2017

HIGHER SCHOOL CERTIFICATE
TRIAL HSC EXAMINATION

Mathematics Extension 2

Examiners Mr J. Dillon, Mr G. Huxley, Mr G. Rawson and Mrs D. Crancher

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided for your use
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 14)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Name: ____________________________

Teacher: __________________________

Section I

10 marks
Attempt Questions 1 and 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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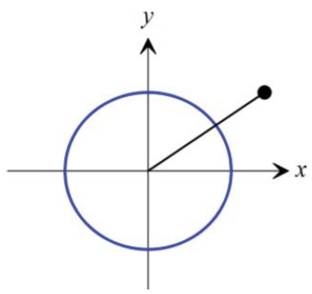
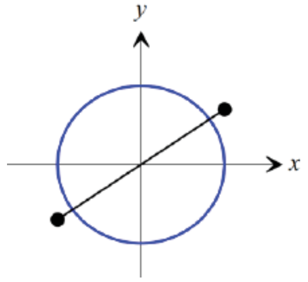
1. The Argand diagram below shows the complex number z, represented by a vector, along with the unit circle.

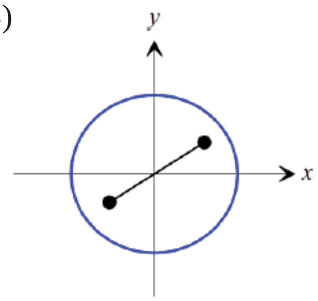


Which diagram best illustrates the vectors representing $\sqrt{z}$?

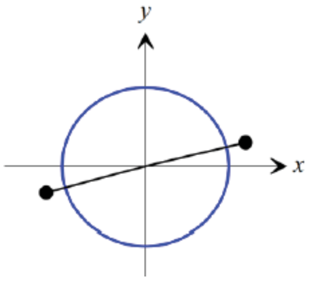
(A)



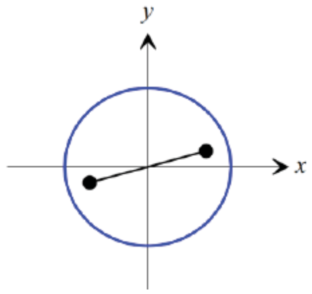
(B)



(C)



(D)



2. Which line intersects the circle $|z-3-2i|=2$ twice?

(A) $|z-3-2i|=|z-5|$ (B) $|z-i|=|z+1|$

(C) Re (z) = 5 (D) Im (z) = 0

3. The polynomial equation $x^3+x^2-x-4=0$ has roots $\alpha,\ \beta$ and γ. Which of the following polynomial equations has roots $\alpha^2,\ \beta^2$ and γ^2?

(A) $x^3-3x^2+9x-16=0$ (B) $x^3-3x^2-8x-16=0$

(C) $x^3-x^2+9x-1=0$ (D) $x^3-x^2-8x-1=0$

4. What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

(A) $p=-2$ and $q=-4$ (B) $p=-2$ and $q=4$

(C) $p=2$ and $q=4$ (D) $p=2$ and $q=4$

5. The equation $x^3-y^3+3xy+1=0$ defines y implicitly as a function of x.
What is the value of $\dfrac{dy}{dx}$ at the point (1, 2)?

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{3}{4}$ (D) 1

6. What is the natural domain of the function $f(x) = \frac{1}{2}\left(x\sqrt{x^2-1} - \ln\left(x + \sqrt{x^2-1}\right)\right)$?

(A) $x \le -1$ or $x \ge 1$

(B) $-1 \le x \le 1$

(C) $x \ge 1$

(D) $x \le -1$

7. The point $P(cp, \frac{c}{p})$ lies on the hyperbola $xy = c^2$

What is the equation of the normal to the hyperbola at P?

(A) $p^2x - py + c - cp^4 = 0$

(B) $p^3x - py + c - cp^4 = 0$

(C) $x + p^2y - 2c = 0$

(D) $x + p^2y - 2cp = 0$

8. What are the co-ordinates of the foci of the graph of $xy = 12$?

(A) $\left(2\sqrt{3}, 2\sqrt{3}\right)$ and $\left(-2\sqrt{3}, -2\sqrt{3}\right)$

(B) $\left(2\sqrt{6}, 2\sqrt{6}\right)$ and $\left(-2\sqrt{6}, -2\sqrt{6}\right)$

(C) $\left(2\sqrt{3}, 0\right)$ and $\left(-2\sqrt{3}, 0\right)$

(D) $\left(2\sqrt{6}, 0\right)$ and $\left(-2\sqrt{6}, 0\right)$

9. The substitution of $x = \sin\theta$ in the integral $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ results in which integral?

(A) $\int_0^{\frac{1}{2}} \frac{\sin^2\theta}{\cos\theta}\,d\theta$

(B) $\int_0^{\frac{1}{2}} \sin^2\theta\,d\theta$

(C) $\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\,d\theta$

(D) $\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$

10. How many ways are there of choosing three different numbers in increasing order from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 so that no two of the numbers are consecutive?

(A) 20

(B) 48

(C) 56

(D) 72

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

***Answer each question in a new answer booklet*.**

All necessary working should be shown in every question.

Question 11 (15 marks) ***Answer this question in a new answer booklet***

(a) (i) Simplify i^{2017} **1**

(ii) Sketch the locus of $\arg(z-1)=\frac{\pi}{4}$ **1**

(b) $z=-\sqrt{3}+i$ and $w=1+i$

(i) Find $\frac{z}{w}$ in Cartesian form. **2**

(ii) Convert both z and w to modulus – argument form. **3**

(iii) Use your answers to (i) and (ii) to find the exact value of $\cos\frac{7\pi}{12}$. **1**

(c) $(x+iy)^2=7-24i$, where x and y are real.

(i) Find the exact values of x and y. **3**

(ii) Hence, solve the equation $2z^2+6z+(1+12i)=0$. **2**

(d) Use De Moivre's Theorem to show that $(\cot\theta+\text{i})^n+(\cot\theta-i)^n=\frac{2\cos n\theta}{\sin^n\theta}$. **2**

Question 12 (15 marks) ***Answer this question in a new answer booklet***

(a) The equation $32x^3 - 16x^2 - 2x + 1 = 0$ has roots α, β, and γ.

(i) What is the value of γ if $\gamma = \alpha + \beta$? **2**

(ii) Fully factorise $P(x) = 32x^3 - 16x^2 - 2x + 1$ **2**

(b) The polynomial $P(z) = z^4 - 5z^3 + az^2 + bz - 10$ where *a* and *b* are real.

Given that $2 + i$ is a zero of $P(z)$, write $P(z)$ as a product of two real quadratic factors. **2**

(c) $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$.

Find *a* and *b*. **2**

(d) When $P(x)$ is divided by $(x-2)$ and $(x+3)$ the respective remainders are −7 and 3.

Find the remainder when $P(x)$ is divided by $(x-2)(x+3)$. **2**

(e) Let $z = 1 + i$ be a root of: $z^2 - biz + c = 0$, where *b* and *c* are real.

(i) Find *b* and *c* **2**

(ii) Find the other root of the polynomial. **1**

(f) Solve the equation $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$ given that it has a triple root. **2**

Question 13 (15 marks) ***Answer this question in a new answer booklet***

(a) (i) By writing $\dfrac{(x-2)(x-5)}{x-1}$ in the form $mx+b+\dfrac{a}{x-1}$, find the equation of the oblique asymptote of $y=\dfrac{(x-2)(x-5)}{x-1}$. **2**

(ii) Hence sketch the graph of $y=\dfrac{(x-2)(x-5)}{x-1}$, clearly indicating all intercepts and asymptotes. **2**

(b) Let $f(x)=3x^5-10x^3+16x$

(i) Show that $f'(x)\geq 1$ for all x. **2**

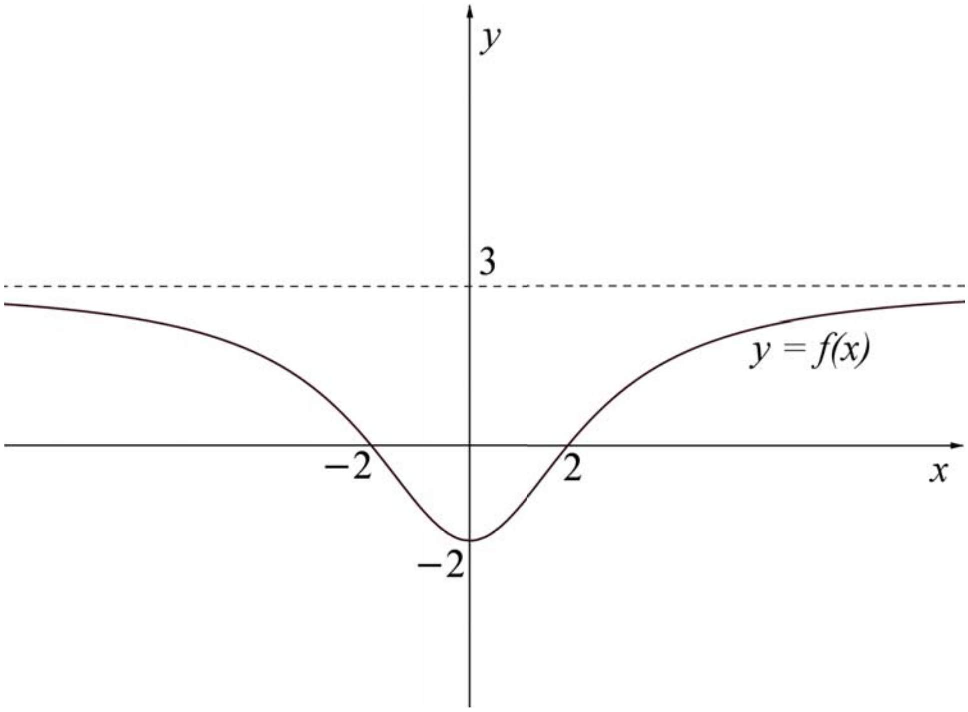
(ii) For what values of x is $f''(x)$ decreasing? **2**

(iii) Sketch the graph of $y=f(x)$, indicating any turning points and points of inflexion. **2**

Question 13 continues on the next page …..

Question 13 continued ….

(c) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **1**

(ii) $y = e^{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

Question 14 (15 marks) ***Answer this question in a new answer booklet***

(a) $A(5\cos\theta, 4\sin\theta)$ is a point on the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

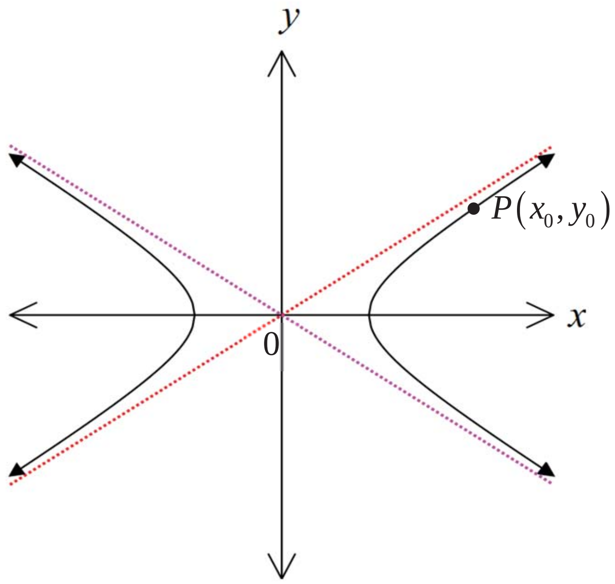
The normal at A meets the x-axis at P and the y-axis at Q.

(i) Show that the normal to the ellipse at A has the equation

$$5x\sin\theta - 4y\cos\theta = 9\sin\theta\cos\theta$$ **2**

(ii) M is the midpoint of PQ. Show that the locus of M is an ellipse. **3**

(b) The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a>b>0$.



(i) Write the equations of the asymptotes to the hyperbola in general form. **1**

(ii) Write an expression for tan θ, where θ is the acute angle between the asymptotes, in terms of a and b. **2**

(iii) Hence, write an expression for sin θ. **1**

Question 14 continues on the next page …..

(iv) If C and D are the feet of the perpendiculars drawn from $P\,(x_0, y_0)$ to the asymptotes show that $CP \times DP = \dfrac{a^2b^2}{a^2+b^2}$ **3**

(v) Prove that $OCPD$, where O is the origin, is a cyclic quadrilateral. **1**

(vi) Calculate the area of ΔPCD. **2**

Question 15 (15 marks) ***Answer this question in a new answer booklet***

(a) Find $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$ using the substitution $x = \sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. **2**

(b) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{1}{3-\cos x - 2\sin x}\,dx = \frac{\pi}{2}$. **4**

(c) (i) Find the real numbers *a*, *b* and *c* such that **3**

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)} \equiv \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

(ii) Hence, find $\int \frac{7x+4}{\left(x^2+1\right)(x+2)}\,dx$ **2**

(d) (i) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n t\,dt$. **2**

Show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ with $n \ge 2$.

(ii) Hence, otherwise, show that the exact value of $\int_0^{\frac{\pi}{2}} \cos^4 t\,dt = \frac{3\pi}{16}$. **2**

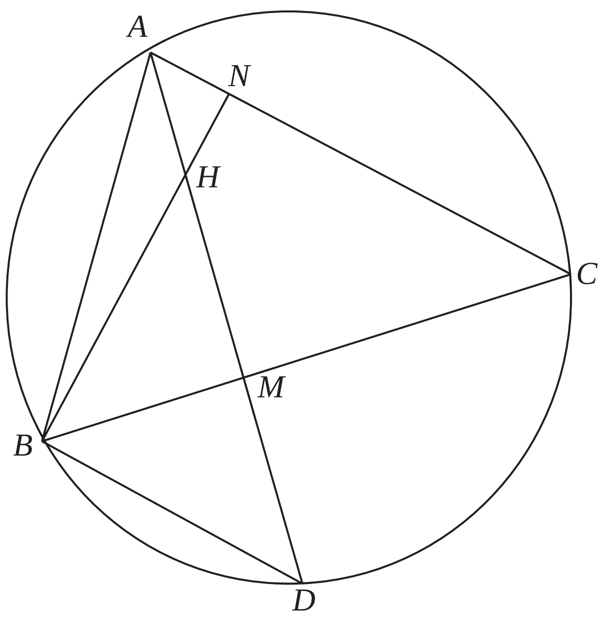
Question 16 (15 marks) ***Answer this question in a new answer booklet***

(a) A School Council consists of six year 12 students and five year 11 students, from whom a committee of five members is chosen at random.

What is the probability that the year 12 students have a majority on the committee? **2**

(b) In the circle below, points *A, B* and *C* lie on the circumference of a circle. The altitudes *AM* and *BN* of an acute angled triangle *ABC* meet at *H*. *AM* produced cuts the circle at *D*.

Prove that $HM = MD$. **3**



Not to scale

(c) The *n*th Fermat number, F_n, is defined by $F_n = 2^{2^n} + 1$ for $n = 0,1,2,3...$.

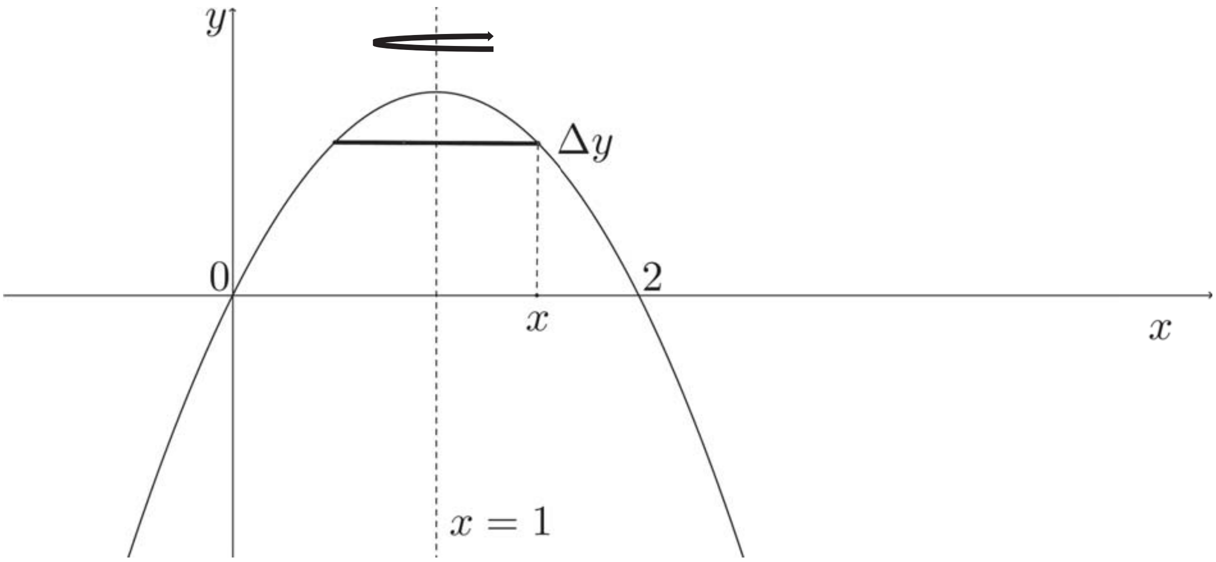
Prove by mathematical induction, that for all positive integers:

$$F_0 \times F_1 \times F_2 \times ... \times F_{n-1} = F_n - 2$$ **4**

Question 16 continues on the next page …..

Question 16 continued ….

(d) The area bounded by the curve $y = 2x - x^2$ and the x-axis is rotated through 180° about the line $x = 1$.



(i) Show that the volume, ΔV, of a representative horizontal slice of width Δy is given by **2**

$$\Delta V = \pi\left(x-1\right)^2 \Delta y$$

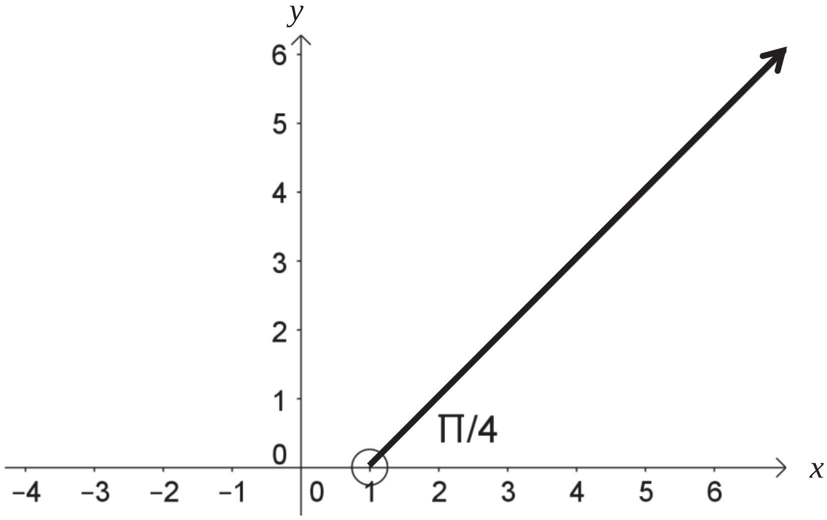
(ii) Hence, show that the volume of the solid of revolution is given by **2**

$$V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi\left(1-y\right)\Delta y$$

(iii) Hence, find the volume of the solid of revolution. **2**

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

Outcome	Solutions	Marking Guidelines
E3	(a) (i) $i^{2017} = \left(i^{2016}\right)(i)$ $= \left(i^{4}\right)^{504}(i)$ $= i$	1 mark for correct solution
E3	(ii) 	1 mark for correct diagram
E3	(b) (i) $\frac{z}{w} = \frac{-\sqrt{3}+i}{1+i} \times \frac{1-i}{1-i}$ $= \frac{-\sqrt{3}+i\sqrt{3}+i+1}{2}$ $= \left(\frac{1-\sqrt{3}}{2}\right) + i\left(\frac{1+\sqrt{3}}{2}\right)$	2 marks for complete correct solution 1 mark for substantial working that could lead to a correct solution

E3	(ii) $z = -\sqrt{3} + 1$ $\lvert z\rvert = \sqrt{\left(-\sqrt{3}\right)^2 + (1)^2}$ $= \sqrt{3+1}$ $= 2$ $Arg(z) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$ $= \frac{5\pi}{6}$ $\therefore z = 2cis\frac{5\pi}{6}$ $w = 1 + i$ $\lvert w\rvert = \sqrt{(1)^2 + (1)^2}$ $= \sqrt{2}$ $Arg(w) = \tan^{-1}\left(\frac{1}{1}\right)$ $= \frac{\pi}{4}$ $\therefore w = \sqrt{2}cis\frac{\pi}{4}$	3 marks for complete correct solution 2 marks for substantial working that could lead to a correct solution with only one error 1 mark for substantial working that could lead to a correct solution
E3	(iii) $\frac{z}{w} = \frac{2cis\frac{5\pi}{6}}{\sqrt{2}cis\frac{\pi}{4}}$ $= \sqrt{2}cis\frac{7\pi}{12}$ $= \sqrt{2}\cos\frac{7\pi}{12} + i\left(\sqrt{2}\sin\frac{7\pi}{12}\right)$ and $\frac{z}{w} = \left(\frac{1-\sqrt{3}}{2}\right) + i\left(\frac{1+\sqrt{3}}{2}\right)$ Equating real parts: $\sqrt{2}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2}$ $\therefore \cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$ or $\frac{\sqrt{2}-\sqrt{6}}{4}$	1 mark for complete correct solution

E3	(c) (i) $(x+iy)^2 = 7-24i$ $x^2+2ixy-y^2 = 7-24i$ $x^2-y^2 = 7$.............(1) $2xy = -24$ $\therefore y = \dfrac{-12}{x}$.........(2) sub (2) into (1): $x^2-\left(\dfrac{-12}{x}\right)^2 = 7$ $x^2-\dfrac{144}{x^2} = 7$ $x^4-144 = 7x^2$ $x^4-7x^2-144 = 0$ $(x^2-16)(x^2+9) = 0$ $\therefore x = \pm 4$ as x is real If $x=4$, $y = \dfrac{-12}{4} = -3$ $x=-4$, $y=3$ Therefore solutions for $(x+iy)^2 = 7-24i$ are $x=4$, $y=-3$ and $x=-4$, $y=3$.	3 marks for complete correct solution 2 marks for substantial working that could lead to a correct solution with only one error 1 mark for substantial working that could lead to a correct solution
E3	(ii) $2z^2+6z+(1+12i) = 0$ $\Delta = b^2-4ac$ $= 6^2-4\times 2(1+12i)$ $= 28-96i$ $= 4(7-24i)$ $\therefore z = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $= \dfrac{-6\pm\sqrt{4(7-24i)}}{2\times 2}$ $= \dfrac{-6\pm 2(4-3i)}{2\times 2}$ $= \dfrac{1-3i}{2}, \dfrac{-7+3i}{2}$	2 marks for complete correct solution 1 mark for substantial work that could lead to a correct solution

E3	(d) $\left(\cot\theta+i\right)^n+\left(\cot\theta-i\right)^n$ $=\left(\frac{\cos\theta+i\sin\theta}{\sin\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\sin\theta}\right)^n$ $=\frac{1}{\sin^n\theta}\left\{\left(\cos\theta+i\sin\theta\right)^n+\left(\cos\left(-\theta\right)+i\sin\left(-\theta\right)\right)^n\right\}$ $=\frac{1}{\sin^n\theta}\left(\cos n\theta+i\sin n\theta+\cos\left(-n\theta\right)+i\sin\left(-n\theta\right)\right)$ using de Moivre's theorem $=\frac{1}{\sin^n\theta}\left(\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta\right)$ $=\frac{2\cos n\theta}{\sin^n\theta}$ **Multiple Choice Answers:** 1. C 2. A 3. A 4. B 5. D 6. C 7. B 8. B 9. D 10. C	2 marks for complete correct solution 1 mark for substantial work that could lead to a correct solution

Year 12	Mathematics Extension 2	Task 4 (Trial) 2017
Question 12	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

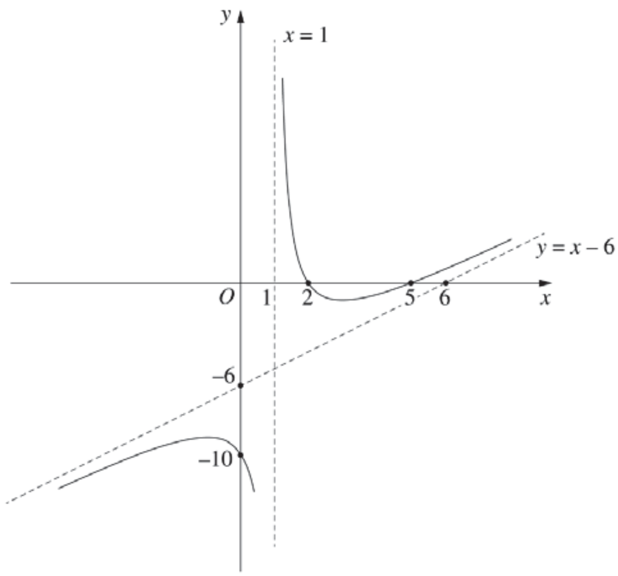
E4 uses efficient techniques for the algebraic manipulation required in dealing with questions involving polynomials

Part	Solutions	Marking Guidelines
(a) (i)	Roots are α, β, α + β. $\Sigma\alpha = 2\alpha + 2\beta = -\left(\frac{-16}{32}\right) = \frac{1}{2}$ $\therefore \alpha + \beta = \delta = \frac{1}{4}$	(a)(i) **2 Marks** ~ Correct solution. **1 Mark** ~ Makes significant progress towards the solution (Uses correct sum)
(ii)	Test $P\left(\frac{1}{4}\right) = 0$ $(4x-1)$ is a factor. The other two roots have $\alpha + \beta = \frac{1}{4}$; $\alpha\beta = \frac{-1}{8}$ from sum and product of roots of *P(x)* $\therefore a\left(x^2 - \frac{1}{4}x - \frac{1}{8}\right) = 0$ $8x^2 - 2x - 1 = 0$ $\therefore P(x) = (4x-1)(4x+1)(2x-1)$	(a) (ii) **2 Marks** ~ Correct factorisation. **1 Mark** ~ Makes significant progress towards solution
(b)	Real coefficients. So $2-i$ is also a root. $2+i+2-i=4$ $(2+i)(2-i)=5$ So one quadratic factor is: (z^2-4z+5) If the other 2 roots are α, β: $4+\alpha+\beta = 5 \rightarrow \alpha+\beta = 1$ $5\alpha\beta = -10 \rightarrow \alpha\beta = -2$ $\therefore P(z) = (z^2-4z+5)(z^2-z-2)$	(b) **2 Marks** ~ Correct solution. Must be quadratic factors. **1 Mark** ~ Makes significant progress towards the solution (One correct quadratic factor)
(c)	$P(2) = 0 \rightarrow 4a + 2b = -44$ $P'(2) = 0 \rightarrow 4a + b = -32$ $\therefore a = -5$ $b = -12$	(c) **2 marks:** Correct solution **1 Mark** ~ Significant progress towards solution.
(d)	$P(x) = Q(x)(x-2)(x+3) + ax + b$ $P(2) = 2a + b = -7$ $P(-3) = -3a + b = 3$ $\therefore a = -2$ $b = -3$ Remainder = $-2x-3$	**(d) 2 Marks** ~ Correct remainder . **1 Marks** ~ Makes significant progress towards the solution.
(e) (i)	$(1+i)^2 - bi(1+i) + c = 0$ $(2-b)i + b + c = 0i + 0.$ $\therefore b = 2$ $c = -2$	**(e) (i) 2 Marks** ~ Correct solution. **1 Mark** ~ Makes significant progress towards solution(substitution completed.)
(ii)	Sum of roots = *bi* =2*i* So, other roots = $2i - (1+i) = i - 1$	(e) (ii) **1 mark**: correct answer.
(f)	$P(x) = x^4 - 5x^3 - 9x^2 + 81x - 108$ $P'(x) = 4x^3 - 15x^2 - 18x + 81$ $P''(x) = 12x^2 - 30x - 18 = 6(2x+1)(x-3)$ Possible triple roots are *x*=3, -½ $P\left(-\frac{1}{2}\right) \neq 0; P(3) = P'(3) = P''(3) = 0$ Product of roots $= -108$, so solutions are: $x = 3, -4$	(f) **2 marks**: Correct roots with justification. **1 mark**: Significant progress towards correct solution.

Year 12	Mathematics Extension 2	Trial Exam 2017 HSC
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E6 - combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

Part	Solutions	Marking Guidelines
(a)	(i) $\frac{(x-2)(x-5)}{x-1} = \frac{x^2-7x+10}{x-1}$ $$\begin{array}{r} x-6 \\ x-1\overline{\smash{)}x^2-7x+10} \\ \underline{x^2-x} \\ -6x+10 \\ \underline{-6x+6} \\ 4 \end{array}$$ $\therefore \lim_{x\to\infty}\frac{(x-2)(x-5)}{x-1}$ $=\lim_{x\to\infty} x-6+\frac{4}{x-1}$ $=x-6+0$ ie the oblique asymptote is $y=x-6$	**2 marks** : correct solution **1 mark** : substantially correct solution
	(ii)  	**2 marks** : correct solution *(need correct intercepts AND asymptotes.)* **1 mark** : substantially correct solution *(NB: the maximum stationary point on the left hand branch is to the left of the y-axis. Marks were not deducted for showing this incorrectly)*
(b)	(i) $f(x)=3x^5-10x^3+16x$ $f'(x)=15x^4-30x^2+16$ $=15(x^2-1)^2+1$ ≥ 1, since $15(x^2-1)^2 \geq 0$ for all x.	**2 marks** : correct solution **1 mark** : substantially correct solution

(ii) $f''(x)$ is decreasing when $f'''(x) < 0$

$$f''(x) = 60x^3 - 60x$$

$$f'''(x) = 180x^2 - 60 < 0$$

$$x^2 < \frac{1}{3}$$

$$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

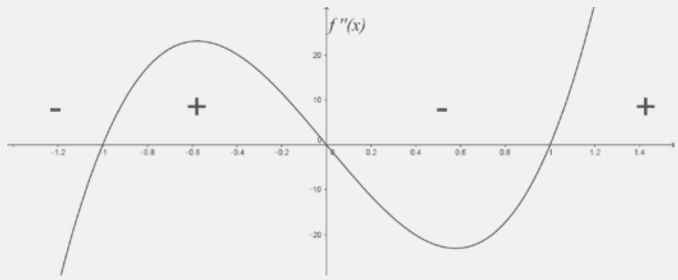
2 marks : correct solution

1 mark : substantially correct solution
(or reading "decreasing" as "negative"… which would give this solution…

(ii) $f''(x) = 60x^3 - 60x$

$$= 60x\left(x^2 - 1\right)$$

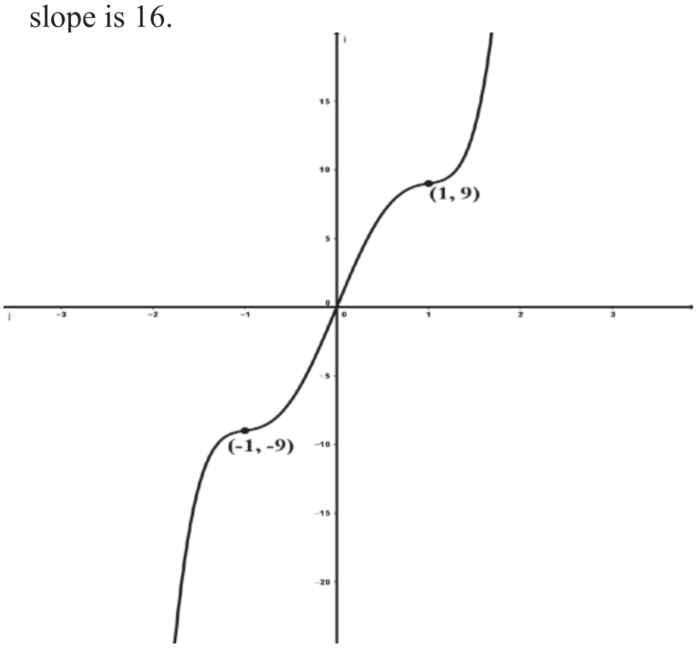
$$= 60x(x-1)(x+1)$$



$f''(x) < 0$ for $x < -1$ and $0 < x < 1$

…which is wrong)

(iii) $f'(x) \geq 1$, so $f(x)$ is monotonic increasing, has no stationary points, and the smallest slope occurs at $x = \pm 1$ where $f'(x) = 1$, ie $(-1,-9)$ & $(1,9)$. These are points inflexion. the othe point of inflexion is $(0,0)$ where the slope is 16.

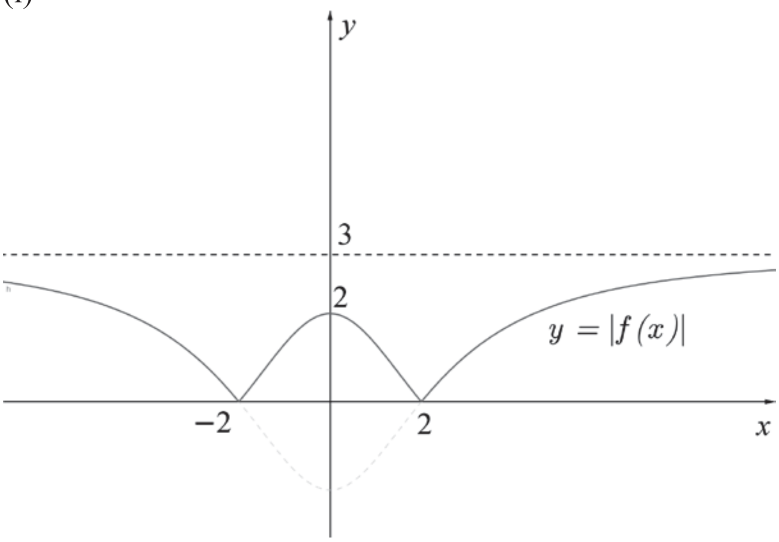


2 marks : correct solution

1 mark : substantially correct solution

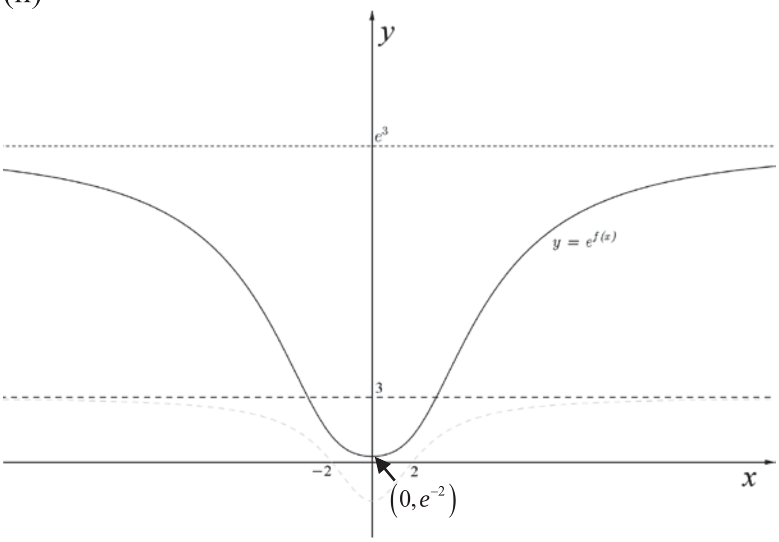
(c)

(i)



1 mark : correct solution

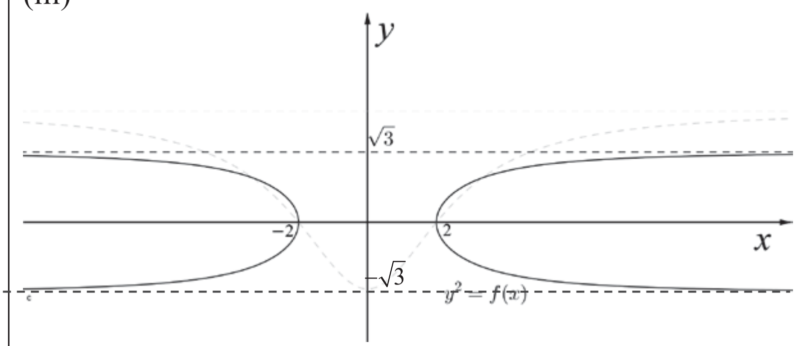
(ii)



2 marks : correct solution

1 mark : substantially correct solution

(iii)



2 marks : correct solution

1 mark : substantially correct solution

Year 12	Mathematics Extension 2	Task 4 (Trial) 2017
Question 14	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of conic sections

Part	Solutions	Marking Guidelines
(a) (i)	$\frac{d}{dx}\left(\frac{x^2}{25}+\frac{y^2}{16}\right)=\frac{d}{dx}(1) \quad \rightarrow \frac{dy}{dx}=\frac{-16x}{25y}$ Gradient of normal= $\frac{25y}{16x}=\frac{5\sin\theta}{4\cos\theta}$ $y-4\sin\theta=\frac{5\sin\theta}{4\cos\theta}(x-5\cos\theta)$ $4y\cos\theta-16\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$	**(a)(i) 2 Marks** ~ Correct with working. **1 Marks** ~ Makes significant progress towards the solution
(ii)	$P=\left(\frac{9\cos\theta}{5},0\right)$; $Q=\left(0,\frac{-9\sin\theta}{4}\right)$; $M=\left(\frac{9\cos\theta}{10},\frac{-9\sin\theta}{8}\right)$ Jusify locus of *M* is an ellipse by eliminating $\sin\theta,\cos\theta$ and showing the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Alternately, it was accepted that M is in parametric form $(a\cos\theta, b\sin\theta)$.	(a)(ii) **3 marks**: Finds *P, Q, M* and justifies the locus of *M*. **2 marks:** Significant progress. **1 mark:** Some relevant progress.
(b) (i)	*bx–ay = 0* *bx + ay = 0*	**(b)(i) 1 mark**: Correct answer in general form. If equations are correct but not in general form, you received this mark if you used general form in part (iv).
(ii)	$\tan\frac{\theta}{2}=\frac{b}{a} \qquad \therefore \tan\theta=\frac{2\left(\frac{b}{a}\right)}{1-\left(\frac{b}{a}\right)^2}=\frac{2ab}{a^2-b^2}$	(b)(ii) **2 marks**: correct solution. **1 Mark** ~ Makes significant progress towards solution *Note: Many people used angle between 2 lines formula, but it was easier to use double angle result which becomes a form of the t result because that led to more easily achieving part (iii)*
(iii)	$\sin\theta=\frac{2\left(\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)^2}=\frac{2ab}{a^2+b^2}$	(b)(iii) **1 mark:** Correct answer.
(iv)	*CP* and *DP* are perpendicular distances from *P* to the asymptotes. $CP\times DP=\frac{\lvert bx_0-ay_0\rvert}{\sqrt{(-a)^2+b^2}}\times\frac{\lvert bx_0+ay_0\rvert}{\sqrt{a^2+b^2}}$ $=\frac{(bx_0)^2-(ay_0)^2}{a^2+b^2}=\frac{b^2x_0{}^2-a^2y_0{}^2}{a^2+b^2}$ $=\frac{a^2b^2}{a^2+b^2}$	(b)(iv) **3 marks:** Correct solution, realising that this is a "show" question. **2 marks:** Significant progress.. **1 mark** : Some relevant progress made.
(v)	$\angle OCP=\angle ODP=90^0$ since *CP, DP* are perpendiculars *OCPD* is cyclic because these angles are opposite and supplementary.	(b)(v) **1 mark:** Indicating which angles are right angles, as well as giving the reason for being cyclic.
(vi)	$\Delta PCD=\frac{1}{2}.CP.PD\times\sin(180^0-\theta)$ $=\frac{1}{2}\times\frac{a^2b^2}{a^2+b^2}\times\frac{2ab}{a^2+b^2} \quad = \quad \frac{a^3b^3}{(a^2+b^2)^2}$	(b)(vi) **2 marks:** Correct solution, including showing the use of sin ratio of supplementary angle. **1 mark:** Significant progress.

Outcomes Addressed in this Question

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

Part	Solutions	Marking Guidelines
(a)	Let $x = \sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $dx = \cos\theta d\theta$ $(1-x^2)^{\frac{3}{2}} = (1-\sin^2\theta)^{\frac{3}{2}}$ $= (\cos^2\theta)^{\frac{3}{2}} = \cos^3\theta$ $\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}} dx = \int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta d\theta$ $= \int \tan^2\theta d\theta = \int (\sec^2\theta - 1) d\theta$ $= \tan\theta - \theta + c = \frac{x}{(1-x^2)^{\frac{1}{2}}} - \sin^{-1}x + c$	**2 Marks** ~ Correct solution. **1 Marks** ~ Makes significant progress towards the solution
(b)	$t = \tan\frac{x}{2}$ $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ or $dx = \frac{2}{1+t^2}dt$ When $x = 0$ then $t = 0$ and when $x = \frac{\pi}{2}$ then $t = 1$ $3 - \cos x - 2\sin x = \frac{3(1+t^2) - (1-t^2) - 4t}{1+t^2}$ $= \frac{3 + 3t^2 - 1 + t^2 - 4t}{1+t^2}$ $= \frac{2(2t^2 - 2t + 1)}{1+t^2}$ $= 2\left[(t-\frac{1}{2})^2 + \frac{1}{4}\right]\frac{2}{1+t^2}$ $\int_0^{\frac{\pi}{2}} \frac{1}{3 - \cos x - 2\sin x} dx = \int_0^1 \frac{1}{2}\left[\frac{1}{(t-\frac{1}{2})^2 + \frac{1}{4}}\right] \times \frac{1+t^2}{2} \times \frac{2}{1+t^2} dt$ $= \int_0^1 \frac{1}{2}\left[\frac{1}{(t-\frac{1}{2})^2 + \frac{1}{4}}\right] dt$ $\quad \left[\text{Let } u = t - \frac{1}{2},\ du = dt\right]$ $= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}\left[\frac{1}{u^2 + \frac{1}{4}}\right] du$ $= \frac{1}{2}\left[2\tan^{-1}u\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= \tan^{-1}1 - \tan^{-1}(-1)$ $= \frac{\pi}{2}$	**4 Marks** ~ Correct answer **3 Marks** ~ Correctly determines the primitive function (in terms of *t* or another variable). **2 Marks** ~ Correctly expresses the integral in terms of *t*. **1 Mark** ~ Correctly finds dx in terms of dt and determines the new limits.

(c) (i)	$\frac{7x+4}{(x^2+1)(x+2)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$ $7x+4=(ax+b)(x+2)+c(x^2+1)$ Let $x=-2$ and $x=0$ $-10=5c$ $\quad 4=b(0+2)-2(0^2+1)$ $c=-2$ $\quad b=3$ Equating the coefficients of x^2 $\mathbf{0=a-2 \text{ or } a=2}$ $\therefore a=2,\ b=3$ and $c=-2$	**3 Marks** ~ Correct answer. **2 Marks** ~ Calculates two of the variables **1 Mark** ~ Makes some progress in finding *a, b* or *c*.
(ii)	$\int\frac{7x+4}{(x^2+1)(x+2)}\,dx=\int\frac{2x+3}{x^2+1}-\frac{2}{x+2}\,dx$ $=\int\frac{2x}{x^2+1}+\frac{3}{x^2+1}-\frac{2}{x+2}\,dx$ $=\ln\left\|x^2+1\right\|+3\tan^{-1}x-2\ln\|x+2\|+c$ $=\ln\left\|\frac{x^2+1}{(x+2)^2}\right\|+3\tan^{-1}x+c$	**2 Marks** ~ Correct answer. **1 Mark** ~ Correctly finds one of the integrals.
(d) (i)	$I_n=\int_0^x\cos^n x\,dx$ $=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ Integration by parts $I_n=\int_0^{\frac{\pi}{2}}\cos^{n-1}t\cos t\,dt$ $=\left[\cos^{n-1}t\sin t\right]_0^{\frac{\pi}{2}}+(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\sin^2 t\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t(1-\cos^2 t)\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}(\cos^{n-2}t-\cos^n t)\,dt$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt-(n-1)\int_0^{\frac{\pi}{2}}\cos^n t\,dt$ Using the original integral $\int_0^{\frac{\pi}{2}}\cos^n t\,dt=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt-n\int_0^{\frac{\pi}{2}}\cos^n t\,dt+\int_0^{\frac{\pi}{2}}\cos^n t\,dt$ $n\int_0^{\frac{\pi}{2}}\cos^n t\,dt=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt$ $\int_0^{\frac{\pi}{2}}\cos^n t\,dt=\frac{(n-1)}{n}\int_0^{\frac{\pi}{2}}\cos^{n-2}t\,dt$ $I_n=\frac{(n-1)}{n}I_{n-2}$	**2 Marks** ~ Correct answer. **1 Mark** ~ Correctly integrates by parts.

(ii)	$I_n = \frac{(n-1)}{n} I_{n-2}$ $I_4 = \frac{(4-1)}{4} I_2$ $= \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2 t dt$ $= \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2t)dt$ $= \frac{3}{8}\left[x + \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}}$ $= \frac{3}{8}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$ $= \frac{3\pi}{16}$	**2 Marks** ~ Correct answer. **1 Mark** ~ Using the result from (d)(i) to obtain the definite integral.

Year 12 2017	Mathematics Extension 2	Task 4 Trial
Question No. 16	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E7 uses the techniques of slicing and cylindrical shells to determine volumes
E9 communicates abstract ideas and relationships using appropriate notation and logical argument

Outcome	Solutions	Marking Guidelines
E9	(a) Total arrangements = ${}^{11}C_5 = 462$ Majority of year 12 = $\left({}^6C_3 \times {}^5C_2\right) + \left({}^5C_1 \times {}^6C_4\right) + {}^6C_5 = 281$ Probability = $\dfrac{281}{462}$	2 marks for complete correct solution 1 marks for finding Total arrangements
E2	(b) In ΔAMC and ΔANH $\angle AMC = \angle ANH = 90°$ ($AM \perp BC$ and BN $\perp AC$ since AM and BN are altitudes of ΔAMC and ΔANH) $\angle MAC = \angle HAN$ (common angle to ΔAMC and ΔANH) $\therefore \Delta AMC \parallel\!\mid \Delta ANH$ (equiangular) $\therefore \angle ACB = \angle AHN$ (corresponding angles in similar triangles, $\Delta AMC \parallel\!\mid \Delta ANH$, are equal) Also, $\angle AHN = \angle BHM$ (vertically opposite angles are equal) $\angle BDA = \angle ACB$ (angles at the circumference on the same arc AB are equal) Now, In ΔBMD and ΔBMH $\angle BDA = \angle BHM$ (from above, i.e. $\angle BDA = \angle ACB = \angle AHN = \angle BHM$) $\angle BMD = \angle AMC = 90°$ (vertically opposite angles equal) $\angle BMH + \angle BMD = 180°$ (angle sum of straight angle, $\angle AMD$, is 180°) $\angle BMH + 90° = 180°$ $\angle BMH = 90°$ $\therefore \angle BMD = \angle BMH = 90°$ MB is common $\therefore \Delta BMD \equiv \Delta BMH$ (AAS) $\therefore HM = MD$ (corresponding sides of congruent triangles, $\Delta BMD \equiv \Delta BMH$, are equal) Note: There are other solutions that were accepted as well.	3 marks for complete correct proof with correct reasoning 2 marks for substantial working with correct reasoning that could lead to a correct proof with only a minor error 1 mark for some substantial working with correct reasoning that could lead to a correct proof

E2	(c) When $n = 1$, $LHS = F_0 = 2^{(2^0)} + 1 = 3$ $RHS = F_1 - 2 = 2^{(2^1)} + 1 - 2 = 3$ $\therefore$ Statement is true when $n = 1$. Assume the statement is true for $n = k$, some fixed positive integer. i.e. $F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} = F_k - 2$ When $n = k + 1$, $LHS = F_0 \times F_1 \times F_2 \times \ldots \times F_{n-1}$ $= F_0 \times F_1 \times F_2 \times \ldots \times F_k$ $= F_0 \times F_1 \times F_2 \times \ldots \times F_{k-1} \times F_k$ $= (F_k - 2) \times F_k$ by assumption $= (F_k)^2 - 2F_k$ $= \left(2^{2^k} + 1\right)^2 - 2\left(2^{2^k} + 1\right)$ $= 2^{2 \times 2^k} + 2 \times 2^{2^k} + 1 - 2 \times 2^{2^k} - 2$ $= 2^{2^{k+1}} + 1 - 2$ $= \left(2^{2n} + 1\right) - 2$ $= F_n - 2$ as required If statement is true for $n = k$, it has been proved true for $n = k + 1$. Since true for $n = 1$, then proved true for $n = 2, 3, 4, \ldots$	4 marks for complete correct solution 3 marks for substantial working that could lead to a complete correct solution with only one error 2 marks for substantial working that could lead to a correct solution after correctly proving true for $n = 1$ with more than one error or an incomplete solution 1 mark for correctly proving true for $n = 1$
E7	(d)(i) Let r be the radius of a typical slice $\therefore r + 1 = x \rightarrow r = x - 1$ Now, $\Delta V = \pi r^{2h} = \pi(x-1)^2 \Delta y$	2 marks for complete correct show 1 mark for correct radius
E7	(ii)When $x = 1, y = 2 - 1 = 1$ $\therefore V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(x-1)^2 \Delta y$ Now $-y = x^2 - 2x$ $\therefore\ 1 - y = x^2 - 2x + 1$ $\therefore\ 1 - y = (x-1)^2$ hence $\therefore\ V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y) \Delta y$	2 marks for complete correct show 1 mark for substantial correct working that could lead to a correct show
E7	(iii) $V = \pi \int_0^1 1 - y\ dy$ $= \pi \left[y - \frac{y^2}{2}\right]_0^1$ $= \frac{\pi}{2}\ u^3$	2 marks for complete correct solution 1 mark for substantial correct working that could lead to a correct solution