STUDENT'S NAME: ______________________________

TEACHER'S NAME: ______________________________

HURLSTONE
AGRICULTURAL
HIGH SCHOOL

2019

HIGHER
SCHOOL
CERTIFICATE
EXAMINATION

Mathematics Extension 2
Assessment Task 4
Trial Examination

Examiners~ G. Huxley, G. Rawson

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided.
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- This test paper must **NOT** be removed from the examination

Total marks: 100

Section I – 10 marks (pages 2–4)

- Attempt Questions 1–10
- Allow about 20 minutes for this section

Section II – 90 marks (pages 5–13)

- Attempt Questions 11–16
- Allow about 2 hour 40 minutes for this section

Section 1
10 marks
Attempt Questions 1 – 10 Allow about 20 minutes for this section.
Use the multiple choice answer sheet provided for Questions 1 – 10

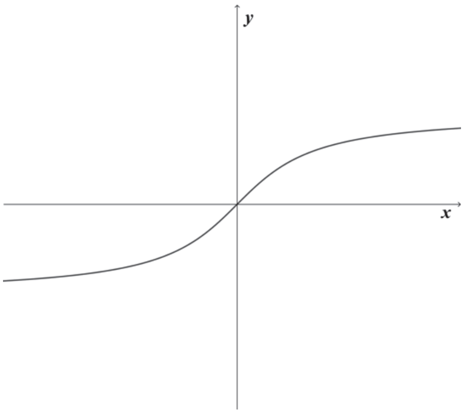
---

1. What is the gradient of the curve $2x^3 - y^2 = 7$ at the point where $y = -3$?

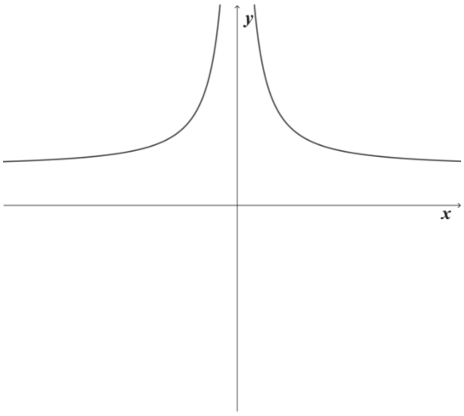
A –4
B –2
C –1
D 4

2. Which graph best describes $|y| = \tan^{-1}(x)$?

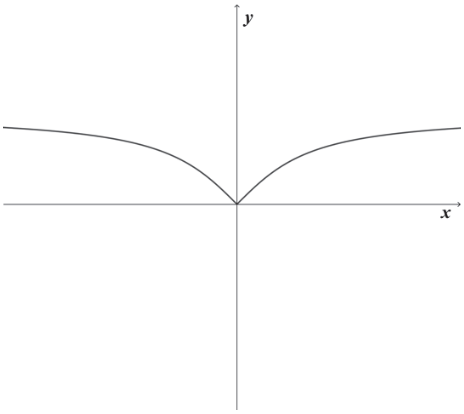
A



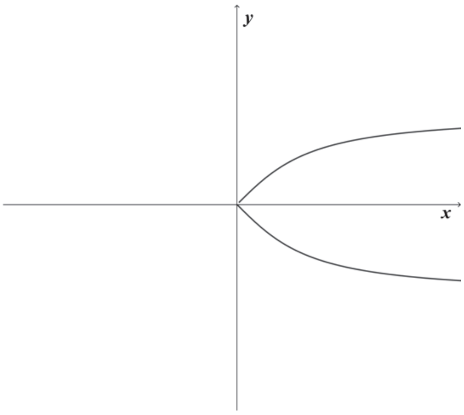
B



C



D



3. The sum of the eccentricities of two conics is $\frac{3}{2}$.
Which of the following could the two conics ***not*** be?

A An ellipse and a hyperbola

B An ellipse and a parabola

C A circle and a hyperbola

D A hyperbola and a parabola

4. The substitution $t = \tan\frac{\theta}{2}$ is used to find $\int \frac{d\theta}{\cos\theta}$.
Which of the following gives the correct expression for the required integral?

A $\int \frac{1}{2\left(1-t^2\right)} dt$

B $\int \frac{2}{1-t^2} dt$

C $\int \frac{2t}{1-t^2} dt$

D $\int \frac{4t}{\left(1+t^2\right)^2} dt$

5. Multiplying a non-zero complex number by $\frac{1+i}{1-i}$ results in a rotation about the origin on an Argand diagram.
What is the rotation?

A Clockwise by $\frac{\pi}{4}$

B Clockwise by $\frac{\pi}{2}$

C Anticlockwise by $\frac{\pi}{4}$

D Anticlockwise by $\frac{\pi}{2}$

6. It is given that $z = 2 + i$ is a root of $z^3 + az^2 - 7z + 15 = 0$, where a is a real number.

What is the value of a?

A -7

B -1

C 1

D 7

7. The equation $|z-3|+|z+3|=10$ defines an ellipse.
What is the length of the semi minor axis?

A	4	B	5
C	8	D	10

8. Under which condition would the graph of $f(x)=\dfrac{1}{x^2+mx-n}$, where m and n are real constants, have no vertical asymptotes?

A	$m^2 < -4n$	B	$m^2 > -4n$
C	$m^2 < 4n$	D	$m^2 > 4n$

9. Consider the region bounded by the y-axis, the line $y = 4$ and the curve $y = x^2$.

If this region is rotated about the line $y = 4$, which expression gives the volume of the solid of revolution?

A	$V=\pi\int_0^4 x^2 dy$	B	$V=\pi\int_0^2 (4-y)^2 dx$
C	$V=2\pi\int_0^2 (4-y)x\,dy$	D	$V=\pi\int_0^4 (4-y)^2 dx$

10. A hostel has four vacant rooms. Each room can accommodate a maximum of four people.
In how many different ways can six people be accommodated in the four rooms?

A	4020	B	4068
C	4080	D	4096

Section 2

90 marks

Attempt Questions 11 – 16 **Allow about 2 hours and 40 minutes for this section**

Answer each question in a separate answer booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) Start a new answer booklet. **Marks**

(a) Given that ω is one of the complex roots of the equation $z^3 - 1 = 0$,

show that $\dfrac{1}{1+\omega} + \dfrac{1}{1+\omega^2} = 1$ **2**

(b) Show that the quadratic equation $(1+i)z^2 + 4iz - 2(1-i) = 0$ has equal roots. **2**

(c) (i) Find the value of $\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)^6$ **2**

(ii) Show that $(\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta) = 1+\cos\theta + i\sin\theta$ **1**

(iii) Hence show that $\left(1+\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)^6 + \left(1+\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)^6 = 0$ **2**

(d) Sketch the region on the Argand diagram defined by

$$-\frac{\pi}{2} \le \arg(z-1-i) \le \pi \text{ and } |z| \le \sqrt{2}$$ **3**

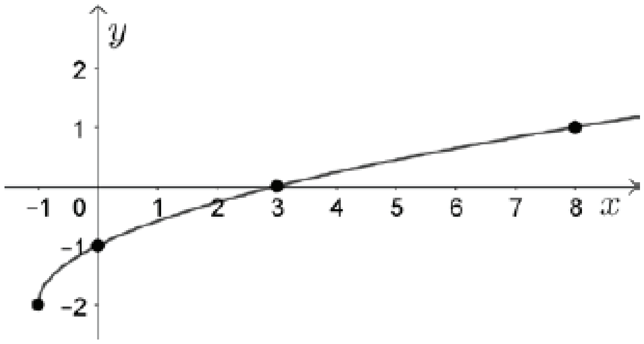
(e) The points A and B represent the complex numbers $z_1 = 2 - i$ and $z_2 = 8 + i$ respectively.

Find all the possible complex numbers z_3, represented by C, such that ΔABC is isosceles *and* right angled at C. **3**

End of Question 11

Question 12 (15 marks) Start a new answer booklet. **Marks**

(a) The diagram below shows the graph of the function $f\left(x\right)=\sqrt{x+1}-2$



On the separate diagrams on the answer sheet provided for this question, sketch the following graphs, showing clearly any intercepts on the co-ordinate axes, key points and the equations of any asymptotes.

(i) $y=\left[f\left(x\right)\right]^2$ **2**

(ii) $y^2=f\left(x\right)$ **2**

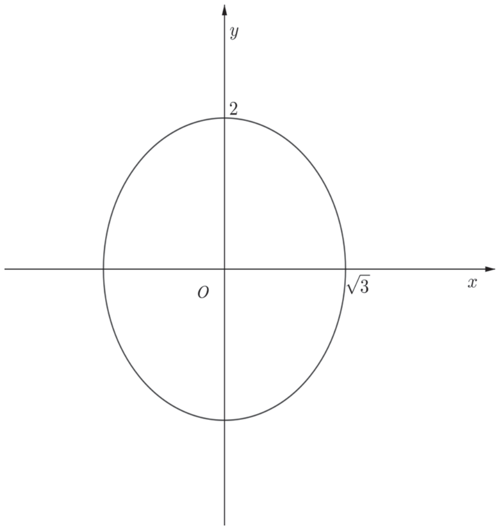
(iii) $y=\dfrac{1}{f\left(x\right)}$ **3**

(iv) $y=x\times f\left(x\right)$ **3**

Question 12 continues on the next page.

Marks

(b) The diagram of the ellipse E with equation $\frac{x^2}{3}+\frac{y^2}{4}=1$ is shown below.



The line $y = mx + 4$, with $m > 0$, is a tangent to the ellipse E at the point P.

(i) Find the value of m. **3**

(ii) Determine the co-ordinates of P **2**

End of Question 12

Question 13 (15 marks) Start a new answer booklet. **Marks**

(a) (i) Find a and b such that $x = 2$ is a double root of $P(x) = x^4 + ax^3 + x^2 + b$. **3**

(ii) For the values of a and b above, factorise $P(x)$ over the real numbers. **1**

(b) Consider the polynomial $P(x) = x^3 - x^2 - 21x + 45$ with roots $\alpha,\ \beta$ and γ.

(i) Find the monic polynomial with roots $\alpha - 3,\ \beta - 3$ and $\gamma - 3$. **3**

(ii) Hence solve $P(x) = 0$. **2**

(c) Suppose $\alpha,\ \beta$ and γ are the roots of the polynomial equation

$$x^3 + x + 12 = 0$$

(i) Find $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Hence explain why only one of the roots is real. **2**

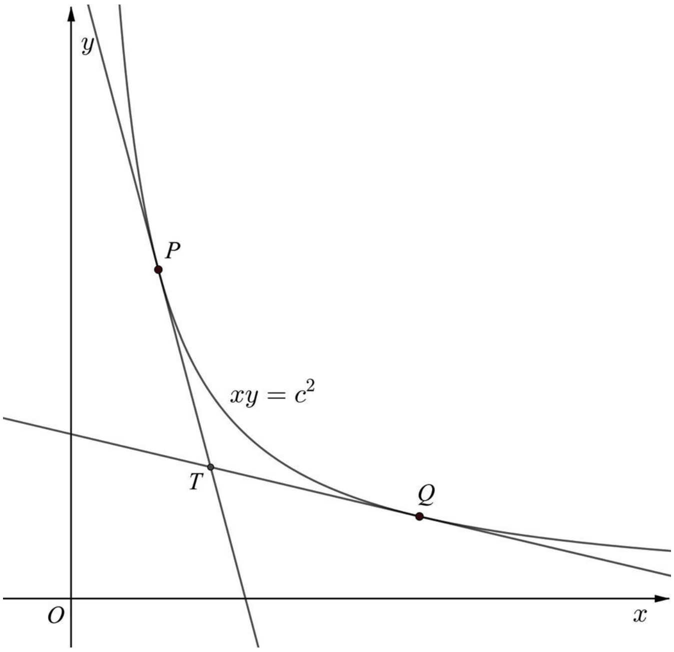
(iii) Let the real root be denoted by α. Show that $-3 < \alpha < -2$. **2**

End of Question 13

Question 14 (15 marks) Start a new answer booklet. **Marks**

(a) Sketch $\frac{x^2}{16}+\frac{y^2}{11}=1$
indicating the co-ordinates of its foci and equations of the directrices. **2**

(b) The distinct points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ are on the same branch of the hyperbola with the equation $xy=c^2$.
The tangent at *P* has the equation $x+p^2y=2cp$. (You do not have to derive this.)
The tangents at *P* and *Q* meet at the point *T*.

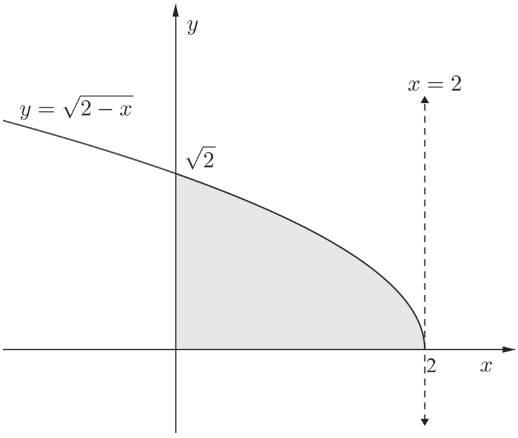


(i) Show that *T* has co-ordinates $\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$ **2**

(ii) Let *P* and *Q* move so that the tangent at *P* intersects the *x*-axis at $(cq,0)$.
Show that the locus of *T* is a hyperbola and state its eccentricity. **4**

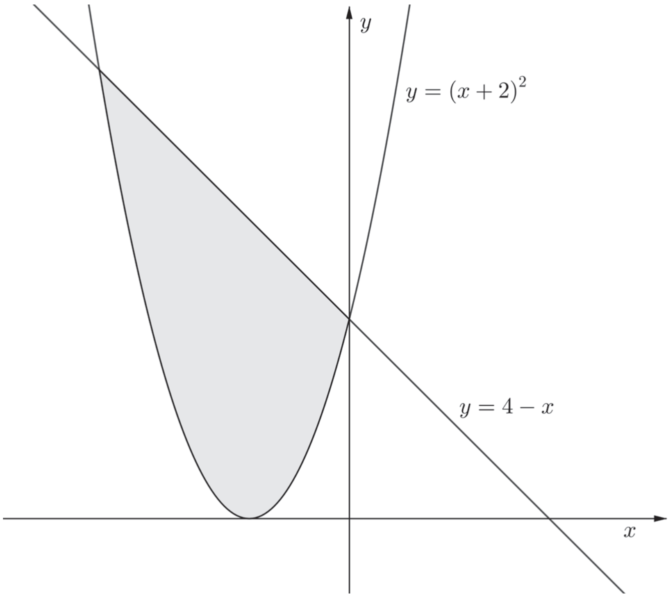
Question 14 continues on the next page.

(c) The region bounded by the curve $y = \sqrt{2-x}$, the x axis and the y axis is rotated About the line $x = 2$ to form a solid.



Calculate the volume of the solid generated. **3**

(d) The diagram shows the region bounded by the curve $y = (x+2)^2$ and the line $y = 4 - x$.



Use the method of cylindrical shells to calculate the volume generated when This region is rotated about the y-axis. **4**

End of Question 14

Question 15 (15 marks) Start a new answer booklet. **Marks**

(a) Find $\int \sin^5\theta\cos^4\theta\,d\theta$ **3**

(b) Find $\int \frac{\ln x}{x^2}\,dx$ **2**

(c) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **4**

(d) (i) Prove that $\frac{x^2}{\left(x^2+1\right)^{n+1}} = \frac{1}{\left(x^2+1\right)^n} - \frac{1}{\left(x^2+1\right)^{n+1}}$ **1**

(ii) Given that $I_n = \int_0^1 \frac{dx}{\left(x^2+1\right)^n}$, prove that $2nI_{n+1} = \frac{1}{2^n} + \left(2n-1\right)I_n$. **2**

(iii) Hence, evaluate $\int_0^1 \frac{dx}{\left(x^2+1\right)^3}$. **3**

End of Question 15

Question 16 (15 marks) Start a new answer booklet. **Marks**

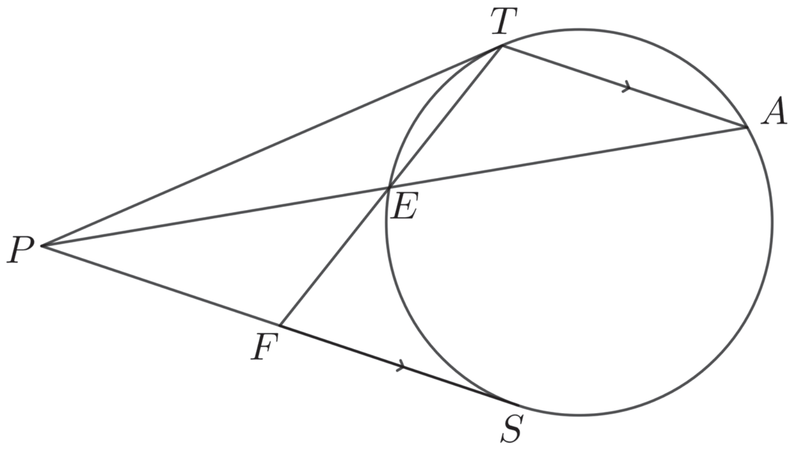
(a) (i) Show that $\sqrt{2}\cos x - \frac{1}{\sqrt{2}}\cos x = \cos\frac{\pi}{4}\cos x$ **1**

(ii) A sequence of numbers T_n , $n = 1, 2, 3, \ldots$. Is defined by $T_1 = 2$, $T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$ for $n = 3, 4, 5, \ldots$.

Use mathematical induction to show that:

$$T_n = \left(\sqrt{2}\right)^{n+2}\cos\left(\frac{n\pi}{4}\right),\ n = 1, 2, 3, \ldots .$$ **3**

(b) The diagram below shows two tangents *PT* and *PS* drawn to a circle from a point *P* outside the circle. Through *T*, a chord *TA* is drawn parallel to the tangent *PS*. The secant *PA* meets the circle at *E*, and *TE* produced meets *PS* at *F*.



(i) Prove that ΔEFP is similar to ΔPFT . **2**

(ii) Hence show that $PF^2 = TF \times EF$ **1**

(iii) Hence or otherwise, prove that *F* is the midpoint of *PS*. **2**

Question 16 continues on the next page.

Marks

(c) Suppose $f\left(x\right)=x-\ln\left(1+x+\frac{x^2}{2}\right)$.

Show that $f(x)$ is an increasing function for all values of x. **2**

(d) Given that $\frac{m+n}{2}\geq\sqrt{mn}$, show that, for $m>0, n>0, p>0, q>0$:

(i) $\left(m+n\right)\left(n+p\right)\left(m+p\right)\geq 8mnp$ **2**

(ii) $\frac{m}{n}+\frac{n}{p}+\frac{p}{q}+\frac{q}{m}\geq 4$ **2**

End of Question 16

END OF EXAMINATION

HURLSTONE AGRICULTURAL HIGH SCHOOL

EXTENSION 2 MATHEMATICS

2019

TRIAL EXAMINATION

SAMPLE SOLUTIONS AND MARKING GUIDELINES

SECTION I ANSWER SHEET

EXTENSION 2 MATHEMATICS

Answers

1	●	ⓑ	ⓒ	ⓓ
2	ⓐ	ⓑ	ⓒ	●
3	ⓐ	ⓑ	ⓒ	●
4	ⓐ	●	ⓒ	ⓓ
5	ⓐ	ⓑ	ⓒ	●
6	ⓐ	●	ⓒ	ⓓ
7	●	ⓑ	ⓒ	ⓓ
8	●	ⓑ	ⓒ	ⓓ
9	ⓐ	●	ⓒ	ⓓ
10	●	ⓑ	ⓒ	ⓓ

Year 12	Mathematics Extension 2	TRIAL 2019
Question No. 11	Solutions and Marking Guidelines	

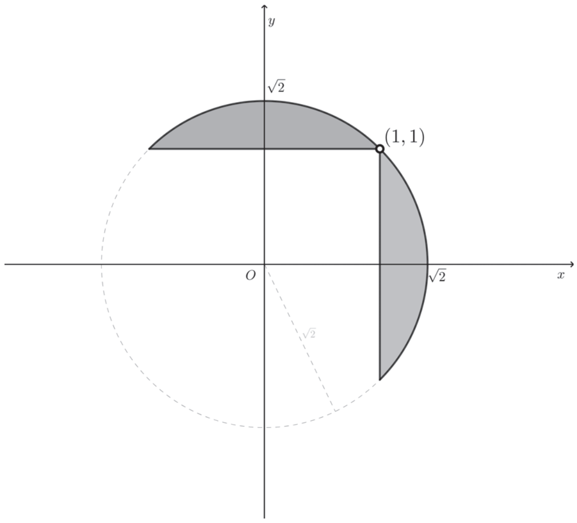
Outcomes Addressed in this Question

E3 - uses the relationship between algebraic and geometric representations of **complex numbers** (and of conic sections)

	Solutions	Marking Guidelines
	(a) $z^3 - 1 = (z - 1)(z^2 + z + 1)$ ω is complex root, so $\omega^2+\omega+1=0$ $\frac{1}{1+\omega}+\frac{1}{1+\omega^2}=\frac{1+\omega^2+1+\omega}{\left(1+\omega\right)\left(1+\omega^2\right)}$ $=\frac{\omega^2+\omega+1+1}{1+\omega^2+\omega+\omega^3}$ $=\frac{0+1}{0+\omega^3}=\frac{1}{1}=1 \qquad \left(\omega^3=1\right)$	**2 marks:** correct solution **1 mark:** substantially correct solution *(apparently there are approximately 374 ways to do this question)*
	(b) $\Delta = b^2-4ac$ $=\left(4i\right)^2-4\left(1+i\right)\left(-2\left(1-i\right)\right)$ $=-16+8\left(1+i\right)\left(1-i\right)$ $=-16+8\left(1-i^2\right)$ $=-16+8\left(2\right)$ $=0$	**2 marks:** correct solution **1 mark:** substantially correct solution *(NB: finding the sum and products of roots was not enough for a mark* ***unless*** *you actually were able to come up with the double root)*
	(c) (i) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6=\left(\cos\pi+i\sin\pi\right)$ $=-1$	**2 marks:** correct solution **1 mark:** substantially correct solution
	(ii) $\text{LHS}=\left(\cos\theta+i\sin\theta\right)\left(1+\cos\theta-i\sin\theta\right)$ $=\cos\theta+\cos^2\theta-i\sin\theta\cos\theta+i\sin\theta+i\sin\theta\cos\theta-i^2\sin^2\theta$ $=\cos\theta+\cos^2\theta+i\sin\theta+\sin^2\theta$ $=1+\cos\theta+i\sin\theta$ $=\text{RHS}$	**1 mark:** correct solution
	(iii) $\text{LHS}=\left(1+\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6+\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6$ $=\left[\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^6+\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6$ $=-1\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6+\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6$ $=0=\text{RHS}$	**2 marks:** correct solution **1 mark:** substantially correct solution *(NB: Hence, but not otherwise… ie MUST use part (i) and/or (ii) to be elligble for marks)*

Question 11 continued…

(d)

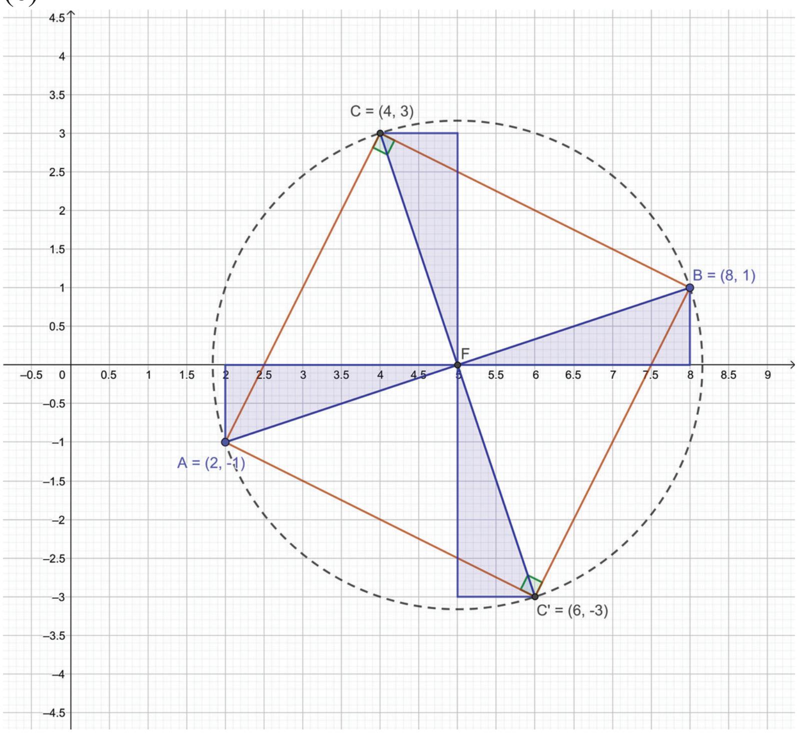


3 marks: correct solution

2 marks: substantially correct solution

1 mark: partial progress towards correct solution

(e)



For ΔABC to be isosceles, with $AC = BC$, C must be on the perpendicular bisector of AB.

For $\angle BCA$ to be a right angle, $ACBC'$ must be a square, inscribed in the circle with AB as diameter.

So C must be either (4, 3) or (6, –3)

The complex numbers are $4 + 3i$ or $6 - 3i$

3 marks: correct solution

2 marks: substantially correct solution

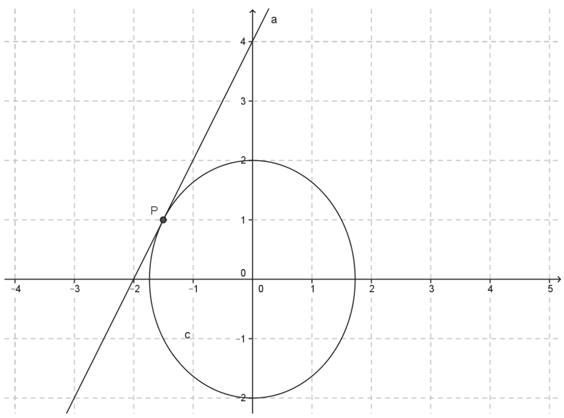
1 mark: partial progress towards correct solution

(NB: algebraic methods are fine for this question, but very few… like ***extremely*** *very few candidates were successful in finding both answers. It was much easier just to draw the square and work around the diagram)*

Higher School Certificate Mathematics Task 4 2019 HSC
Extension 2
Question No. 12 Solutions and Marking Guidelines

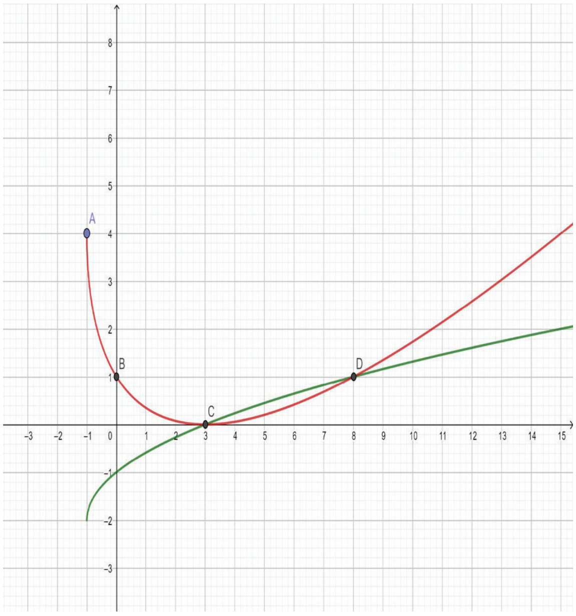
Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

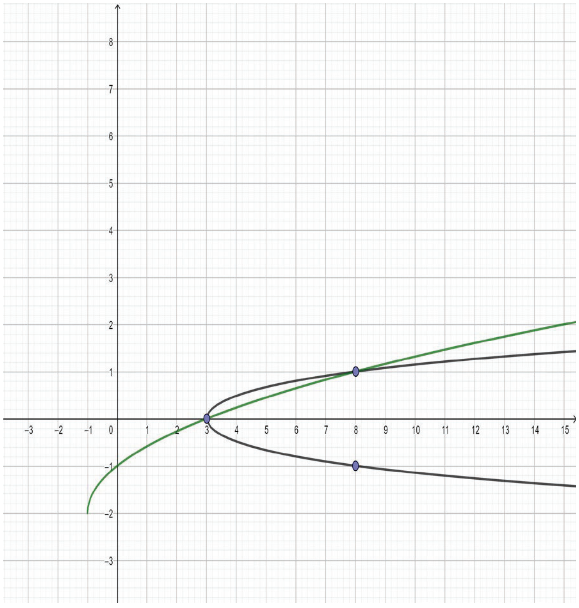
	Solutions	**Marking Guidelines**
(a)	See next page.	(a) (i) 2 marks: Correct graph including shape, intercepts on axes and original function, and end point. 1 mark: One feature not included. (ii) 2 marks: Correct graph including co-ordinates indicated and parabolic shape. 1 mark: One feature not included. (iii) 3 marks: Correct graph including endpoint, intercepts on original function, and asymptote. 2 marks: One feature not included. 1 mark: Some relevant progress made. (iv) 3 marks: Correct graph including co-ordinates indicated, intercepts on axes and original function, and endpoint. 2 marks: One feature not included. 1 mark: Some relevant progress made.
(b)	 This diagram is not entirely necessary to be able to answer the question, but it would have been of assistance to illustrate how the line with positive gradient touches the ellipse in the second quadrant.	(b)

HAHS Mathematics Extension 2
2019 Trial Examination
Question 12(a) answer sheet

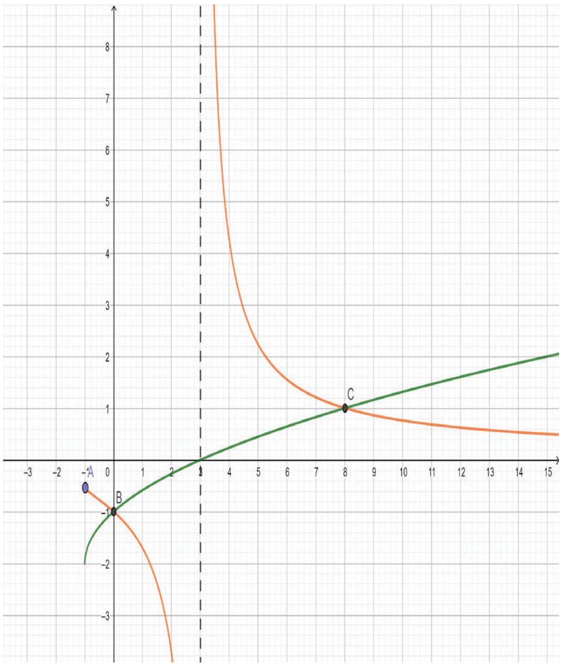
(i) $y=\left[f\left(x\right)\right]^2$



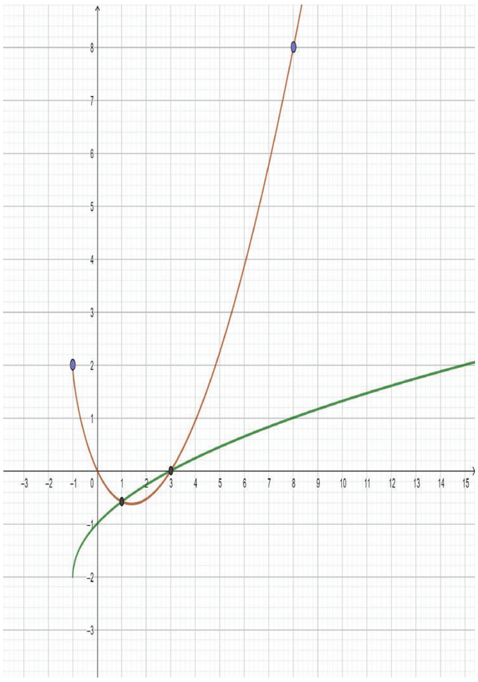
(ii) $y^2=f\left(x\right)$



(iii) $y=\dfrac{1}{f\left(x\right)}$



(iv) $y=x\times f\left(x\right)$



Higher School Certificate Mathematics Extension 2 Task 4 2019 HSC
Question No. 12 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

	Solutions	**Marking Guidelines**
(b)	(i) $\frac{x^2}{3}+\frac{(mx+4)^2}{4}=1$ $\rightarrow 4x^2+3m^2x^2+24mx+36=0$ This will have equal roots because of the tangent property. So its discriminant will equal 0. $(24m)^2-4(4+3m^2)(36)=0$ $\rightarrow 144m^2-576=0$ $144(m-2)(m+2)=0\rightarrow m=\pm 2$ However, we need $m > 0$, so m =2 is the only solution. (ii) Substituting m = 2 into the quadratic equation above: $16x^2+48x+36=0$ $4(2x+3)^2=0$ $x=-\frac{3}{2}$.Find the corresponding point on the ellipse. $\frac{-3/2}{3}+\frac{y^2}{4}=1\rightarrow y^2=1$ But in the 2nd quadrant, $y > 0$. $\therefore P=\left(-\frac{3}{2},1\right)$ Alternately, $x=-\frac{3}{2}$ can be substituted into the straight line equation for the same result.	(b) (i) 3 marks: Complete solution with working 2 marks: Significant progress. 1 mark : Some relevant progress (ii) 2 marks: correct solution with working. CFPA 1 mark : significant progress towards correct solution

Year 12	Mathematics Extension 2	TRIAL 2019
Question No. 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E4 - uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving (conic sections and) polynomials.

	Solutions	Marking Guidelines
	(a)(i) $P(x)=x^4+ax^3+x^2+b$ $P'(x)=4x^3+3ax^2+2x$ now, $P'(2)=0$ ie $4(2)^3+3a(2)^2+2(2)=0$ $32+12a+4=0 \quad \Rightarrow a=-3$ and $P(2)=0$ ie $(2)^4+-3(2)^3+(2)^2+b=0$ $16-24+4+b=0 \quad \Rightarrow b=4$ $\therefore a=-3,\quad b=4$	**3 marks:** correct solution **2 marks:** substantially correct solution **1 mark:** partial progress towards correct solution
	(ii) $P(x)=x^4-3x^3+x^2+4$ $=(x-2)^2(Ax^2+Bx+C)$ $=(x^2-4x+4)(Ax^2+Bx+C)$ $=(x^2-4x+4)(x^2+Bx+1)$ $=(x-2)^2(x^2+x+1)$	**1 mark:** correct solution *NB:* ***should*** *be factorised fully…*
	(b)(i) $P(x)=x^3-x^2-21x+45$ roots of form $y=x-3$, so let $x=y+3$ $(y+3)^2=y^2+6y+9$ $(y+3)^3=y^3+9y^2+27y+27$ so, $P(y+3)=(y+3)^3-(y+3)^2-21(y+3)+45$ $=y^3+9y^2+27y+27-y^2-6y-9-21y-63+45$ $P(y+3)=y^3+8y^2$	**3 marks:** correct solution **2 marks:** substantially correct solution **1 mark:** partial progress towards correct solution
	(ii) $y^3+8y^2=0$ $y^2(y+8)=0$ $y=0,0,-8$ $\therefore \quad x=3,3,-5 \quad$ (from (i), $x=y+3$)	**2 marks:** correct solution **1 mark:** substantially correct solution

Question 13 continued…

(c) (i) $x^3+0x^2+x+12=0$

$$\alpha+\beta+\gamma=-\tfrac{b}{a}=0$$

$$\alpha\beta+\alpha\gamma+\beta\gamma=\tfrac{c}{a}\quad=1$$

$$\alpha\beta\gamma=-\tfrac{d}{a}=-12$$

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$=0-2(1)$$
$$=-2$$

2 marks: correct solution

1 mark: substantially correct solution

(ii) The sum of squares is < 0, so at least one root is unreal.
Coefficients are real, so unreal roots occur in conjugate pairs.
∴ there are two unreal roots.
Degree three polynomial has 3 roots, one of which must be real. (odd degree).

∴ there must be one real root.

2 marks: correct solution

1 mark: substantially correct solution

(iii) $P(x)=x^3+x+12$

$$P(-3)=(-3)^3+(-3)+12=-18<0$$

$$P(-2)=(-2)^3+(-2)+12=2\quad>0$$

∴ the graph crosses the x-axis between –3 and –2
ie $-3<\alpha<-2$.

2 marks: correct solution

1 mark: substantially correct solution

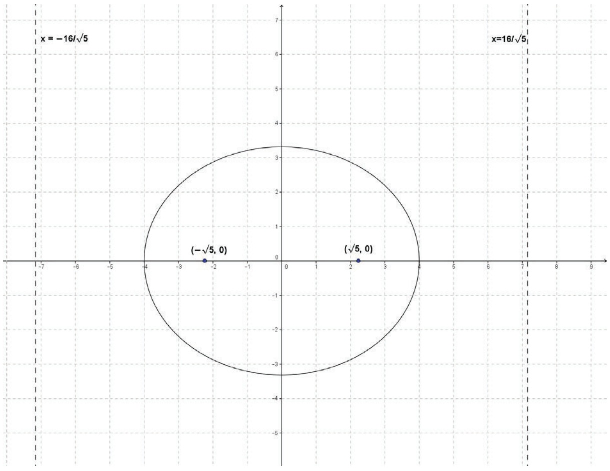
NOTE:
Hence v Hence or otherwise.

- Hence means you **must** use the previous part(s) of question to answer this part. If you use alternate means, you are not answering the question, and therefore making yourself ineligible to receive marks for this part – regardless of whether your method works or not. This was the case for many candidates in this question.
- Hence or otherwise allows you to use whatever means necessary to answer the question. Usually, the previous parts are the most convenient stepping stone to the next part, but it's up to you.

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of conic sections.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

Outcome	Solutions	Marking Guidelines
E3 (a)		(a) (i) 2 marks: Complete solution with foci and directrices labelled. 1 mark : Relevant progress
(b)	(i) At T: $x+p^2y=2cp.....(1)$ $x+q^2y=2cq.....(2)$ $(1)-(2):\left(p^2-q^2\right)y=2c(p-q)$ $(p-q)(p+q)y=2c(p-q)$ $\therefore y=\dfrac{2c}{p+q} \quad ; p\neq q$ Substitute into $(1): x+p^2\left(\dfrac{2c}{p+q}\right)=2cp$ $x(p+q)+2cp^2=2cp^2+2cpq$ $x=\dfrac{2cpq}{p+q} \qquad \therefore T=\left(\dfrac{2cpq}{p+q},\dfrac{2c}{p+q}\right)$ (ii)Substitute *(cq, 0)* into equation of tangent through *P*. $cq+p^2(0)=2cp \qquad \rightarrow q=2p$ Substitute into co-ordinates of *T* $T=\left(\dfrac{2cp(2p)}{p+2p},\dfrac{2c}{p+2p}\right)$ $=\left(\dfrac{4cp}{3},\dfrac{2c}{3p}\right)$ The paramater *p* can be eliminated by multiplication: $xy=\left(\dfrac{4cp}{3}\right)\times\left(\dfrac{2c}{3p}\right)=\dfrac{8c^2}{9}$ which is constant for the hyperbola $xy=c^2$. Therefore the locus of *T* is a rectangular hyperbola, so it has eccentricity $=\sqrt{2}$.	(b) (i) 2 marks: Correct solutions with working. 1 mark: Relevant progress. (ii) 4 marks: Full solution with working. 3 marks: One element of solution incorrect. 2 marks: Significant progress. 1 mark: Some relevant progress.

E7 (c)	Using the method of slicing, we get washers, with thickness Δy. Washer: Outer radius = 2; Inner radius = $2-x$ Area = $\pi\left(2^2-\left(2-x\right)^2\right)$ But $y=\sqrt{2-x} \rightarrow y^2=2-x$ $\therefore A(y)=\pi\left(4-y^4\right)$ $\Delta V=\pi\left(4-y^4\right)\Delta y$ $V=\lim_{\Delta y\to 0}\sum_{y=0}^{\sqrt{2}}\pi\left(4-y^4\right)\Delta y$ $=\pi\int_0^{\sqrt{2}}4-y^4\,dy$ $=\pi\left[4y-\frac{y^5}{5}\right]_0^{\sqrt{2}}=\frac{16\sqrt{2}\,\pi}{5}$ units^3 Alternately, the method of cylindrical shells would give the following integral: $V=2\pi\int_0^2\left(2-x\right)^{\frac{3}{2}}dx$ with the same final result.	(c) 3 marks: Full solution with working. 2 marks: Significant progress. 1 mark: Some relevant progress.
(d)	Cylindrical shells will have height equal to the difference between the line and parabola. Since each shell's radius is positive, the radius will be the negative of the *x* value at each point in the given region. $A(x)=2\pi rh$ $r=-x$; $h=(4-x)-(x+2)^2$ $=-5x-x^2$ $\therefore A(x)=2\pi(-x)\left(-5x-x^2\right)$ $=-2\pi\left(-5x^2-x^3\right)$ $\Delta V=-2\pi\left(-5x-x^3\right)\Delta x$ Points of intersection of curves: $(x+2)^2=4-x$ $x(x+5)=0 \quad \rightarrow x=0,-5$ $\therefore V=\lim_{\Delta x\to 0}\sum_{x=-5}^{0}-2\pi\left(-5x^2-x^3\right)\Delta x$ $=2\pi\int_{-5}^{0}5x^2+x^3\,dx$ $=2\pi\left[\frac{5x^3}{3}+\frac{x^4}{4}\right]_{-5}^{0}=\frac{625\pi}{6}$ units^3	(d) 4 marks: Full solution with working. 3 marks: One element of solution incorrect. 2 marks: Significant progress. 1 mark: Some relevant progress.

Year 12	Mathematics Extension 2	TRIAL 2019
Question No. 15	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E8 - applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

	Solutions	Marking Guidelines
	(a) $\begin{aligned}\int \sin^5\theta\cos^4\theta\,d\theta &= \int \sin^4\theta\cos^4\theta\sin\theta\,d\theta \\ &= \int\left(1-\cos^2\theta\right)^2\cos^4\theta\sin\theta d\theta \\ &= -\int\left(1-u^2\right)^2 u^4\,du \\ &= -\int\left(u^4-2u^6+u^8\right)du \\ &= -\left(\frac{u^5}{5}-\frac{2u^7}{7}+\frac{u^9}{9}\right)+C \\ &= -\tfrac{1}{5}\cos^5\theta+\tfrac{2}{7}\cos^7\theta-\tfrac{1}{9}\cos^9\theta+C\end{aligned}$ $u=\cos\theta$ $du=-\sin\theta d\theta$	**3 marks:** correct solution **2 marks:** substantially correct solution **1 mark:** partial progress towards correct solution
	(b) $\begin{aligned}\int\frac{\ln x}{x^2}dx &= \int \ln x.\frac{1}{x^2}dx \\ &= uv-\int v\,du \\ &= \ln x\left(-\frac{1}{x}\right)-\int\left(-\frac{1}{x}\right)\frac{1}{x}dx \\ &= -\frac{\ln x}{x}+\int\frac{1}{x^2}dx \\ &= -\frac{\ln x}{x}-\frac{1}{x}+C \\ &= \frac{-\ln x-1}{x}+C\end{aligned}$ $u=\ln x$ $du=\frac{1}{x}dx$ $dv=\frac{1}{x^2}dx$ $v=-\frac{1}{x}$	**2 marks:** correct solution **1 mark:** substantially correct solution
	(c) $t=\tan\frac{x}{2}$ $\quad \cos x=\frac{1-t^2}{1+t^2}$ $\quad x=\frac{\pi}{2}\ \Rightarrow\ t=1$ $dx=\frac{2dt}{1+t^2}$ $\quad x=0\ \Rightarrow\ t=0$ $\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x} &= \int_0^1\frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}dt \\ &= \int_0^1\frac{2}{3+t^2}dt \\ &= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}}-\tan^{-1}0\right] \\ &= \frac{2}{\sqrt{3}}.\frac{\pi}{6}=\frac{\pi}{3\sqrt{3}}\end{aligned}$	**4 marks:** correct solution **3 marks:** substantially correct solution **2 marks:** significant progress towards correct solution **1 mark:** limited progress towards correct solution

Question 15 continued...

(d) (i) $$\text{LHS} = \frac{x^2}{\left(x^2+1\right)^{n+1}} = \frac{x^2+1-1}{\left(x^2+1\right)^{n+1}}$$

$$= \frac{x^2+1}{\left(x^2+1\right)^{n+1}} - \frac{1}{\left(x^2+1\right)^{n+1}}$$

$$= \frac{1}{\left(x^2+1\right)^{n}} - \frac{1}{\left(x^2+1\right)^{n+1}} = \text{RHS}$$

1 mark: correct solution

(ii) $$I_n = \int_0^1 \frac{dx}{\left(x^2+1\right)^n} = uv - \int_0^1 v\,du$$

$$\text{let } u = \frac{1}{\left(x^2+1\right)^n}$$

$$du = \frac{-2nx}{\left(x^2+1\right)^{n+1}}$$

$$\text{and } dv = dx$$

$$v = x$$

$$= \left[\frac{x}{\left(x^2+1\right)^n}\right]_0^1 - \int_0^1 \frac{-2nx^2}{\left(x^2+1\right)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{\left(x^2+1\right)^{n+1}}\,dx$$

$$= \frac{1}{2^n} + 2n\int_0^1 \left(\frac{1}{\left(x^2+1\right)^n} - \frac{1}{\left(x^2+1\right)^{n+1}}\right)dx$$

$$I_n = \frac{1}{2^n} + 2n\left(I_n - I_{n+1}\right)$$

$$2nI_{n+1} = \frac{1}{2^n} + \left(2n-1\right)I_n$$

2 marks: correct solution

1 mark: substantially correct solution

(iii) $$\int_0^1 \frac{dx}{\left(x^2+1\right)^3} = I_3 \quad \text{so let } n = 2 \qquad \text{and } n = 1$$

$$2\left(2\right)I_{2+1} = \frac{1}{2^2} + \left(2\left(2\right)-1\right)I_2 \qquad 2\left(1\right)I_{1+1} = \frac{1}{2^1} + \left(2\left(1\right)-1\right)I_1$$

$$4I_3 = \frac{1}{4} + 3I_2 \qquad 2I_2 = \frac{1}{2} + I_1$$

$$I_1 = \int_0^1 \frac{dx}{x^2+1} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}$$

$$2I_2 = \tfrac{1}{2} + \tfrac{\pi}{4} \qquad 4I_3 = \tfrac{1}{4} + 3\left(\tfrac{1}{4} + \tfrac{\pi}{8}\right)$$

$$I_2 = \tfrac{1}{4} + \tfrac{\pi}{8} \qquad I_3 = \tfrac{1}{16} + \tfrac{3}{4}\left(\tfrac{2+\pi}{8}\right)$$

$$= \frac{2+6+3\pi}{32} = \frac{8+3\pi}{32}$$

3 marks: correct solution

2 marks: substantially correct solution

1 mark: partial progress towards correct solution

Higher School Certificate Mathematics Extension 2 Task 4 2019 HSC
Question No. 16 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

	Solutions	**Marking Guidelines**
E2(a)	**(i)** $\sqrt{2}\cos x-\frac{1}{\sqrt{2}}\cos x=\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\cos x$ $=\frac{1}{\sqrt{2}}\cos x=\cos\frac{\pi}{4}\cos x$ **(ii)** RTP: $T_n=\left(\sqrt{2}\right)^{n+2}\cos\left(\frac{n\pi}{4}\right)$ Step 1: $n=1; LHS=2 \quad RHS=\left(\sqrt{2}\right)^{3}\times\frac{1}{\sqrt{2}}$ $=2=LHS$ $n=2; LHS=0 \quad RHS=\left(\sqrt{2}\right)^{4}\cos\frac{\pi}{2}$ $=0=LHS$ Statement is true for n = 1, 2.. Step 2: Assume true for all n up to $n=k$:i.e.$T_k=\left(\sqrt{2}\right)^{k+2}\cos\left(\frac{k\pi}{4}\right)$ Prove true for $n=k+1$: i.e. $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\cos\left(\frac{(k+1)\pi}{4}\right)$ $LHS=T_{k+1}=2T_k-2T_{k-1}$ $=2\left(\sqrt{2}\right)^{k+2}\cos\left(\frac{k\pi}{4}\right)-2\left(\sqrt{2}\right)^{k+1}\cos\left(\frac{(k-1)\pi}{4}\right)$ $=\left(\sqrt{2}\right)^{k+4}\cos\left(\frac{k\pi}{4}\right)-\left(\sqrt{2}\right)^{k+3}\cos\left(\frac{(k-1)\pi}{4}\right)$ $=\left(\sqrt{2}\right)^{k+3}\left[\sqrt{2}\cos\left(\frac{k\pi}{4}\right)-\cos\left(\frac{k\pi}{4}-\frac{\pi}{4}\right)\right]$ $=\left(\sqrt{2}\right)^{k+3}\left[\sqrt{2}\cos\left(\frac{k\pi}{4}\right)-\cos\frac{k\pi}{4}\cos\frac{\pi}{4}-\sin\frac{k\pi}{4}\sin\frac{\pi}{4}\right]$ $=\left(\sqrt{2}\right)^{k+3}\left[\sqrt{2}\cos\frac{k\pi}{4}-\frac{1}{\sqrt{2}}\cos\frac{k\pi}{4}-\sin\frac{k\pi}{4}\sin\frac{\pi}{4}\right]$ $=\left(\sqrt{2}\right)^{k+3}\left[\cos\frac{\pi}{4}\cos\frac{k\pi}{4}-\sin\frac{k\pi}{4}\sin\frac{\pi}{4}\right]$ $=\left(\sqrt{2}\right)^{k+3}\cos\left(\frac{(k+1)\pi}{4}\right)$ $=RHS$ as required.	(a) (i) 1 mark: Correct working. (ii) 3 marks: Complete solution with reasoning. Step 1 needs to consider both n=1,2. 2 marks: Error made in solution. 1 mark: Substantial relevant progress.

(b)	(i) $\angle PTF=\angle TAP$ Alternate segment theorem $\angle \mathrm{EPF}=\angle \mathrm{TAP}$ Alternate angles, $TA\Vert PS$ $\therefore \angle PTF=\angle EPF$ Both equal $\angle TAP$ In $\Delta EFP,\Delta PFT$ $\angle EPF=\angle PTF$ shown above $\angle EFP=\angle TFP$ common angle $\therefore \Delta EFP \vert\vert\vert \Delta TFP$ equiangular	(b)(i) 2 marks: Correct proof with reasons. 1 mark: Significant progress.
	(ii) In $\Delta EFP,\Delta PFT$ $\frac{PF}{TF}=\frac{EF}{PF}$ corresponding sides in similar triangles are in proportion $\therefore\ PF^2=TF\times EF$	(ii) 1 mark: Correct statement of the relationship, both in words and algebraically.
	(iii) For the tangent *FS* and secant *FT,* $FS^2=TF\times EF$ Intercept properties of tangent and secant. From (ii) $PF^2=TF\times EF$ $\therefore PF^2=FS^2$ $i.e. PF=FS$;so F is the midpoint of PS	(iii) 2 marks: Statement of tangent-secant relationship, and applying it to (ii) for the solution. 1 mark: Significantly correct.
(c)	$f'(x)=1-\frac{x+1}{1+x+\frac{x^2}{2}}$ $=\frac{1+x+\frac{x^2}{2}-x-1}{1+x+\frac{x^2}{2}}$ $=\frac{x^2}{2+2x+x^2}=\frac{x^2}{1+(1+x)^2}$ $i.e. f'(x)\geq 0$ Because the numerator is non-negative and the denominator is positive definite. So *f(x)* is non-decreasing for all values of *x*.	(c) 2 marks: Correctly shown. 1 mark: Significantly correct.
(d)	(i) $(m+n)(n+p)(m+p)\geq(2\sqrt{mn})(2\sqrt{np})(2\sqrt{mp})$ $=8\sqrt{m^2n^2p^2}$ $=8mnp$	(d)(i) 2 marks: Correct working. 1 mark: 1 step omitted.
	(ii) $\frac{m}{n}+\frac{n}{p}+\frac{p}{q}+\frac{q}{m}\geq 2\sqrt{\left(\frac{m}{n}\right)\left(\frac{n}{p}\right)}+2\sqrt{\left(\frac{p}{q}\right)\left(\frac{q}{m}\right)}$ $=2\left(\sqrt{\frac{m}{p}}+\sqrt{\frac{p}{m}}\right) \qquad =2\left(2\sqrt{\frac{mp}{pm}}\right) \qquad =4$ $\therefore LHS\geq 4$	(ii) 2 marks: Correct working. 1 mark: 1 step omitted.