INTERNATIONAL GRAMMAR SCHOOL
Concordia per Diversitatem

Name:.............................

INTERNATIONAL GRAMMAR SCHOOL

MATHEMATICS

Extension 2

YEAR 12

TRIAL EXAMINATION

31st JULY, 2001

Time allowed —3 hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL eight questions.
- ALL questions are of equal value.(8 @ 15 marks = 120 marks)
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Start each question on a *new page*. Number each question clearly.
- Label each page with your name.
- A table of Standard Integrals is attached.

QUESTION 1 (Start a new page) — MARKS

a) Find $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$ — 2

b) i) Find a, b and c such that — 2

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$$

ii) Find $\int \frac{16}{(x^2+4)(2-x)}\,dx$ — 2

c) Find $\int \frac{\ln x}{x^2}\,dx$ — 4

d) Use the substitution $t = \tan\frac{\theta}{2}$ to show that — 5

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

QUESTION 2 (Start a new page) — MARKS

a) The complex number Z moves such that $\text{Im}\left(\frac{1}{\overline{Z}-i}\right) = 1$. — 3

Show that the locus of Z is a circle and find its centre and radius.

b) i) Find the square root of the complex number $5 - 12i$ — 2

ii) Given that $Z = \frac{1+\sqrt{5-12i}}{2+2i}$ and is purely imaginary, — 2

find Z^{400}

c) i) Shade the region in the argand diagram containing all points representing the complex numbers Z such that — 3

$|Z-1-i| \leq 1$ and $-\frac{\pi}{4} \leq \text{Arg}(Z-i) \leq \frac{\pi}{4}$

ii) Let ϕ be the complex number of minimum modulus satisfying the inequalities of i). — 1

Express ϕ in the form $x + yi$

d) Express $\phi = \frac{-1+i}{\sqrt{3}+i}$ in modulus / argument form. — 4

Hence, evaluate $\cos\frac{7\pi}{12}$ in surd form.

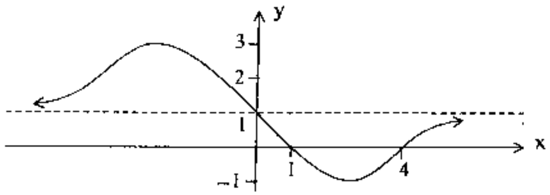
QUESTION 3 *(Start a new page)* **MARKS**

a) Consider the equation $x^3 + 7x - 6i = 0$.

i) Given that this equation has no purely real root, show that none of the roots is a conjugate of any of the others. 1

ii) If $2i$ is one of the roots and the other two roots are purely imaginary, find the other two roots. 2

b)



The above diagram shows the graph of $y = f(x)$.
Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes.

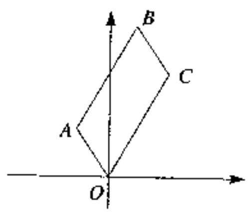
i) $y = \dfrac{1}{f(x)}$ 2

ii) $y = f(|x|)$ 2

iii) $y = \ln f(x)$ 2

iv) $y = \sin^{-1}(f(x))$ 3

c)



In the diagram above, $OABC$ is a parallelogram with $OA = \frac{1}{2}OC$.

The point A represents the complex number $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

If $\angle AOC = 60°$, what complex number does C represent?

QUESTION 4 *(Start a new page)* **MARKS**

a) Factorise $P(x) = x^4 - 5x^3 + 4x^2 + 2x - 8$ over 3
(i) R (all real numbers)
(ii) C (all complex numbers)

b) Write down all polynomials that have degree 4, 3 as a single zero and -1 as a zero of multiplicity 3. 1

c) If $\alpha, \beta, \wp$ are the roots of $x^3 - 2x^2 + x + 3 = 0$ evaluate:

(i) $\alpha^2 + \beta^2 + \wp^2$ (ii) $\alpha^3 + \beta^3 + \wp^3$ 3

d) If $\alpha, \beta, \wp$ are the roots of $x^3 + 2x^2 - 2x + 3 = 0$ form the equation whose roots are:

(i) $2\alpha, 2\beta, 2\wp$ (ii) $\alpha^2, \beta^2, \wp^2$ 4

e) The roots of the polynomial $P(x) = x^3 + ax^2 + bx + c = 0$ are in arithmetic progression. Show that the relationship between the coefficients of $P(x)$ is $2a^3 = 9ab - 27c$ 3

f) Prove that if α is a root of multiplicity r of $P(x)$ then it is a root of multiplicity $(r-1)$ of $P'(x)$. 1

QUESTION 5 *(Start a new page)* **MARKS**

a) i) Show that the equation of the chord of contact of the tangents from a point (x_o, y_o) to the rectangular hyperbola $xy = c^2$ is $xy_o + x_oy = 2c^2$. 5

ii) Hence find the chord of contact of the tangents from the point (2,1) to the hyperbola $xy = 4$ and determine the points of contact.

b) i) Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2 = a^2m^2 + b^2$. 5

ii) Hence show that the pair of tangents from the point (3,4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to one another.

c) i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is $a\sin\theta\, x + by = (a^2+b^2)\tan\theta$. 5

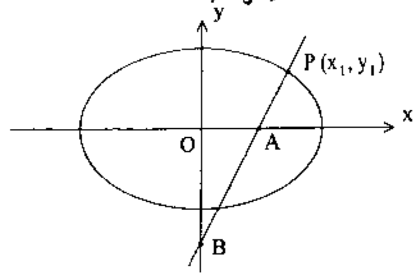
ii) The normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the x-axis at G.

PN is the perpendicular from P to the x-axis.

Prove that $OG = e^2 \cdot ON$, where O is the origin.

QUESTION 6 *(Start a new page)* **MARKS**

a)



The point $P(x_1, y_1)$, where $x_1 > 0$ and $y_1 > 0$, lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The normal at P intersects the x axis at A and the y axis at B.

i) Show that the equation of the normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ 3

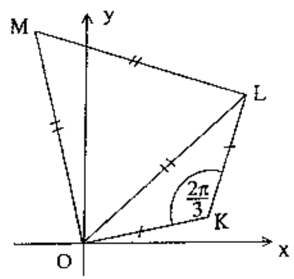
ii) Explain why the point A cannot be the focus of the ellipse. 2

iii) Find the ratio in which A divides the interval BP internally. 2

iv) Find the midpoint M of AB in terms of x_1 and y_1. 1

v) Given that H divides the interval OM in the ratio 4:1, show that the locus of H is an ellipse. 3

b)

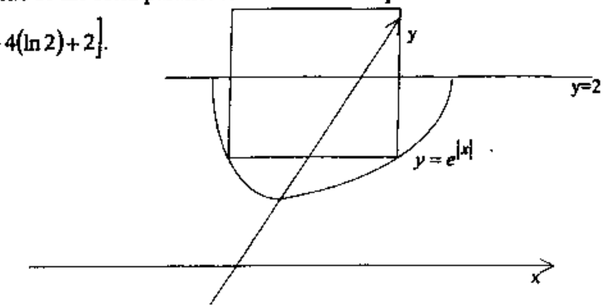


The points K and M in a complex plane represent the complex numbers α and β respectively. The triangle OKL is isosceles and $\angle OKL = \frac{2\pi}{3}$. The triangle OLM is equilateral. 4

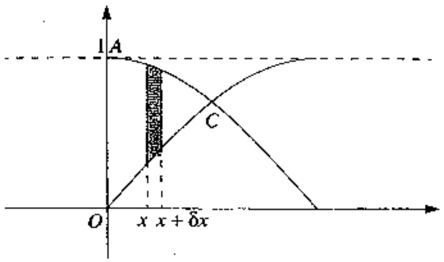
Show that $3\alpha^2 + \beta^2 = 0$

QUESTION 7 (Start a new page) MARKS

a) The base of a solid is formed by the segment cut off by the line $y = 2$ of the curve $y = e^{|x|}$. Cross sections of the solid parallel to the x axis are squares. Show that the volume is given by $4\left[2(\ln 2)^2 - 4(\ln 2) + 2\right]$. **8**



b) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



The region OAC is to be rotated about the line $y = 1$.

(i) Write down the coordinates of the point C. **1**

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by **3**

$$\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x.$$

(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. **3**

QUESTION 8 (Start a new page) MARKS

a) Let $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$, where n is a positive integer.

i) Using integration, show that **4**

$$(n-1)\, I_n = 2^{n-2}\sqrt{3} + (n-2)\, I_{n-2}$$

ii) Evaluate $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$ **3**

b) Consider the polynomial $x^5 - i = 0$

i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$ **2**

ii) Show that **4**

$$(x - i)\left(x^2 - 2i\sin\frac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\frac{3\pi}{10}x - 1\right) = 0$$

iii) Show that $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$ **2**

IGS - Year 12 Trial (4U) Extension 2 Solutions.

Question 1

a) $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

let $u = \tan x$
$du = \sec^2 x\,dx$

$\therefore \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1} u + C$

$= \sin^{-1}(\tan x) + C$

b) (i) $\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$

$16 = (ax+b)(2-x) + c(x^2+4)$

when $x = 2$

$16 = c(8)$

$\therefore c = 2$

equating coefficients of x^2

$-a + c = 0$

$c = a$

$a = 2$

equating coefficients of x

$2a - b = 0$

$b = 2a$

$b = 4$

$\int \frac{16}{(x^2+4)(2-x)}\,dx = \int \left(\frac{2x+4}{x^2+4} + \frac{2}{2-x}\right) dx$

$I = \int \left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x}\right) dx$

$= \ln|x^2+4| + \frac{4}{2}\tan^{-1}\frac{x}{2} - 2\ln|2-x| + c$

$= 2\tan^{-1}\frac{x}{2} + \ln\left(\frac{x^2+4}{(2-x)^2}\right) + C$

c) $I = \int \frac{\ln x}{x^2}\,dx$

let $u = \ln x$ $\quad v' = \frac{1}{x^2}$

$u' = \frac{1}{x}$ $\quad v = -\frac{1}{x}$

$\therefore I = uv - \int v u'$

$= \ln x\left(-\frac{1}{x}\right) + \int \frac{1}{x^2}\,dx$

$= -\frac{\ln x}{x} - \frac{1}{x} + C$

d) $I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6}$

let $t = \tan\frac{\theta}{2}$ $\quad dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

$\frac{2\,dt}{1+t^2} = d\theta$

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$I = \int_0^1 \frac{\frac{2\,dt}{1+t^2}}{4\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) + 6}$

$= \int_0^1 \frac{2\,dt}{8t - 2 + 2t^2 + 6 + 6t^2}$

$= \int_0^1 \frac{2\,dt}{8t^2 + 8t + 4}$

Question 1 (continued)

$I = \int_0^1 \frac{dt}{4t^2+4t+2}$

$= \int_0^1 \frac{dt}{(2t+1)^2+1}$

$= \left[\frac{1}{2}\tan^{-1}(2t+1)\right]_0^1$

$= \frac{1}{2}\left(\tan^{-1}3 - \tan^{-1}1\right)$

let $\tan^{-1}3 = A$, $\tan^{-1}1 = B$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$\frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$

$\tan(A - B) = \frac{1}{2}$

$A - B = \tan^{-1}\left(\frac{1}{2}\right)$

$\therefore I = \frac{1}{2}\tan^{-1}\frac{1}{2}$

Question 2

a) let $z = x + iy$ [illegible]

$\frac{1}{z+i} = $ [illegible]

$= x + i(y+1)$ [illegible]

$\Im\left(\frac{1}{z+i}\right) = 1$

$\therefore \frac{y+1}{x^2+(y+1)^2} = 1$

$y + 1 = x^2 + (y+1)^2$

$x^2 + y^2 + y = 0$

$x^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{4}$

The locus of z is a circle centre $\left(0, -\frac{1}{2}\right)$, radius $\frac{1}{2}$

b) let $\sqrt{5-12i} = x + iy$

$5 - 12i = x^2 - y^2 + 2ixy$

equating real and imaginary parts

$x^2 - y^2 = 5$ ① and $-12 = 2xy$

$-\frac{6}{x} = y$

[illegible]

$x^4 - 5x^2 - 36 = 0$

[illegible] $x^2 = 9$ [illegible]

[illegible]

$\therefore \sqrt{5-12i} = 3-2i$ or $-3+2i$

Question 2 (continued)

(b) (ii) $z = \dfrac{1+\sqrt{5-12i}}{2+2i}$

$z = \dfrac{1+3-2i}{2+2i}$ or $z = \dfrac{1-3+2i}{2+2i}$

$z = \dfrac{4-2i}{2+2i}$ or $z = \dfrac{-2+2i}{2+2i}$

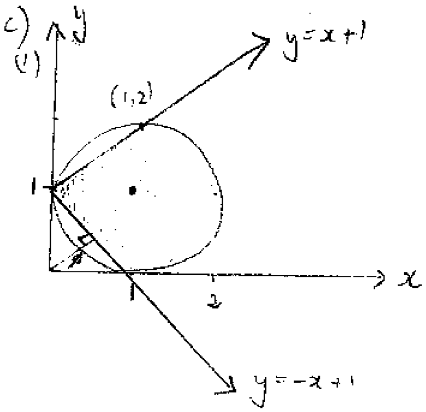
$z = \dfrac{2-i}{1+i} \times \dfrac{1-i}{1-i}$ or $z = \dfrac{-1+i}{1+i} \times \dfrac{1-i}{1-i}$

$z = \dfrac{1-3i}{2}$ or $z = i$

choose $z = i$ (as it is purely imaginary)

$z^{400} = i^{400} = (i^4)^{100} = 1$

c) (i)



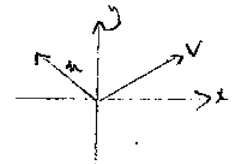
(ii) ϕ is complex number of minimum modulus.

∴ shortest distance to line $y=-x+1$ is the complex number $\phi = \frac{1}{2}+\frac{1}{2}i$

d) $\phi = \dfrac{-1+i}{\sqrt{3}+i} = \dfrac{u}{v}$

let $u = -1+i = \sqrt{2}\operatorname{cis}\frac{3\pi}{4}$

$v = \sqrt{3}+i = 2\operatorname{cis}\frac{\pi}{6}$



$\therefore \phi = \frac{\sqrt{2}}{2}\operatorname{cis}\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)$

$\phi = \frac{1}{\sqrt{2}}\operatorname{cis}\frac{7\pi}{12}$ (in mod-arg form)

If $\phi = \dfrac{-1+i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i}$

$\dfrac{1-\sqrt{3}+i(1+\sqrt{3})}{4}$

equating real parts of ϕ

$\frac{1}{\sqrt{2}}\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{4}$

$\cos\frac{7\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$

Question 3

a) (i) For the equation $x^3+7x-6i=0$

let α, β, γ be the roots

sum of roots $\alpha+\beta+\gamma=0$

if we let $\alpha = x+iy$, $\beta = x-iy$ } are conjugates

$\gamma = a+ib$

$\therefore \alpha+\beta+\gamma = 2x+a+ib=0$

$\Rightarrow b=0$

If none of the roots are purely real, then we can't assume the roots are conjugates

(ii) let $\alpha = 2i$

$\beta = bi$

$\gamma = ci$

then sum of roots

$i(2+b+c)=0$

$\therefore b+c=-2$ … ①

product of roots

$-i(2bc) = 6i$

$bc=-3$ … ②

sub (2) into (1)

$\frac{-3}{c}+c=-2$

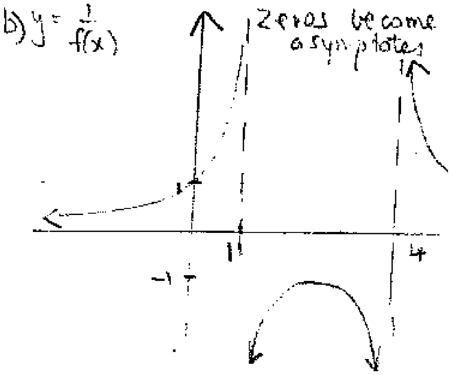
$c^2+2c-3=0$

$(c-1)(c+3)=0$

$c=1$ or $c=-3$

∴ roots are $x=2i, x=-3i, x=i$

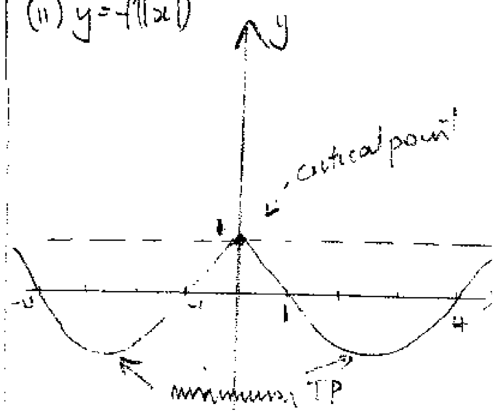
b) $y = \frac{1}{f(x)}$



max becomes min

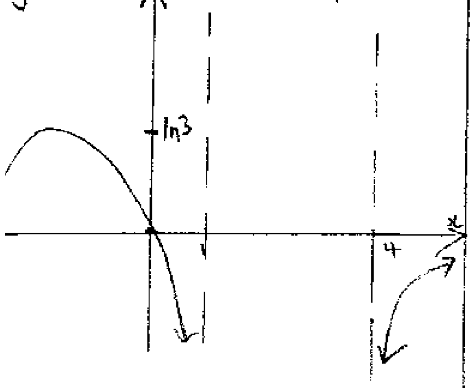
increasing becomes decreasing

(ii) $y = f(|x|)$

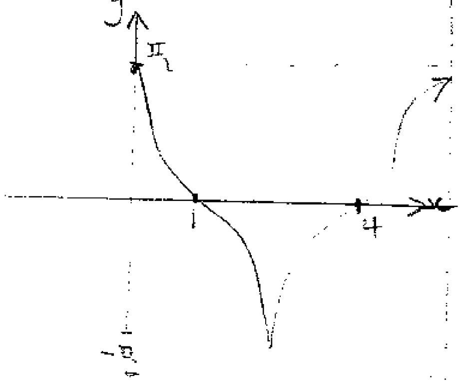


Positive x values are reflected about the y axis

$y = \ln f(x)$ y is defined only for $f(x) > 0$



) $y = \sin^{-1}(f(x))$ is only defined for $-1 \le y \le 1$



c) $|OA| = \frac{1}{2}|OC|$

$|OC| = 2\left(\frac{1}{4}+\frac{3}{4}\right) = 2$

$\arg(OA) = \tan^{-1}\frac{\sqrt{3}}{-1} = 120°$

$\therefore \angle COA = 60°$

$\therefore C = 2(\cos 60° + i\sin 60°)$

$= 2 \times \frac{1}{2} + i\, 2 \times \frac{\sqrt{3}}{2}$

$\underline{C = 1 + \sqrt{3}i}$

Question 4

a) $P(x) = x^4 - 5x^3 + 4x^2 + 2x - 8$

$P(-1) = 1 + 5 + 4 - 2 - 8 = 0$

$\therefore (x+1)$ is a factor

$$\begin{array}{r} x^3 - 6x^2 + 10x - 8 \\ x+1 \overline{) x^4 - 5x^3 + 4x^2 + 2x - 8} \\ x^4 + x^3 \\ -6x^3 + 4x^2 \\ -6x^3 - 6x^2 \\ 10x^2 + 2x \\ 10x^2 + 10x \\ -8x - 8 \\ -8x - 8 \end{array}$$

$Q(x) = x^3 - 6x^2 + 10x - 8$

$Q(4) = 64 - 96 + 40 - 8 = 0$

$\therefore (x-4)$ is a factor

$$\begin{array}{r} x^2 - 2x + 2 \\ x-4 \overline{) x^3 - 6x^2 + 10x - 8} \\ x^3 - 4x^2 \\ -2x^2 + 10x \\ -2x^2 + 8x \\ 2x - 8 \\ 2x - 8 \end{array}$$

$R(x) = x^2 - 2x + 2$

$\therefore$ over $\mathbb{R}$

$P(x) = (x-4)(x+1)(x^2 - 2x + 2)$

over $\mathbb{C}$ $x = \frac{2 \pm \sqrt{-4}}{2}$

$= 1 \pm i$

$P(x) = (x-4)(x+1)(x-1-i)(x-1+i)$

over $\mathbb{C}$

b) $P(x) = k(x+1)^3(x-3)$

c) $\alpha + \beta + \gamma = 2$

$\alpha\beta + \beta\gamma + \alpha\gamma = 1$

$\alpha\beta\gamma = 3$

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 2^2 - 2(1)$

$= 2$

(ii) as α, β, γ are the roots of $x^3 - 2x^2 + x + 3 = 0$

then $\alpha^3 = 2\alpha^2 + \alpha + 3$

$\beta^3 = 2\beta^2 + \beta + 3$

$\gamma^3 = 2\gamma^2 + \gamma + 3$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) + (\alpha+\beta+\gamma)$

$= 2(2) + 2 + 9$

$= 15$

d) $x^3 + 2x^2 + 2x + 3 = 0$

$\left(\frac{y}{2}\right)^3 + 2\left(\frac{y}{2}\right)^2 + 2\frac{y}{2} + 3 = 0$

$\frac{y^3}{8} + \frac{y^2}{2} + \frac{y}{1} + 3 = 0$

$y^3 + 4y^2 + 8y + 24 = 0$

(iii) $x^3+2x^2+2x+3=0$

$(\sqrt{y})^3+2(\sqrt{y})^2+2\sqrt{y}+3=0$

$y\sqrt{y}+2\sqrt{y}=-2y-3$

$\sqrt{y}(y+2)=-(2y+3)$

$y(y^2+4y+4)=4y^2+12y+9$

$y^3+4y^2+4y=4y^2+12y+9$

$y^3-8y^2+8y-9=0$

e) $P(x)=x^3+ax^2+bx+c=0$

let roots be

$\alpha-d,\ \alpha,\ \alpha+d$

$\therefore$ sum, $\alpha-d+\alpha+\alpha+d=-a$

$3\alpha=-a$

products in pairs

$\alpha(\alpha-d)+\alpha(\alpha+d)+\alpha^2-d^2=b$

$3\alpha^2-d^2=b$

$3\left(\frac{a^2}{9}\right)-d^2=b$

$a^2-3d^2=3b$

$d^2=\frac{a^2-3b}{3}$

product

$\alpha(\alpha^2-d^2)=-c$

$-\frac{a}{3}\left(\frac{a^2}{9}-\frac{3a^2-9b}{9}\right)=-c$

$-a^3+3a^3-9ab=-27c$

$2a^3-9ab=-27c$

f) $P(x)=(x-\alpha)^r Q(x)$

$P'(x)=r(x-\alpha)^{r-1}Q(x)+Q'(x)(x-\alpha)^r$

$=(x-\alpha)^{r-1}\left(rQ(x)+Q'(x)(x-\alpha)\right)$

$P'(x)=(x-\alpha)^{r-1}S(x)$

$\therefore$ α is a root of multiplicity

$(r-1)$ of $P'(x)$

Question 5

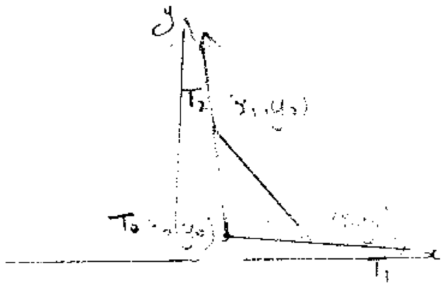
a) i) $y=\frac{c^2}{x}$

$\frac{dy}{dx}=-\frac{c^2}{x^2}=-\frac{y}{x}$

at a point (x_1,y_1)

$\frac{dy}{dx}=-\frac{y_1}{x_1}$

at a point (x_2,y_2) $\frac{dy}{dx}=-\frac{y_2}{x_2}$



Eqn of T_1: $(y-y_1)=-\frac{y_1}{x_1}(x-x_1)$

Eqn of T_2: $(y-y_2)=-\frac{y_2}{x_2}(x-x_2)$

T_1: $x_1y+y_1x=2y_1x_1=2c^2$

T_2: $x_2y+y_2x=2c^2$

Question 5 (continued)

(i) The tangents T_1 and T_2 intersect at T_0 (x_0,y_0)

$\therefore x_1y_0+y_1x_0=2c^2$

$\therefore x_2y_0+y_2x_0=2c^2$

and hence (x_1,y_1) and (x_2,y_2) satisfy $\underline{x_0y+y_0x=2c^2}$

(ii) The chord of contact to the hyperbola $xy=4$ at $(2,1)$ has equation

$2y+x=8$ (from above)

Thus chord of contact intersects $y=\frac{4}{x}$ when

$2\left(\frac{4}{x}\right)+x=8$

$8+x^2=8x$

$x^2-8x+8=0$

$x=\frac{8\pm\sqrt{64-32}}{2}$

$=\frac{8\pm\sqrt{32}}{2}$

$x=4\pm2\sqrt{2}$

when $x=4+2\sqrt{2}$ $y=\frac{4}{4+2\sqrt{2}}$

$x=4-2\sqrt{2}$ $y=\frac{4}{4-2\sqrt{2}}$

$\therefore$ pts of contact are:

$\left(4+2\sqrt{2},\ \frac{2}{2+\sqrt{2}}\right)\left(4-2\sqrt{2},\ \frac{2}{2-\sqrt{2}}\right)$

rationalising denominator

$\frac{2}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}}=\frac{4-2\sqrt{2}}{2}=2-\sqrt{2}$

$\left(4+2\sqrt{2},\ 2-\sqrt{2}\right)\left(4-2\sqrt{2},\ 2+\sqrt{2}\right)$

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) If $y=mx+c$ is a tangent to the ellipse then on substitution there should only be one solution i.e $\Delta=0$

$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$b^2x^2+a^2(m^2x^2+2mcx+c^2)=a^2b^2$

$x^2(b^2+a^2m^2)+x(2a^2mc)+a^2(c^2-b^2)$

if $\Delta=0$, then

$(2a^2mc)^2-4(b^2+a^2m^2)a^2(c^2-b^2)=0$

$4a^4m^2c^2-4a^2b^2c^2-4a^4m^2c^2+4a^2b^4+4a^4m^2b^2$

$4a^2b^2c^2=4a^2b^4+4a^4m^2b^2$

$(\div 4a^2b^2)$ $c^2=b^2+a^2m^2$

$\underline{c^2=a^2m^2+b^2}$ geo

$b^2 = a^2(e^2-1)$ $\quad \frac{b^2}{a^2} - a^2 = e^2$

Question 5 (continued)

(c) (i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

has gradient $\frac{dy}{dx} = \frac{x b^2}{y a^2}$

at $(a\sec\theta, b\tan\theta)$

$\frac{dy}{dx} = \frac{a\sec\theta\, b^2}{b\tan\theta\, a^2}$

$= \frac{b\sec\theta}{a\tan\theta}$

$= \frac{b}{a}\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}$

$\frac{dy}{dx} = \frac{b}{a\sin\theta}$

$\therefore$ gradient of normal is $-\frac{a\sin\theta}{b}$

Equ^n of normal is

$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$

$by - b^2\tan\theta = -a\sin\theta\, x + a^2\tan\theta$

$a\sin\theta\, x + by = (a^2+b^2)\tan\theta$

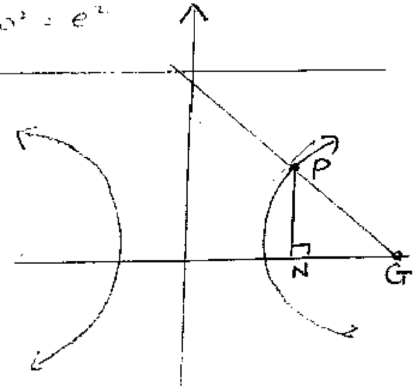
(ii) at G, $y=0$

$\therefore x = \frac{(a^2+b^2)\tan\theta}{a\sin\theta}$

$x = \frac{a^2+b^2}{a\cos\theta}$ $\quad G = \left(\frac{a^2+b^2}{a\cos\theta}, 0\right)$

at N, $y=0$

$N = (a\sec\theta, 0)$



$OG = \frac{a^2+b^2}{a\cos\theta}$

$ON = a\sec\theta = \frac{a}{\cos\theta}$

$OG = \frac{a^2+b^2}{a} \cdot \frac{ON}{a}$

$OG = \frac{a^2+b^2}{a^2} \cdot ON$

where $b^2 = a^2(e^2-1)$

$\therefore \frac{a^2+b^2}{a^2} = \frac{a^2 + a^2(e^2-1)}{a^2}$

$= \frac{a^2}{a^2}(1+e^2-1)$

$= e^2$

$\therefore OG = e^2 \cdot ON$

Question 5

a) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

at (x_1, y_1)

$\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$

grad of normal is $\frac{a^2 y_1}{b^2 x_1}$

equation of normal is

$(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

$b^2 x_1 y - b^2 y_1 x_1 = a^2 y_1 x - a^2 y_1 x_1$

$(a^2 - b^2)x_1 y_1 = a^2 x y_1 - b^2 y x_1$

$a^2 - b^2 = a^2\frac{x}{x_1} - b^2\frac{y}{y_1}$

$\therefore \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

is the equation of the normal.

At A, on the x-axis $y=0$.

$\therefore \frac{a^2 x}{x_1} = a^2 - b^2 = a^2 e^2$

$x = \frac{a^2 - b^2}{a^2} x_1 = e^2 x_1$

If A is the focus then $x = ae$

$\therefore e^2 x_1 = ae$ $\quad x_1 = \frac{a}{e}$

$x_1 = \frac{a}{e}$ is the equation of the directrix, which does not lie on the ellipse

$\therefore$ A is not the focus

(ii) at B, $x = 0$.

$\therefore y = -\frac{a^2 - b^2}{b^2} y_1 = -\frac{a^2 e^2 y_1}{b^2}$

$\therefore B\left(0, -\frac{a^2 e^2 y_1}{b^2}\right)$

and $A(x_1 e^2, 0)$

now A divides BP in the ratio $m:n$ where

$x_1 e^2 = \frac{m x_1 + n \cdot 0}{m+n}$ ①

not required $\left(0 = \frac{m y_1 + n\left(-\frac{a^2 e^2 y_1}{b^2}\right)}{m+n} \right)$ ②

from ① $e^2(m+n) = m$

$e^2 n = m(1-e^2)$

$\frac{e^2}{1-e^2} = \frac{m}{n}$

$\therefore m : n = e^2 : 1-e^2$

(iii) $A(x_1 e^2, 0)$ $\quad B\left(0, -\frac{a^2 e^2 y_1}{b^2}\right)$

$\therefore M\left(\frac{x_1 e^2}{2}, -\frac{a^2 e^2 y_1}{2b^2}\right)$

Question 6 (continued)

(c) H divides OM in the ratio 4 : 1

$$= \left(\frac{4\left(\frac{x_1 e^2}{2}\right) + 1(0)}{5}, \frac{4\left(\frac{-a^2e^2y_1}{2b^2}\right)+1(0)}{5} \right)$$

$$= \left(\frac{2}{5}e^2x_1, \frac{-2}{5}\frac{a^2e^2}{b^2}y_1 \right)$$

$$= (X, Y)$$

to the a locus we find a relationship between X and Y

$$\frac{5X}{2e^2} = x_1 \qquad \frac{5Yb^2}{-2a^2e^2} = y_1$$

as (x_1, y_1) lies on the ellipse then $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

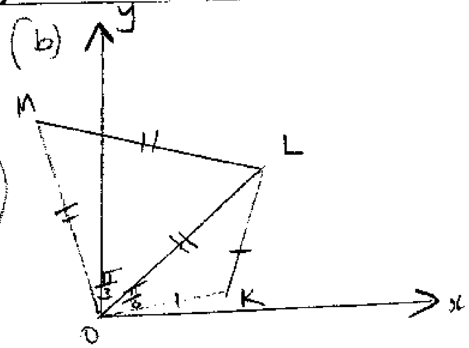
$$\frac{25X^2}{4a^2e^4} + \frac{25Y^2b^2}{4a^4e^4} = 1$$

$\therefore b^2 = a^2(1-e^2)$ and $a^2e^2 = a^2 - b^2$

$$\frac{X^2}{\frac{4}{25}\frac{(a^2-b^2)^2}{a^2}} + \frac{Y^2}{\frac{4}{25}\frac{(a^2-b^2)^2}{b^2}}$$

This is the locus of an ellipse

(b)



given $\angle OKL = \frac{2\pi}{3}$

$\therefore \angle OLK = \angle LOK = \frac{\pi - \frac{2\pi}{3}}{2} = \frac{\pi}{6}$

$\angle LOM = \frac{\pi}{3}$ ($\triangle OML$ is equilateral)

now $\angle KOM$ is $\frac{\pi}{2}$

$\therefore$ OK to OM is a rotation through $\frac{\pi}{2}$ and an enlargement by k.

To find k.

$|\alpha| = |OK| = |LK|$

and $|OL| = \sqrt{3}|\alpha|$

$\therefore k = \sqrt{3}$

let α is rotated through $\frac{\pi}{2}$ means a multiplication by i

$\therefore \beta = \alpha i\sqrt{3}$

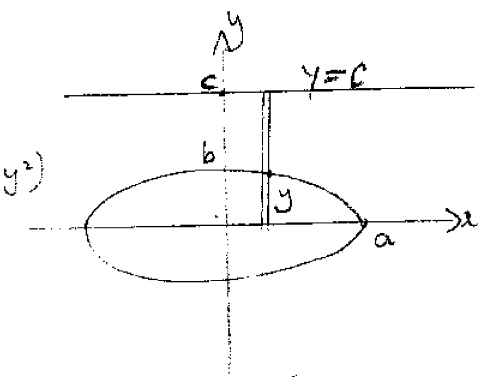
square both sides

$\beta^2 = -3\alpha^2$

$3\alpha^2 + \beta^2 = 0$ qed

Question 7

a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = \frac{a^2b^2 - b^2x^2}{a^2} = b^2 - \frac{b^2}{a^2}x^2$



consider slices ⊥ to x-axis

$A = \pi((c+y)^2 - (c-y)^2)$

$= \pi(c^2 + 2cy + y^2 - c^2 + 2cy - y^2)$

$= 4\pi cy$

$\therefore \delta V = 4\pi cy\,\delta x$

$V = 2\int_0^a \delta V.$

$= 8\pi\int_0^a cy \cdot dx$

$= 8\pi c\int_0^a \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}} \cdot dx$

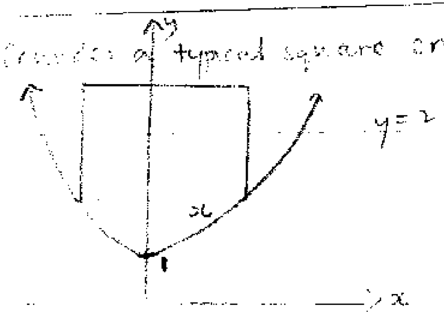
$= \frac{8\pi cb}{a}\underbrace{\int_0^a \sqrt{a^2 - x^2}\,dx}_{\frac{1}{4}\text{ circle}}$

$= \frac{8\pi cb}{a}\frac{1}{4}\pi a^2$

$= 2\pi^2 abc$ units^3

Deleted Q

b) Consider a typical square cross-section Area $= (2|x|)^2$



$A = 4x^2$

$\delta V = 4x^2\,\delta y$

$V = \int_1^2 4x^2\,dy$

$y = e^{|x|}$

$\log_e y = |x|$

$(\ln y)^2 = x^2$

Question 7

b) $V = 4\int_1^2 (\ln y)^2 \cdot dy$

let $u' = 1 \quad v = (\ln y)^2$

$u = y \quad v' = 2(\ln y)\frac{1}{y}$

$V = 4\left[y(\ln y)^2 \Big|_1^2 - \int_1^2 \frac{2}{y} \ln y \cdot y \cdot dy \right]$

$= 8(\ln 2)^2 - 8\int_1^2 \ln y \cdot dy$

let $u = \ln y \quad v' = 1$

$u' = \frac{1}{y} \quad v = y$

$V = 8(\ln 2)^2 - 8\left[[y \ln y]_1^2 - \int_1^2 y \cdot \frac{1}{y} \cdot dy \right]$

$= 8(\ln 2)^2 - 16(\ln 2) + 8[y]_1^2$

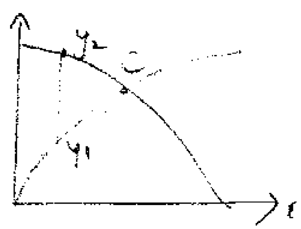
$= 4(2(\ln 2)^2 - 2(\ln 2) + 2)$ qed

) (i) $y = \cos x$ and $y = \sin x$

$\cos x = \sin x$

$\tan x = 1$

$x = \frac{\pi}{4}$ Hence $C \equiv \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$



(ii) $\partial V = \pi\left((1 - y_1)^2 - (1 - y_2)^2\right) \partial x$

$= \pi\left((1 - \sin x)^2 - (1 - \cos x)^2\right) \partial x$

$= \pi(1 - 2\sin x + \sin^2 x - 1 + 2\cos x - \cos^2 x) \partial x$

$\partial V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x) . \partial x$ qed

(iii) $V = \lim_{\partial x \to 0} \sum_{x=0}^{\frac{\pi}{4}} \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x) . \partial x$

$= \pi \int_0^{\frac{\pi}{4}} (2\cos x - 2\sin x + \sin^2 x - \cos^2 x) dx$

$= \pi \int_0^{\frac{\pi}{4}} (2\cos x - 2\sin x - \cos 2x) dx$

$= \pi \left(2\sin x + 2\cos x - \frac{\sin 2x}{2}\right)_0^{\frac{\pi}{4}}$

$= \pi\left(\left(2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}} - \frac{1}{2}\right) - (0 + 2 - 0)\right)$

$= \pi\left(2\sqrt{2} - \frac{5}{2}\right)$

$= \frac{\pi}{2}(4\sqrt{2} - 5)$ cub units.

Question 8:

a) $I_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x \operatorname{cosec}^{n-2} x \, dx$

let $u' = \operatorname{cosec}^2 x$

$u = -\cot x$

$v = \operatorname{cosec}^{n-2} x$

$v' = (n-2) \operatorname{cosec}^{n-3} x \, (-\operatorname{cosec} x \cot x)$

$v' = -(n-2) \operatorname{cosec}^{n-2} x \cot x$

$I_n = -\cot x \operatorname{cosec}^{n-2} x \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (n-2) \operatorname{cosec}^{n-2} x \cot^2 x \, dx$

but $\cot^2 x + 1 = \operatorname{cosec}^2 x$

and $\cot \frac{\pi}{2} = 0$

$\cot \frac{\pi}{6} = \sqrt{3}$

$\operatorname{cosec} \frac{\pi}{6} = 2$

$I_n = \sqrt{3} (2)^{n-2} - (n-2) \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$

$I_n = \sqrt{3} (2)^{n-2} - (n-2) \underbrace{\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^n x \, dx}_{I_n} + (n-2) \underbrace{\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^{n-2} x \, dx}_{I_{n-2}}$

$I_n (1 + n - 2) = \sqrt{3} (2)^{n-2} + (n-2) I_{n-2}$

$(n-1) I_n = \sqrt{3} (2)^{n-2} + (n-2) I_{n-2}$ qed

Question 8 (continued)

(ii) $J = \int_0^{\frac{\pi}{3}} \sec^4 x \, dx$

using $\sec x = \operatorname{cosec}(\frac{\pi}{2} - x)$ let $u = \frac{\pi}{2} - x$, $du = -dx$

$J = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^4 u \, du = I_4$

$3 I_4 = 2^2 \sqrt{3} + 2 I_2$

$I_2 = \sqrt{3}$

$\therefore 3 I_4 = 4\sqrt{3} + 2\sqrt{3}$

$I_4 = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

$\therefore J = 2\sqrt{3}$

b) $x^5 - i = 0$

(i) $(x - i)(x^4 + i x^3 + i^2 x^2 + i^3 x + i^4) = 0$

$(x - i)(x^4 + i x^3 - x^2 - i x + 1) = 0$

but since $x \neq i$ then

$x^4 + i x^3 - x^2 - i x + 1 = 0$

(ii) $x^5 = i$ let $x = r \operatorname{cis} \theta$

$r^5 \operatorname{cis} 5\theta = i$ (by de Moivre's theorem)

$r^5 \operatorname{cis} 5\theta = \operatorname{cis} \frac{\pi}{2}$ $\Rightarrow r = 1$

$\operatorname{cis} 5\theta = \operatorname{cis} \frac{\pi}{2}$

$5\theta = \frac{\pi}{2} + 2k\pi$, $\theta = \frac{\pi}{10} + \frac{2k\pi}{5}$

Question 3. (continued)

$\theta = \frac{\pi}{10} + \frac{2k\pi}{5} \qquad k = 0, 1, 2, 3, 4$

when $k = 0$, $\quad \theta = \frac{\pi}{10} \quad = \frac{\pi}{10}$

$k = 1$, $\quad \theta = \frac{\pi}{10} + \frac{2\pi}{5} \quad = \frac{\pi}{2}$

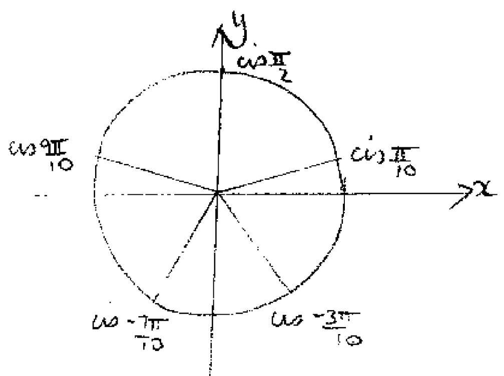
$k = 2$, $\quad \theta = \frac{\pi}{10} + \frac{4\pi}{5} \quad = \frac{9\pi}{10}$

$k = 3$, $\quad \theta = \frac{\pi}{10} + \frac{6\pi}{5} \quad = \frac{13\pi}{10} = -\frac{3\pi}{10}$

$k = 4$, $\quad \theta = \frac{\pi}{10} + \frac{8\pi}{5} \quad = \frac{17\pi}{10} = -\frac{7\pi}{10}$

$\therefore x^5 - i = 0$ can be expressed a

$(x - \text{cis}\frac{\pi}{10})(x - \text{cis}\frac{\pi}{2})(x - \text{cis}\frac{9\pi}{10})(x - \text{cis}\frac{3\pi}{10})(x - \text{cis}\frac{7\pi}{10}) = 0$



from the diagram let $\alpha = \text{cis}\frac{\pi}{10}$

$-\bar{\alpha} = \text{cis}\frac{9\pi}{10}$

let $\beta = \text{cis}-\frac{3\pi}{10}$

$-\bar{\beta} = \text{cis}-\frac{7\pi}{10}$

$\therefore$ pairing

$(x - i)(x - \alpha)(x + \bar{\alpha})(x - \beta)(x + \bar{\beta}) = 0$

$(x - i)(x^2 + (\bar{\alpha} - \alpha)x - \alpha\bar{\alpha})(x^2 + (\bar{\beta} - \beta)x - \beta\bar{\beta})$

but $\alpha\bar{\alpha} = 1 \quad \beta\bar{\beta} = 1$

$\bar{\alpha} - \alpha = \text{cis}\frac{9\pi}{10} - \text{cis}\frac{\pi}{10}$

$\bar{\alpha} - \alpha = \cos\frac{\pi}{10} - i\sin\frac{\pi}{10} - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}$

$\bar{\alpha} - \alpha = -2i\sin\frac{\pi}{10}$

similarly

$\bar{\beta} - \beta = -2i\sin\left(-\frac{3\pi}{10}\right) = -2i\sin\frac{3\pi}{10}$

Altogether,

$(x - i)(x^2 - 2i\sin\frac{\pi}{10}x - 1)(x^2 + 2i\sin\frac{3\pi}{10}x - 1)$

To show

(iii) $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$

we equate coefficients of x^2

Given,

$(x^2 - 2i\sin\frac{\pi}{10}x - 1)(x^2 - 2i\sin\frac{3\pi}{10}x - 1)$

$= x^4 + ix^3 - x^2 - ix + 1$

$-1 + (-2i\sin\frac{\pi}{10})(2i\sin\frac{3\pi}{10}) + -1 = -1$

$4i^2\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = 1$

$\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$