INTERNATIONAL GRAMMAR SCHOOL

Concordia per Diversitatem

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

1999

MATHEMATICS

4 UNIT (ADDITIONAL)

Time Allowed - Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- Write your student Name / Number on every page of the question paper and your answer sheets.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board approved calculators may be used.
- The answers to the eight questions are to be handed in separately clearly marked Question 1, Question 2, etc..
- *The question paper must be handed to the supervisor at the end of the examination.*

STUDENT NUMBER / NAME......................................

QUESTION 1. Use a *separate* Writing Booklet. **Marks**

(a) Find $\int \frac{x+7}{x^2+16}dx$. **2**

(b) Find $\int xe^{3x}dx$. **2**

(c) (i) Find real constants A and B such that **4**

$$\frac{7x-4}{2x^2-3x-2} = \frac{A}{2x+1} + \frac{B}{x-2}.$$

(ii) Hence find $\int \frac{7x-4}{2x^2-3x-2}dx$.

(d) Using an appropriate diagram, or otherwise, evaluate $\int_0^{1\frac{1}{2}} \sqrt{9-x^2}dx$. **3**

(e) (i) Use the substitution $t = \tan\frac{x}{2}$ to show that $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x} = \int_0^1 \frac{dt}{t^2+t+1}$. **4**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\sin x}$.

QUESTION 2. Use a *separate* Writing Booklet. **Marks**

(a) Given $z = \sqrt{6} - \sqrt{2}i$, find: **6**

(i) $\text{Re}(z^2)$;

(ii) $|z|$;

(iii) $\arg z$;

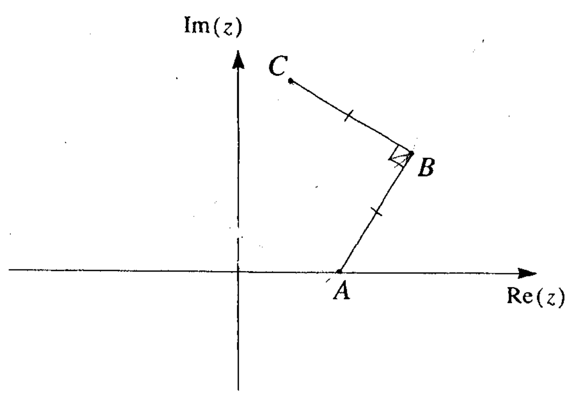
(iv) z^4 in the form $x + iy$.

(b) The equations $|z - 8 - 6i| = 2\sqrt{10}$ and $\arg z = \tan^{-1}2$ both represent loci on the Argand plane. **5**

(i) Write down the Cartesian equations of the loci, and hence show that the points of intersection of the loci are $2 + 4i$ and $6 + 12i$.

(ii) Sketch both loci on the same diagram, showing their points of intersection. (You need not show the intercepts with the axes.)

(c) **4**



The diagram above shows the fixed points A, B and C in the Argand plane, where $AB = BC$, $\angle ABC = \frac{\pi}{2}$, and A, B and C are in anticlockwise order. The point A represents the complex number $z_1 = 2$ and the point B represents the complex number $z_2 = 3 + \sqrt{5}i$.

(i) Find the complex number z_3 represented by the point C.

(ii) D is the point on the Argand plane such that $ABCD$ is a square. Find the complex number z_4 represented by D.

QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) (i) Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$. **8**

(ii) Show that the equation of the normal at P is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$.

(iii) The line through P parallel to the y-axis meets the asymptote $y = \frac{bx}{a}$ at Q. The tangent at P meets the same asymptote at R. The normal at P meets the x-axis at G. Prove that $\angle RQG$ is a right angle.

(iv) What sort of quadrilateral is $RQPG$?

(b) The tangents at two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ intersect at $T(x_0, y_0)$. **7**

(i) Show that the equation of the chord of contact PQ is $\frac{xx_0}{16} + \frac{yy_0}{9} = 1$.

(You may assume that the tangent at P has equation $\frac{xx_1}{16} + \frac{yy_1}{9} = 1$, and similarly for Q.)

(ii) If the chord PQ touches the circle $x^2 + y^2 = 9$, then by considering the distance of the chord from the origin, or otherwise, show that the point $T(x_0, y_0)$ satisfies $\frac{9{x_0}^2}{256} + \frac{{y_0}^2}{9} = 1$.

(iii) Give a geometrical description of the locus of T.

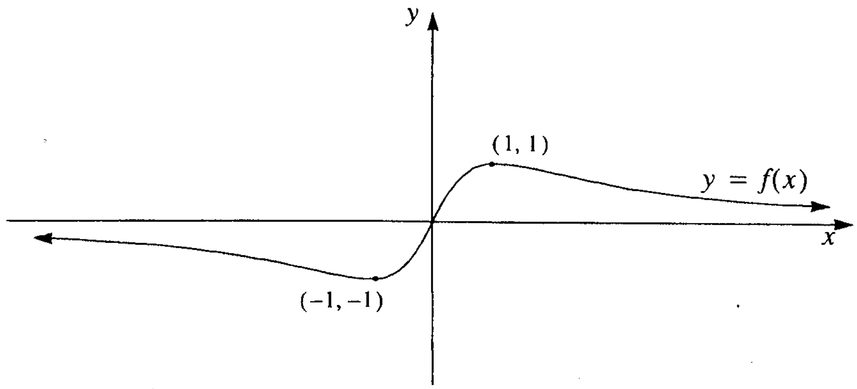
QUESTION 4. Use a *separate* Writing Booklet. **Marks**

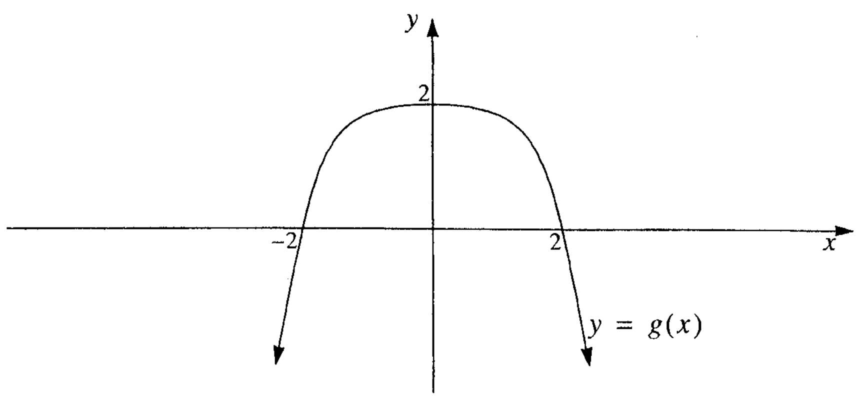
(a) If $P(x)$ is an odd function and $Q(x)$ is an even function, determine whether the following are odd, even or neither: **2**

(i) $P(Q(x))$;

(ii) $Q(P(x))$.

(b) The diagrams represent the curves $f(x) = \dfrac{2x}{x^2+1}$ and $g(x) = 2 - \dfrac{x^4}{8}$. **8**





Use these diagrams to sketch the following functions without calculus, showing all essential features:

(i) $y = |f(x)|$;

(ii) $y = \dfrac{1}{g(x)}$;

(iii) $y = [g(x)]^2$.

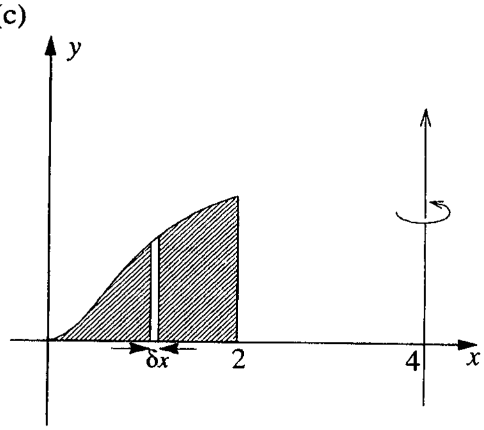
For parts (iv) and (v), you do not need to show the coordinates of any stationary points.

(iv) $y = g(f(x))$;

(v) $y = f(g(x))$.

Marks

(c) The region shown below, bounded by the curve $y = \dfrac{x^2}{x^2 + 1}$, the x-axis and the line $x = 2$, is rotated about the line $x = 4$. **5**



(i) Using the method of cylindrical shells, show that the volume δV of a shell distant x from the origin and with thickness δx is given by

$$\delta V \doteqdot 2\pi(4 - x)\left(1 - \frac{1}{1 + x^2}\right)\delta x$$

(ii) Hence find the volume of the solid.

QUESTION 5. Use a *separate* Writing Booklet. **Marks**

(a) Solve $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$, given that it has a root of multiplicity 3. **3**

(b) Consider the equation **12**

$$p\cos^2 x + 2q\cos x\sin x + r\sin^2 x = s \qquad (1)$$

where $r \neq s$.

(i) Show that $p\cos^2 x + r\sin^2 x = \frac{p-r}{2}\cos 2x + \frac{p+r}{2}$

(ii) Show that equation (1) above can be rewritten as $R\cos(2x - \alpha) = 2s - r - p$, where $R = \sqrt{(p-r)^2 + 4q^2}$ and $\tan\alpha = \frac{2q}{p-r}$.

(iii) Show that the condition for equation (1) above to have real solutions is $q^2 \geq (s-p)(s-r)$.

(iv) Show that if $q^2 > (s-p)(s-r)$, then there are two solutions in the domain $0 \leq x < \pi$.

(v) If θ and ϕ are the solutions found in part (iv), show that $\tan(\theta + \phi) = \frac{2q}{p-r}$.

QUESTION 6. Use a *separate* Writing Booklet.

Marks

(a) A stone of mass 80 grams is attached to a string of length 50 cm. The string is twirled so that the stone moves in a horizontal circle with a speed of 70 revolutions per minute. **4**

(i) Draw a force diagram of the situation.

(ii) By resolving forces vertically and horizontally, find the tension in the string.

(b) An object of mass m kg is thrown vertically upwards. Air resistance is given by $R = 0.05mv^2$ where R is in newtons and v ms^{-1} is the speed of the object. **6**

(Take $g = 9.8$ ms^{-2}.)

(i) Explain why the equation of motion is $\ddot{x} = -\left(\frac{196 + v^2}{20}\right)$, where x is the height of the object t seconds after it is thrown.

(ii) If the velocity of projection is 50 ms^{-1}, find the time taken to reach the highest point.

(c) A particle is describing simple harmonic motion in a straight line. Its centre of motion is not at the origin O, but at $x = x_0$. Its speeds at distances 2, 3 and 5 metres from O are 0, 4 and 2 ms^{-1} respectively. **5**

(i) Given that the particle's equation of motion is $\ddot{x} = -n^2(x - x_0)$, where n is a constant, show that its velocity is given by $v^2 = n^2(a^2 - (x - x_0)^2)$, where a is the amplitude.

(ii) Find the distance of the centre of the motion from the origin.

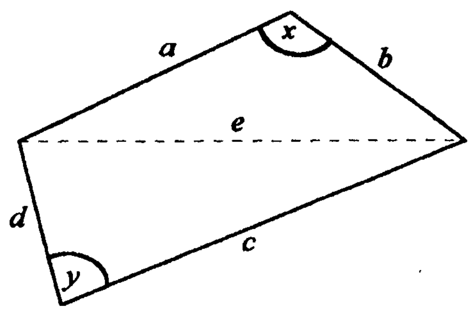
QUESTION 7. Use a *separate* Writing Booklet. **Marks**

(a) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **5**

(ii) Hence show that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)dx = \int_0^{\frac{\pi}{2}} (a\sin^2 x + b\cos^2 x)dx$.

(ii) Deduce that $\int_0^{\frac{\pi}{2}} (a\cos^2 x + b\sin^2 x)dx = \frac{\pi(a+b)}{4}$.

(b) **6**



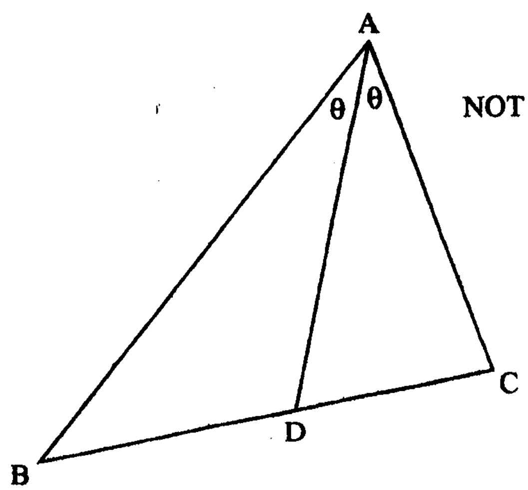
NOT TO SCALE

(i) By using the cosine rule in both triangles show that

$$\frac{dy}{dx} = \frac{ab\sin x}{cd\sin y}$$

(ii) Hence show that the area of the quadrilateral is a maximum when it is cyclic quadrilateral.

(c) **4**



NOT TO SCALE

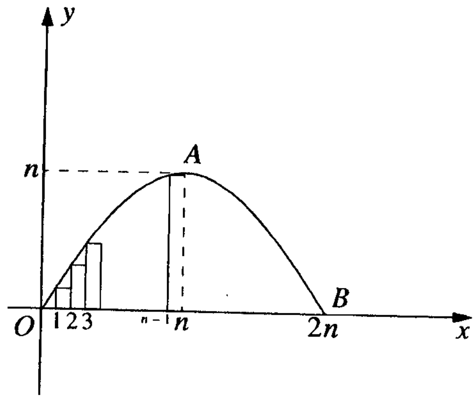
In the triangle ABC it is given that AD is the bisector of angle BAC.

Prove that $\frac{AB}{BD} = \frac{AC}{DC}$.

QUESTION 8. Use a *separate* Writing Booklet.

Marks

(a) 5



The diagram above represents the curve $y = n\sin\frac{\pi x}{2n}$, $0 \le x \le 2n$, where n is any integer $n \ge 2$. The points $O(0, 0)$, $A(n, n)$ and $B(2n, 0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$

(ii) Hence or otherwise, explain why

$$2n\sum_{r=1}^{n-1} \sin\frac{\pi r}{2n} < \frac{\pi n^2}{2}.$$

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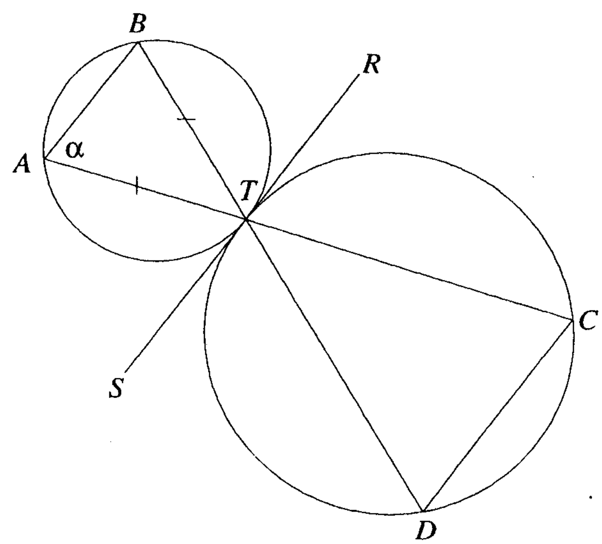
QUESTION 8. (continued)

Marks

(b) Two circles touch externally at a point T. **10**

A and B are points on the first circle such that $AT = BT$, and AT and BT produced meet the second circle at C and D respectively. RS is the common tangent at T. Let $\angle BAT = \alpha$.



(i) Copy the diagram and include the information above.

(ii) Prove that $\angle BAC = \angle ACD$.

(iii) Prove that $ABCD$ is a trapezium with two equal sides.

The line BC cuts the first circle again in V and the second circle again in W, and the line AD cuts the first circle again in U and the second circle again in X.

(iv) Prove that the points U, V, W and X are concyclic.

(v) Prove that $UT = TX$, and hence show that T is the centre of the circle passing through U, V, W and X.