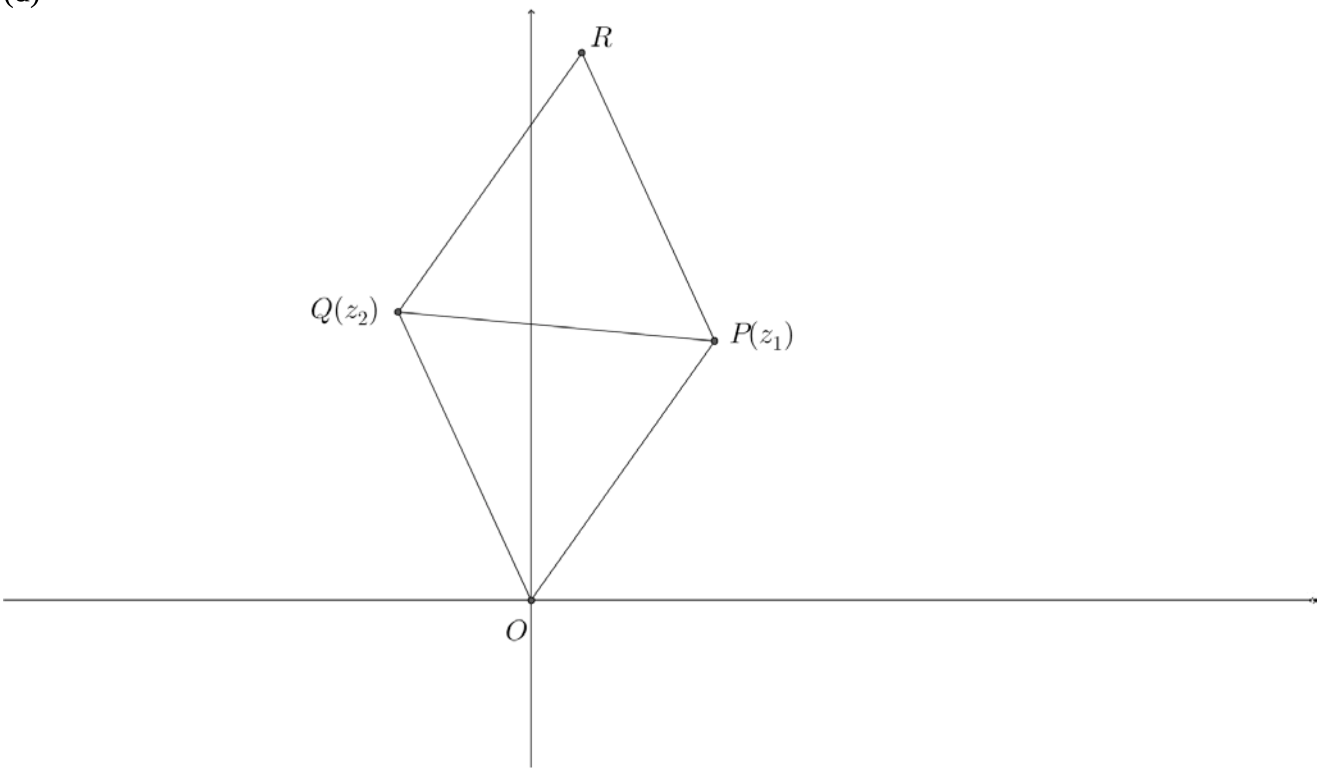
(c) Let $z = 2 + 2i$.

(i) Express z in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Hence express z^{16} in the form $a + ib$ where a and b are real. **2**

Question 11 continues on the next page ……

(d)



The point P on the Argand diagram corresponds to the complex number $z_1 = 1 + \sqrt{2}i$.
The triangles OPQ and PQR are equilateral triangles.

Show that $z_2 = \dfrac{1-\sqrt{6}}{2} + i\left(\dfrac{\sqrt{3}+\sqrt{2}}{2}\right)$. **2**

(e) The complex numbers represented by 0, z, $z + \dfrac{1}{z}$ and $\dfrac{1}{z}$ form a parallelogram.

Describe the locus of z if this parallelogram is a rhombus. **2**

Question 12 (15 marks) Start a new booklet **Marks**

(a) Find $\int \sqrt{x} \ln x \, dx$. **3**

(b) (i) Find real numbers a, b, c and d such that **2**

$$\frac{x^3 - 8x^2 + 9x}{\left(1 + x^2\right)\left(9 + x^2\right)} = \frac{ax + b}{1 + x^2} + \frac{cx + d}{9 + x^2}$$

(ii) Hence, evaluate in simplest form,

$$\int_0^{\sqrt{3}} \frac{x^3 - 8x^2 + 9x}{\left(1 + x^2\right)\left(9 + x^2\right)} \, dx$$ **2**

(c) Use the substitution $u = \sqrt{x - 1}$ to evaluate $\displaystyle\int_2^3 \frac{1 + x}{\sqrt{x - 1}} \, dx$. **3**

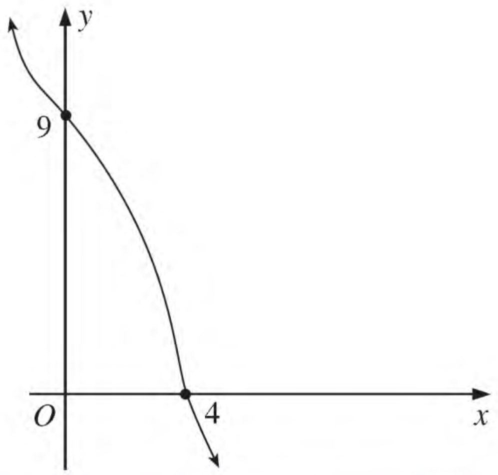
(d) (i) Let $I_n = \int_0^1 \sqrt{x}(1 - x)^n dx$ for $n = 0,\ 1,\ 2,\ 3,...$

Show that $I_n = \dfrac{2n}{2n + 3} I_{n-1}$. **3**

(ii) Hence, or otherwise, find the value of $\int_0^1 \sqrt{x}(1 - x)^3 dx$. **2**

Question 13 (15 marks) Start a new booklet **Marks**

(a) The diagram shows the graph of the (decreasing) function $y = f(x)$.



Draw separate one-third page sketches of graphs of the following:

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) the inverse function $y = f^{-1}(x)$ **2**

(b) The function $y = f(x)$ is defined by $f(x) = \sqrt{3 - \sqrt{x}}$

(i) State the domain of the function $f(x)$ **1**

(ii) Show that $y = f(x)$ is a decreasing function
and determine the range of $y = f(x)$ **2**

(iii) Sketch the graph of $y = f(x)$ for the domain and range determined above. **1**

Question 13 continues on the next page ……

(c) A plane curve is defined by the equation $x^2 + 2xy + y^5 = 4$.

The curve has a horizontal tangent at the point $P(X, Y)$.

By using implicit differentiation or otherwise, show that X is the unique solution to the equation $X^5 + X^2 + 4 = 0$. [**Do not solve this equation**] **4**

Question 14 (15 marks) Start a new booklet **Marks**

(a) When $P(x) = x^4 + ax^3 + b$ is divided by $x^2 + 4$, the remainder is $-x + 13$.
Find the values of *a* and *b*. **3**

(b) The polynomial equation $x^3 + 2x + 1 = 0$ has roots α, β and γ.

(i) Find the equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

(ii) Find the equation with roots $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. **2**

(c) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 3$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$
$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2$$

(i) Find the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and explain why α, β and γ are the roots of $x^3 - 3x^2 + 4x - 2 = 0$. **3**

(ii) Find the values of α, β and γ. **3**

(d) Given $Q(x)$ is a real polynomial, show that if α is a root of $Q(x) - x = 0$,
then α is also a root of $Q(Q(x)) - x = 0$. **2**

Question 15 (15 marks) Start a new booklet **Marks**

(a) Given the hyperbola $16x^2 - 9y^2 = 144$, find:

(i) the eccentricity **1**

(ii) the coordinates of the foci **1**

(iii) the equations of the asymptotes. **1**

(b) (i) Show that the tangent to the curve $xy = c^2$ at $T\left(ct, \frac{c}{t}\right)$ is given by:

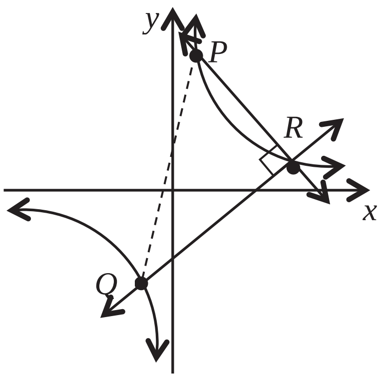
$$x + t^2 y = 2ct$$ **2**

(ii) The tangent cuts the x and y axes at A and B respectively.

Prove that T is the centre of the circle that passes through O, A and B, where O is the origin **2**

(c) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$.

The chord PQ subtends a right angle at another point $R\left(cr, \frac{c}{r}\right)$ on the hyperbola.



Show that the normal at R is parallel to PQ. **4**

Question 15 continues on the next page ……

(d) The point $P\left(a\cos\theta, b\sin\theta\right)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The tangent at P cuts the y – axis at B and M is the foot of the perpendicular from P to the y – axis .

(i) Show that the equation of the tangent to the ellipse at the point P is given by $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$. **2**

(ii) Show that $OM \times OB = b^2$, where O is the origin. **2**

Question 16 (15 marks) Start a new booklet **Marks**

(a) (i) Show that $\dfrac{1}{1 + x^2} \geq 1 - x^2$ for all values of $x \geq 0$. **2**

(ii) The line $y = x$ is a tangent to both functions $y = \tan^{-1} x$ and $y = x - \dfrac{1}{3}x^3$ at the origin.
It is given that $y = \tan^{-1} x$ is monotonically increasing for all values of x.
Use the answer from (i), or otherwise, to show that

$$\tan^{-1}x > x - \frac{1}{3}x^3 \text{ for all values of } x > 0.$$ **1**

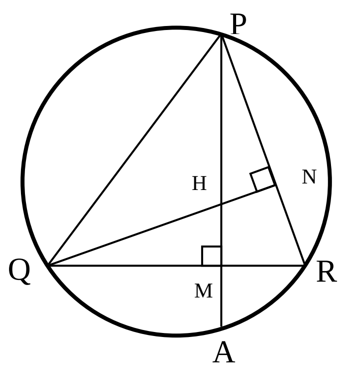
(b) (i) Sketch $y = \dfrac{1}{x+2}$ for $x > -2$ **1**

(ii) Use the graph to show that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + ... + \dfrac{1}{p-1} > \ln(p+2)$ **3**

(c) The altitudes PM and QN of an acute angled triangle PQR meet at H.

PM produced cuts the circle PQR at A.

[A larger diagram is included on page 17, use it and submit it with your solutions]



(i) Explain why $PQMN$ is a cyclic quadrilateral. **1**

(ii) Prove that $HM = MA$. **3**

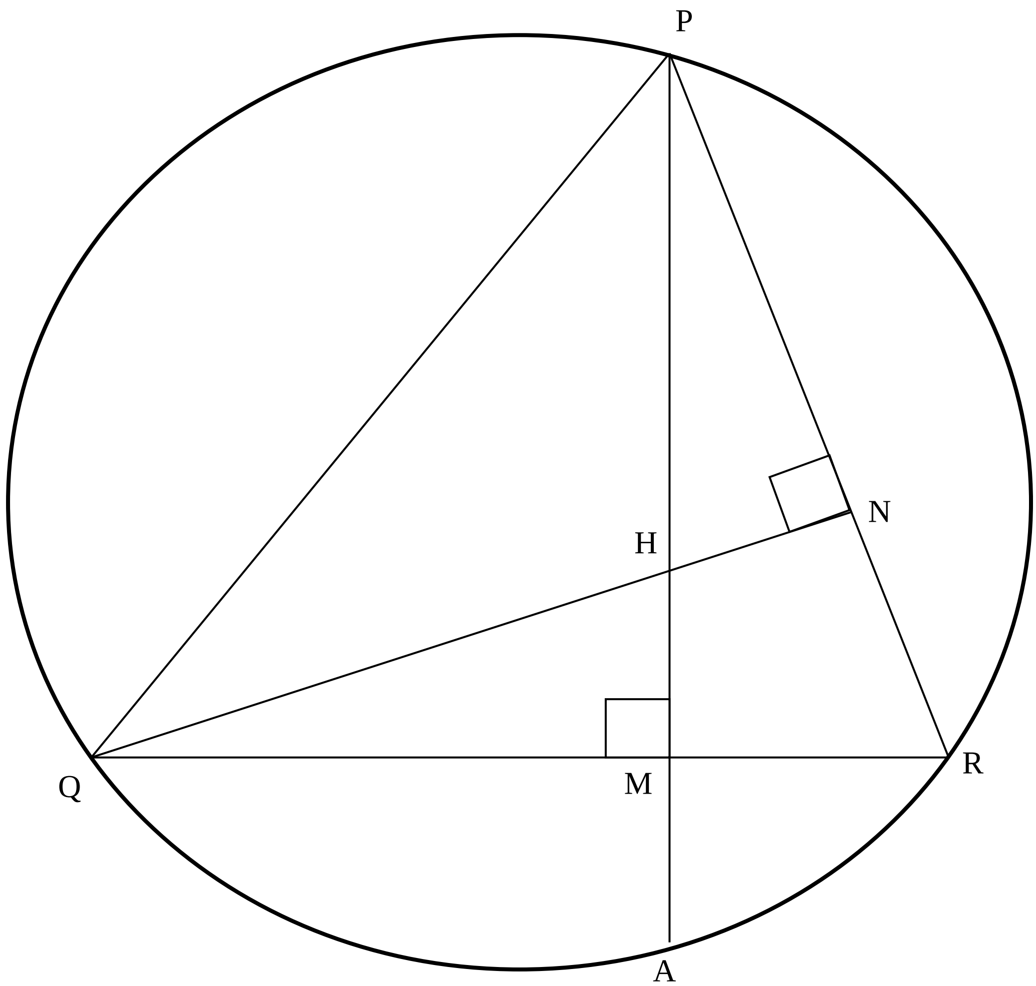
Question 16 continues on the next page ……

(d) (i) Write the first 4 terms of the series $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$ **1**

(ii) Prove, by induction, that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$ for all integers $n \geq 1$ **3**

Use this diagram to help you answer Question 16 (c).

Submit this with your answer to the question, in the appropriate answer booklet.

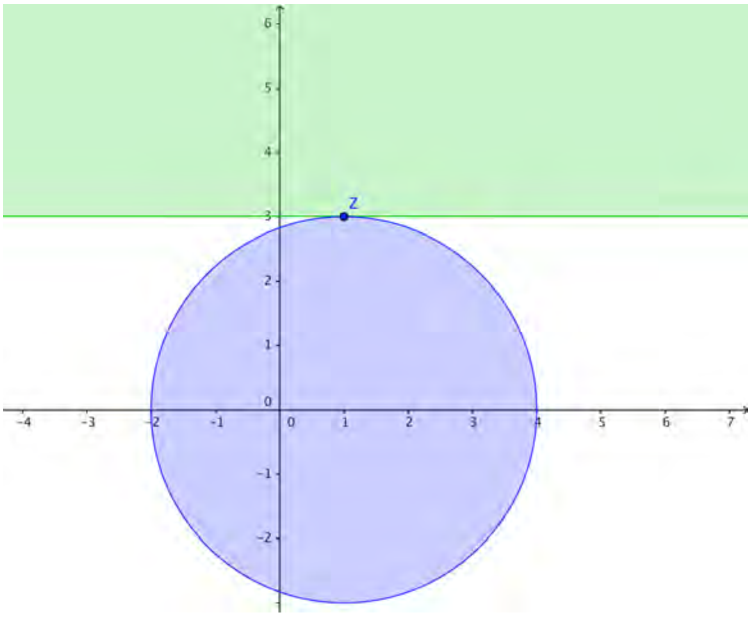


Name: ____________________________________

Year 12	Mathematics Extension 2	Task 4 2015
Question 11	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

Outcomes	**Solutions**	**Marking Guidelines**
(a) (i)	$z+4w=3+2i+4(2-i)$ $=3+2i+8-4i$ $=11-2i$	**1 Mark**: Correct answer.
(ii)	$z^2w=(3+2i)^2(2-i)$ $=(5+12i)(2-i)$ $=10-5i+24i+12$ $=22+19i$	**1 Mark**: Correct answer.
(iii)	$\frac{2}{\overline{w}}=\frac{2}{2+i}\times\frac{2-i}{2-i}$ $=\frac{2(2-i)}{5}$ $=\frac{4}{5}-\frac{2}{5}i$	**2 Marks**: Correct answer. **1 Mark**: Finds the conjugate of *w*.
(b)	 $\lvert z-1\rvert\le 3$ represents a region with a centre is (1, 0) and radius less than or equal to 3. $\text{Im}(z)\ge 3$ represents a region above the horizontal line $y=3$. The point (1, 3) is where the two inequalities hold.	**3 Marks**: Correct answer. **2 Marks**: Correctly graphs one of the inequalities. **1 Mark**: Makes progress in graphing one of the inequalities. Multiple Choice 1. D 2. D 3. A 4. B 5. D 6. B 7. C 8. A 9. C 10. C

(c) (i)	$r^2 = 2^2 + 2^2$ $= 8$ $r = \sqrt{8}$ $\tan\theta = \frac{2}{2}$ $\theta = \frac{\pi}{4}$ $\therefore z = \sqrt{8}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$	**2 Marks**: Correct answer. **1 Mark**: Correctly calculates θ or r.
(ii)	De Moivre's theorem $z^{16} = \left(\sqrt{8}\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{16}$ $= \sqrt{8}^{16}\left(\cos 16\times\frac{\pi}{4} + i\sin 16\times\frac{\pi}{4}\right)$ $= 8^8(\cos 4\pi + i\sin 4\pi)$ $= 16\ 777\ 216\ (1+0i)$ $= 16\ 777\ 216 + 0i$ or $16\ 777\ 216$ or 2^{24}	**2 Marks**: Correct answer. **1 Mark**: Makes progress in using De Moivre's theorem.
(d)	ΔOPQ is equilateral $\angle POQ = \frac{\pi}{3}$ The coordinates of z_2 are obtained by multiplying z_1 by $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. $z_2 = (1+i\sqrt{2})\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ $= \frac{1}{2} + i\frac{\sqrt{3}}{2} + i\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}$ $= \frac{1-\sqrt{6}}{2} + i\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$	**2 Marks**: Correct answer. **1 Mark**: Makes progress to calculate z_2.
(e)	Addition of complex numbers involves the completion of a parallelogram. If this parallelogram is a rhombus then the adjacent sides are equal. $\lvert z\rvert = \left\lvert\frac{1}{z}\right\rvert$ $\lvert z\rvert^2 = 1$ $\lvert z\rvert = 1$ Locus of z is a circle centre (0, 0) and radius 1.	**2 Marks**: Correct answer. **1 Mark**: Makes some progress to describe the locus.

Year 12	Mathematics Extension 2	Task 4 2015
Question 12	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

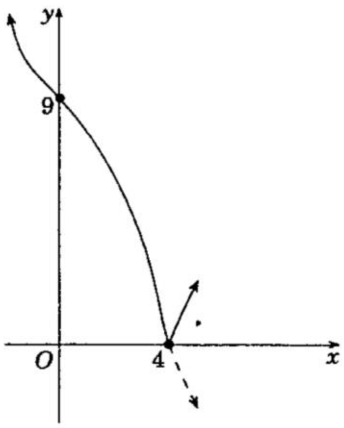
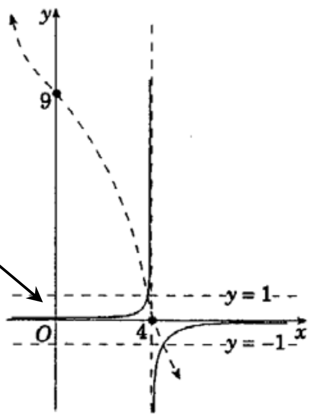
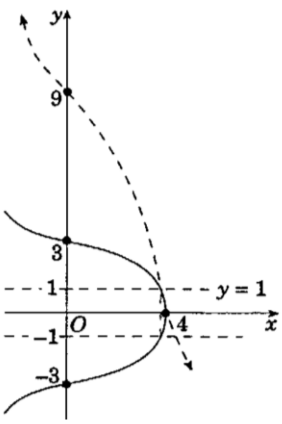
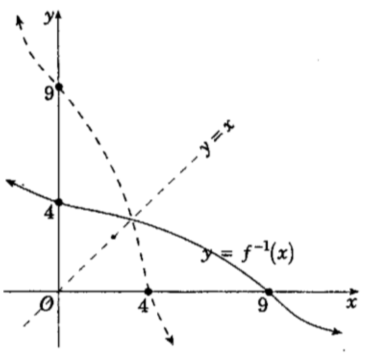
Outcomes	Solutions	Marking Guidelines
(a)	$\int \sqrt{x} \ln x dx = \frac{2}{3}x^{\frac{3}{2}} \ln x - \frac{2}{3}\int x^{\frac{3}{2}} \times \frac{1}{x} dx$ $= \frac{2}{3}x\sqrt{x} \ln x - \frac{2}{3} \times \frac{2}{3} x^{\frac{3}{2}} + c$ $= \frac{2x\sqrt{x}}{9}(3\ln x - 2) + c$	**3 Marks**: Correct answer. **2 Marks**: Makes significant progress. **1 Mark**: Sets up the integration by parts.
(b) (i)	$\frac{x^3 - 8x^2 + 9x}{\left(1+x^2\right)\left(9+x^2\right)} = \frac{ax+b}{1+x^2} + \frac{cx+d}{9+x^2}$ $x^3 - 8x^2 + 9x = (ax+b)(9+x^2) + (cx+d)(1+x^2)$ $= ax^3 + bx^2 + 9ax + 9b + cx^3 + dx^2 + cx + d$ $a + c = 1$ (coefficients x^3) $9a + c = 9$ (coefficients x) Therefore $a = 1$ and $c = 0$ Also $b + d = -8$ (coefficients x^2) When $x = 0$ then $9b + d = 0$ Hence $b = 1$ and $d = -9$ $\therefore a = 1,\ b = 1,\ c = 0$ and $d = -9$	**2 Marks**: Correct answer. **1 Mark**: Makes some progress in finding *a,b,c* or *d*.
(ii)	$\int_0^{\sqrt{3}} \frac{x^3 - 8x^2 + 9x}{\left(1+x^2\right)\left(9+x^2\right)} dx = \int_0^{\sqrt{3}} \frac{x+1}{1+x^2} + \frac{-9}{9+x^2} dx$ $= \int_0^{\sqrt{3}} \frac{x}{1+x^2} + \frac{1}{1+x^2} - \frac{9}{9+x^2} dx$ $= \left[\frac{1}{2}\ln(1+x^2) + \tan^{-1} x - 3\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$ $= \frac{1}{2}(\ln 4 - \ln 1) + (\tan^{-1}\sqrt{3} - \tan^{-1} 0) - 3(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0)$ $= \ln 2 + \frac{\pi}{3} - 3 \times \frac{\pi}{6} = \ln 2 - \frac{\pi}{6}$	**2 Marks**: Correct answer. **1 Mark**: Correctly finds one of the integrals.
(c)	$u = (x-1)^{\frac{1}{2}}$ $du = \frac{1}{2}(x-1)^{-\frac{1}{2}} dx$ $\qquad u^2 = x - 1$ $\qquad 1 + x = u^2 + 2$ $2du = \frac{dx}{\sqrt{x-1}}$ When $x = 2$ then $u = 1$ and when $x = 3$ then $u = \sqrt{2}$	**3 Marks**: Correct answer. **2 Marks**: Correctly expresses the integral in terms of u **1 Mark**: Correctly finds du in terms of dx and determines the new limits.

	$\int_2^3 \frac{1+x}{\sqrt{x-1}}dx = \int_1^{\sqrt{2}} (u^2+2)2\,du$ $= 2\left[\frac{u^3}{3}+2u\right]_1^{\sqrt{2}}$ $= 2\left[(\frac{2\sqrt{2}}{3}+2\sqrt{2})-(\frac{1}{3}+2)\right]$ $= \frac{2}{3}(8\sqrt{2}-7)$	
(d) (i)	$I_n = \int_0^1 \sqrt{x}(1-x)^n\,dx$ $= \left[\frac{2}{3}x^{\frac{3}{2}}(1-x)^n\right]_0^1 - \int_0^1 \frac{2}{3}x^{\frac{3}{2}}\{-n(1-x)^{n-1}\}dx$ $= 0 - \frac{2n}{3}\int_0^1\{x^{\frac{1}{2}}\times -x\times(1-x)^{n-1}\}dx$ $= -\frac{2n}{3}\int_0^1\{x^{\frac{1}{2}}\times(1-x-1)\times(1-x)^{n-1}\}dx$ $= -\frac{2n}{3}\int_0^1\{x^{\frac{1}{2}}\times(1-x)^n - x^{\frac{1}{2}}(1-x)^{n-1}\}dx$ $= -\frac{2n}{3}(I_n - I_{n-1})$ $3I_n = -2n(I_n - I_{n-1})$ $(2n+3)I_n = 2nI_{n-1}$ $I_n = \frac{2n}{(2n+3)}I_{n-1}$	**3 Marks**: Correct answer. **2 Marks**: Makes significant progress towards the solution. **1 Mark**: Sets up the integration and shows some understanding.
(ii)	The given integral is I_3 where $I_3 = \frac{6}{9}I_2,\ I_2 = \frac{4}{7}I_1,$ $I_1 = \frac{2}{5}I_0$ Also $I_0 = \int_0^1 \sqrt{x}dx$ $= \frac{2}{3}\left[x^{\frac{2}{3}}\right]_0^1$ $= \frac{2}{3}$ Therefore $I_3 = \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}\times\frac{2}{3}$ $= \frac{32}{315}$	**2 Marks**: Correct answer. **1 Mark**: Using the result from (d)(i) to obtain the definite integral.

Year 12	Mathematics Extension 2	TRIAL 2015
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E6 - combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

Outcome	Solutions	Marking Guidelines
	(a) (i) $y = \lvert f(x) \rvert$ 	**<u>1 mark:</u>** correct solution
	(ii) $y = \dfrac{1}{f(x)}$ y-intercept is $\frac{1}{9}$ 	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution
	(iii) $y^2 = f(x)$ $y = \pm\sqrt{f(x)}$ 	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution
	(iv) 	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution

(b) $f(x)=\sqrt{3-\sqrt{x}}$

(i) Domain $3-\sqrt{x}\ge 0$ and $\sqrt{x}\ge 0$

$$\sqrt{x}\le 3 \qquad x\ge 0$$
$$x\le 9$$
$$\therefore 0\le x\le 9$$

1 mark: correct solution

(ii) $f(x)=\left(3-x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$$f'(x)=\frac{1}{2}\left(3-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}.-\frac{1}{2}x^{-\frac{1}{2}}$$
$$=\frac{-1}{4\sqrt{x}\sqrt{3-\sqrt{x}}}$$
$$<0 \quad \text{for} \quad 0<x<9 \quad \left(\text{as } 4\sqrt{x} \text{ and } \sqrt{3-\sqrt{x}}<0\right)$$

$\therefore$ decreasing function

Gradient also undefined for x = 0, 9 (ie vertical)

at $x=0\Rightarrow f(0)=\sqrt{3}$

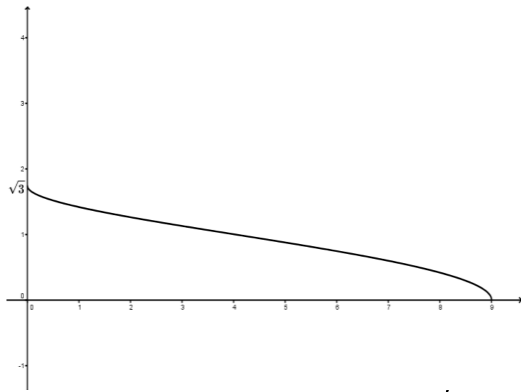
at $x=9\Rightarrow f(9)=0$

$\therefore$ range is $0\le y\le\sqrt{3}$

2 marks: correct solution

1 mark: substantially correct solution

(iii)



1 mark: correct solution

(c) Horizontal tangent at $x=X \quad\rightarrow\quad \frac{dy}{dx}=0$.

$$\frac{d}{dx}\left(x^2+2xy+y^5\right)=\frac{d}{dx}(4)$$
$$2x+2\left(y.1+x.\frac{dy}{dx}\right)+5y^4.\frac{dy}{dx}=0$$
$$\left(2x+5y^4\right)\frac{dy}{dx} = -2(x+y)$$
$$\therefore \quad \frac{dy}{dx} = \frac{-2(x+y)}{\left(2x+5y^4\right)}$$
$$\therefore\ -2(x+y) =0 \quad\rightarrow y=-x \quad \text{where } x=X$$
$$\therefore\ X^2+2X(-X)+(-X)^5=4$$
$$X^2-2X^2-X^5=4$$
$$\therefore\ X^5+X^2+4=0$$

4 marks: correct solution

3 marks: substantially correct solution

2 marks: partial progress towards correct solution

1 mark: implicit diff

Year 12	Mathematics Extension 2	TRIAL 2015
Question No. 14	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Outcome	Solutions	Marking Guidelines
	(a) $x^4+ax^3+b=\left(x^2+4\right)Q(x)-x+13$ Let $x=2i$ $(2i)^4+x(2i)^3+b=\left((2i)^2+4\right)Q(x)-(2i)+13$ $16-8ai+b=0-2i+13$ $a=\frac{1}{4}$ $b=-3$	**<u>3 marks:</u>** correct solution **<u>2 marks:</u>** substantially correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(b) The equation $x^3+2x+1=0$ has roots α,β,γ. (i) $x=\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ $\Rightarrow$ replace x with $\frac{1}{x}$ $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)+1=0$ $1+2x^2+x^3=0$ ie equation is $x^3+2x^2+1=0$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution
	(ii) $x=\frac{1}{\alpha^2},\frac{1}{\beta^2},\frac{1}{\gamma^2}$ $\Rightarrow$ replace x with $\frac{1}{\sqrt{x}}$ $\left(\frac{1}{\sqrt{x}}\right)^3+2\left(\frac{1}{\sqrt{x}}\right)+1=0$ $x^{-\frac{3}{2}}+2x^{-\frac{1}{2}}+1=0$ $x^0+2x^1+x^{\frac{3}{2}}=0$ $x^{\frac{3}{2}}=-2x-1$ $x^3=(-2x-1)^2$ $x^3-4x^2-4x-1=0$	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution

	$(c)(i)\quad \alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $1=3^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)$ $\alpha\beta+\beta\gamma+\alpha\gamma=4$ also, $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$ $2=\dfrac{4}{\alpha\beta\gamma}$ $\alpha\beta\gamma=2$ The polynomial with roots α, β, γ is $(x-\alpha)(x-\beta)(x-\gamma)$ $=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma$ $=x^3-3x^2+4x-2$	**<u>3 marks:</u>** correct solution **<u>2 marks:</u>** substantially correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(ii) $P(x)=x^3+3x^2+4x-2$ $P(1)=0$ $\therefore\quad x=1$ is a root $\therefore\quad P(x)=(x-1)(ax^2+bx+c)$ $=(x-1)(x^2-2x+2)$ $=(x-1)(x-1-i)(x-1+i)$ $\therefore\ \alpha,\beta,\gamma=1,\ 1+i,\ 1-i$	**<u>3 marks:</u>** correct solution **<u>2 marks:</u>** substantially correct solution **<u>1 mark:</u>** partial progress towards correct solution
	(d) If α is a root, $Q(\alpha)-\alpha=0$ $Q(\alpha)=\alpha$ sub $x=\alpha$ into $Q(Q(x))-x$ $=Q(Q(\alpha))-\alpha$ $=Q(\alpha)-\alpha=0$ $\therefore \alpha$ is a root	**<u>2 marks:</u>** correct solution **<u>1 mark:</u>** substantially correct solution

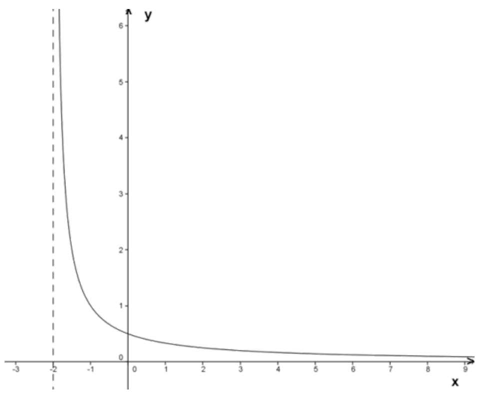
Mathematics Extension 2 Solutions and Marking Guidelines		Trial Examination 2015
Question 15: Outcomes Addressed in this Question: E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials		
	Solutions	**Marking Guidelines**
(a)	$a=3$, $b=4$ (i) $e=\frac{5}{3}$ (ii) Focii: $(\pm 5,0)$ (iii) Asymptotes: $y=\pm\frac{4x}{3}$	(a) (i) **<u>1 mark:</u>** Correct answer. (ii) **<u>1 mark:</u>** Correct answer. (iii) **<u>1 mark:</u>** Correct answer.
(b)	(i) $\frac{dy}{dx}=\frac{-1}{t^2}$ $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct)$ $t^2y-ct=-x+ct$ (ii) $A=(2ct,0)$ $B=\left(0,\frac{2c}{t}\right)$ Midpoint $AB=\left(ct,\frac{c}{t}\right)=T$ Since $\angle BOA=90°$, *AB* is a diameter of a circle passing through *O*. *T* is the midpoint of this diameter, and therefore the centre of the circle passing through *O, A, B.*	(b) (i) **<u>2 marks:</u>** Correct solution to "show that". **<u>1 mark:</u>** Partially correct. (ii) **<u>2 marks:</u>** Correct solution with explanation. **<u>1 mark:</u>** Significant progress
(c)	Gradient of tangent at R is $\frac{-1}{r^2}$ (determined in part (b)) So gradient of normal is r^2. Gradient of $PQ=\frac{-1}{pq}$ $PQ\perp QR$ $\therefore \frac{-1}{pr}\times\frac{-1}{qr}=-1$ $r^2=\frac{-1}{pq}$ Thus the 2 gradients are equal, so the lines are parallel.	(c) **<u>4 marks:</u>** Correct solution. **<u>3 marks:</u>** Correct reasoning with some links omitted. **<u>2 marks:</u>** Some relevant factors omitted in the reasoning. **<u>1 mark:</u>** Some relevant progress.

(d)	(i) $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=-\frac{b\cos\theta}{a\sin\theta}$$ $$\therefore\ y-b\sin\theta=-\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$ $$bx\cos\theta+ay\sin\theta=ab\sin^2\theta+ab\cos^2\theta$$ $$bx\cos\theta+ay\sin\theta=ab$$ $$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$	(d) (i) **<u>2 marks:</u>** Satisfaction of "show that" **<u>1 mark:</u>** Some relevant progress.
	(ii) Sub $x = 0$: $B=\left(0,\frac{b}{\sin b}\right)$ PM is horizontal $\therefore M=(0,b\sin\theta)$ $\therefore OM\times OB=b\sin\theta\times\frac{b}{\sin\theta}=b^2$	(ii) **<u>2 marks:</u>** Correct solution, including the co-ordinates and the respective lengths. **<u>1 mark:</u>** Significant progress.

Question 16: Outcomes Addressed in this Question:

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument

	Solutions	Marking Guidelines
(a)	(i) Consider the difference: $$\frac{1}{1+x^2}-\left(1-x^2\right)=\frac{1-\left(1-x^2\right)\left(1+x^2\right)}{1+x^2}=\frac{1-\left(1-x^4\right)}{1+x^2}=\frac{x^4}{1+x^2}\geq 0$$ Due to positive numerator and denominator. Therefore statement is true.	(i) **<u>2 mark:</u>** Correct solution with working. **<u>1 mark:</u>** Relevant progress.
	(ii) Since $y=\tan^{-1}x$ is monotonic increasing, and we have found in (i) that $\frac{1}{1+x^2}\geq 1-x^2$, we can integrate both sides for positive *x*, which gives: $\tan^{-1}x\geq x-\frac{x^3}{3}$ (We can ignore the constants of integration because we've been told the curves both pass through the origin.)	(ii) **<u>1 mark:</u>** Recognising the derivative relationship between (i) and (ii).
(b)	(i) 	(b) (i) **<u>1 mark:</u>** Correct diagram.
	(ii) $$1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{p+1}>\int_{-1}^{p}\frac{1}{x+2}dx=\ln(x+2)\Big]_{-1}^{p}=\ln(p+2)$$ *This question was not typed as intended. The marking scheme*	(ii) **<u>3 marks:</u>** Any combination of upper rectangles and integrating $\frac{1}{x+2}$ with any sensible boundaries. You did not have to prove the question true. **<u>2 marks:</u>** Significant progress **<u>1 mark:</u>** some relevant progress.

	was adjusted as indicated on the right hand panel..	
(c)	(i) $PQMN$ is cyclic because two equal angles are subtended by an interval **on the same side** of that interval. (ii) $\angle APR = \angle AQR$ Angles at the circumference subtended by arc AR $\angle MQN = \angle MPN$ Angles at the circumference subtended by arc MN. (PQMN proven cyclic in (i)) $\therefore \angle HQM = \angle APR = \angle AQM$ In ΔHQM, ΔAQM QM is common $\angle HQM = \angle AQM$ Proven above $\angle HMQ = \angle AMQ = 90°$ PM is an altitude. $\therefore \Delta HQM \equiv \Delta AQM$ AAS congruency test. $\therefore HM = MA$ corresponding sides in congruent triangles are equal. An alternative solution is to use $HMRN$ is a cyclic quadrilateral and $\angle QHM = \angle NRM$ Exterior angle of a cyclic quadrilateral is equal to the opposite interior angle, etc.	(c) (i) **<u>1 mark:</u>** Correct reason. You had to give a response that indicated that the equal angles are on the same side of the interval. (ii) **<u>3 marks:</u>** Correct solution including all reasoning. **2 marks:** A relevant factor omitted in the reasoning. **<u>1 mark:</u>** Considerable relevant progress.
(d)	(i) $\frac{1}{2\times3}+\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}$ (ii) (Abbreviated solution:) <u>Step 1:</u> when $n = 1$, LHS = RHS = $\frac{1}{6}$ <u>Step 2:</u> When $n = k+1$ $\frac{k}{2(k+2))}+\frac{1}{(k+2)(k+3)}=\frac{k^2+3k+2}{2(k+2)(k+3)}$ $=\frac{(k+1)(k+2)}{2(k+2)(k+3)}=\frac{k+1}{2(k+2)}$ As required. Therefore true by mathematical induction.	(d) (i) **<u>1 mark:</u>** Only needed form as shown. (ii) **<u>3 marks:</u>** Correct induction process. **<u>2 marks:</u>** Most concepts of induction satisfied **<u>1 mark:</u>** Some relevant progress.