International Grammar School

2008
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE answer booklet **Marks**

a) Find $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$ **2**

b) Find $\displaystyle\int 5\cos x \sin^2 x \, dx$ **2**

c) Evaluate $\displaystyle\int_1^e x \ln x \, dx$ **3**

d) Evaluate $\displaystyle\int_2^3 \frac{dx}{x^2-1}$ **4**

e) Using the substitution $t = tan\dfrac{\theta}{2}$ or otherwise find **4**

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Let $A = 3 + 4i$ and $B = 2 - 2i$. Find in the form $x + iy$ (x and y real).

i) $\frac{A}{B}$ **2**

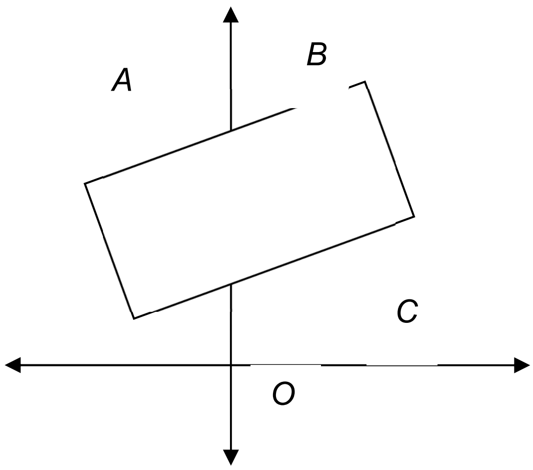
ii) $\sqrt{A}$ **3**

iii) $A - \overline{B}$ **1**

b) i) Write $1 + \sqrt{3}i$ in the form $r(\cos\theta + i\sin\theta)$ **2**

ii) Hence write $\left(1 + \sqrt{3}i\right)^6$ showing that it is real. **2**

c)



The points $OABC$ are the vertices of a rectangle on the Argand diagram with $|OA| = 2|OC|$. If OC represents the complex number $p + iq$, write down the complex numbers represented by:

i) OA **1**

ii) OB **1**

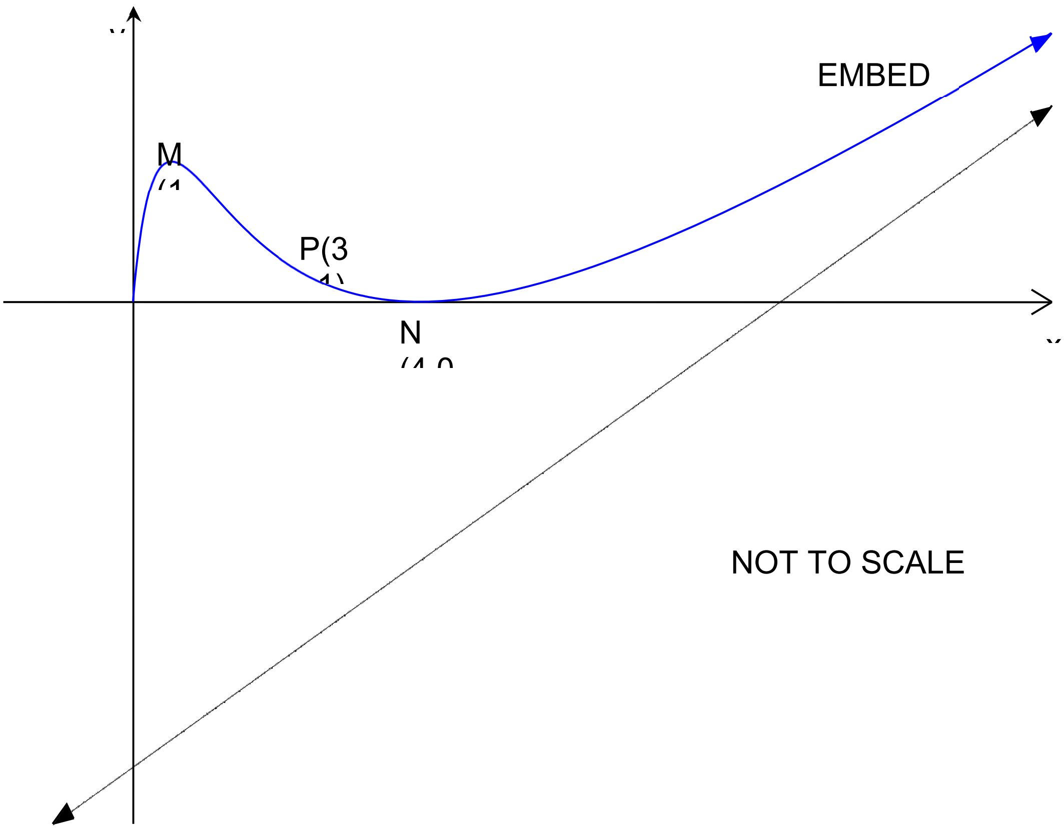
iii) BC **1**

iv) AC **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

a)



The diagram shows the graph of $y = f(x)$ for $x \geq 0$.
M(1, 3) and N(4, 0) are stationary points of $y = f(x)$ and P(3, 1) is a point of inflexion of $y = f(x)$. The line $y = x - 9$ is an asymptote as $x \to \infty$. Draw separate one third page sketches showing any special features for the following:

i) $f'(x)$ **2**

ii) $\dfrac{1}{f(x)}$ **2**

iii) $-\left(f(x)\right)^2$ **2**

Question 3 continues on the next page

b) Determine the gradient of the tangent to the curve $x^2 + 2xy - y^2 = 17$ at the point $(3,\ 2)$ **2**

c) The zeros of $x^3 - 3x^2 - 2x + 4$ are α, β and γ

i) Find a cubic polynomial whose zeros are α^2, β^2 and γ^2 **2**

ii) Hence or otherwise find the value of $\alpha^2 + \beta^2 + \gamma^2$ **1**

iii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$ **2**

d) The equation $P(x) = x^3 + 3x^2 - 24x + k = 0$ has a double root. Find the possible values of *k*. **2**

End of Question 3

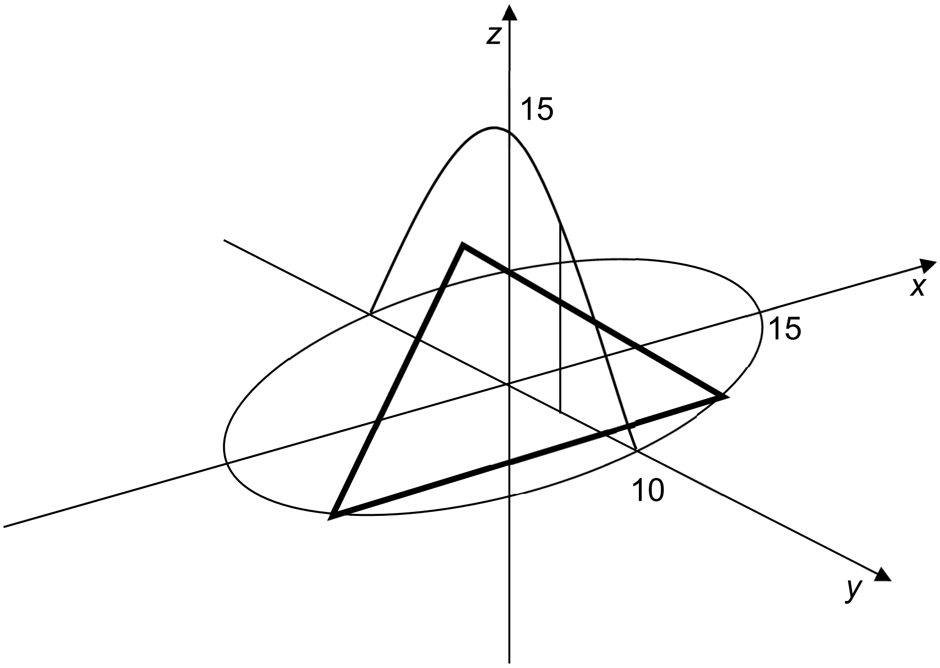
Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) i) Show that a reduction formula for $I_n = \int (ln\,x)^n dx$ is $I_n = x(ln\,x)^n - nI_{n-1}$ **3**

ii) Hence evaluate $\int_1^{e^4} (ln\,x)^3\,dx$ **4**

b) The arc of the curve $y = 6x - x^2 - 8$ where $y \geq 0$ is rotated about the line $x = 1$. By applying the technique of cylindrical shells determine the exact volume of the solid formed **4**

c)



A cake is made with base in the shape of an ellipse, with semi-major axis 15 cm and semi-minor axis 10 cm. Slices of the cake parallel to the major axis of the base are isosceles triangles, whose vertices trace out a semi-elliptical path with the same semi-major axis and semi-minor axis lengths as in the diagram below.

i) Show that the volume of a 'typical' triangular slice is given by:

$$V_{slice} \approx xz\delta y\ cm^3$$ **1**

ii) Find the exact volume of the cake in cm^3. **3**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The line $y = mx + a$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points which have x coordinates x_1 and x_2. **3**

i) Express x_2 in terms of *m, a, b* and x_1. **1**

ii) Hence or otherwise show that the line is a tangent to the ellipse at the point where $\frac{-a^3 m}{b^2 + a^2 m^2}$.

b) A parabola has parametric equations $x = 2at$ and $y = at^2$.

i) Find the equation of the normal to the parabola at the point where $t = p$. **2**

ii) Hence show that, through the point (x_1, y_1), it is possible to draw up to three normals to the parabola. **2**

c) Given the complex number $z = cos\,\theta + i\,sin\,\theta$

i) Use De Moivres Theorem and the binomial expansion find an expression for $cos\,4\theta$ in terms of $cos\,\theta$ **3**

ii) Also, using $z^n + \frac{1}{z^n} = 2\,cos\,n\theta$ determine an expansion for $cos^4\,\theta$ in terms of $cos\,n\theta$ **2**

iii) Hence evaluate $\int_0^{\frac{\pi}{2}} cos^4\theta\; d\theta$ **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) On a suitably labelled Argand diagram sketch the region determined by **2**

$$[\text{Re}(z)]^2 + \text{Im}\,z < 0$$

(b) Consider the function

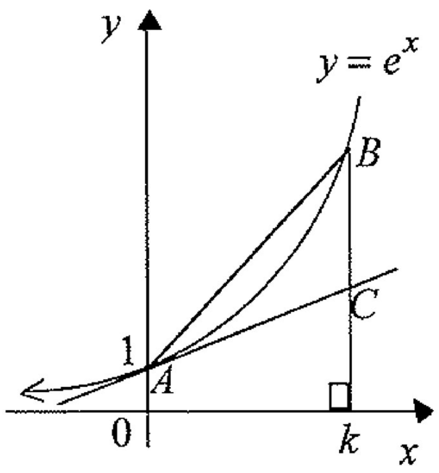
$$f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(i) Use differentiation to show that $e^{-x} + x - 1 \geq 0$ for all values of x. Hence show that $f(x)$ is an increasing function for $x \neq 0$ **3**

(ii) Show that $f(x)$ is continuous at $x = 0$. **2**

(iii) Sketch the graph of $y = f(x)$. **1**

(c)



The curve $y = e^x$ cuts the y axis at A. B is a second point on the curve such that $x = k$, where $k > 0$. The tangent to the curve $y = e^x$ at A cuts the vertical line $x = k$ at the point C.

(i) By considering areas, show that $\frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1+e^k)$. Hence deduce that $2.5 < e < 3$. **4**

(ii) Show that the curve $y = e^x$ bisects the area of ΔABC for some value of k such that $2 < k < 3$. Taking $k = 2.7$ as a first approximation, apply Newton's method once to obtain a second approximation. Give your answer to one decimal place. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Given that $z^5 - 1 = 0$

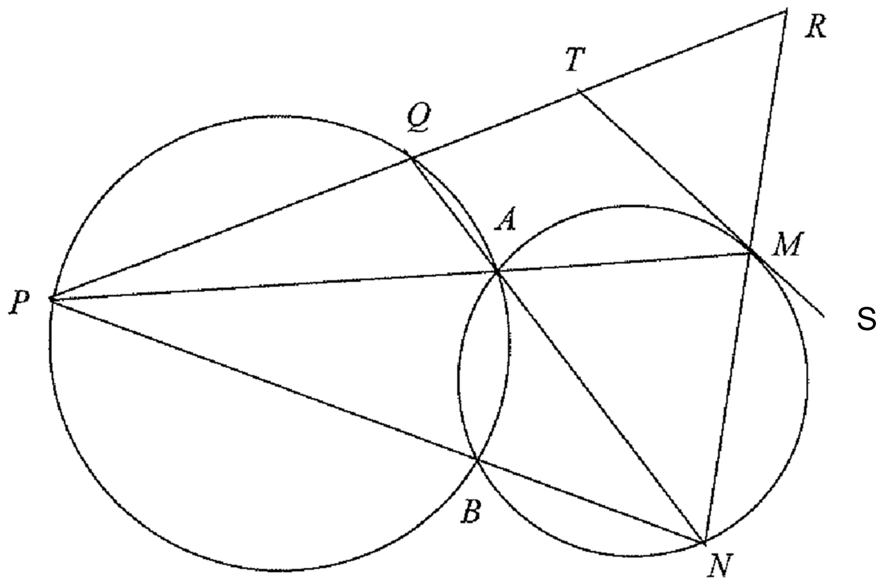
i) Solve for Z over the complex field in the form $cos\,\theta + i\,sin\,\theta$. **3**

i) Hence express $z^5 - 1$ as the product of linear and quadratic factors. **2**

iii) Write down the complex roots of $z^4 + z^3 + z^2 + z + 1 = 0$. **1**

iv) Without evaluating, show that $cos\frac{2\pi}{5} + cos\frac{4\pi}{5} = -\frac{1}{2}$ **2**

b)



In the diagram, the two circles intersect at A and B. P is a point on one circle. PA and PB produced meet the other circle at M and N respectively. NA produced meets the first circle at Q. PQ and NM produced meet at R. The tangent at M to the second circle meets PR at T.
Copy or trace the diagram into your answer booklet.

(i) Show that QAMR is a cyclic quadrilateral. **3**

(ii) Show that TM=TR. **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

a) i) Show that for all values of x and y: **1**

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y$$

ii) Use mathematical induction to show that for all positive integers n: **4**

$$\cos x+\cos 2x+\cos 3x+\cdots\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

iii) Hence show that: **4**

$$\cos 2x+\cos 4x+\cos 6x+\cdots+\cos 16x \;=\; 8.\cos 9x.\cos 4x.\cos 2x.\cos x$$

b) Show that a relationship between the coefficients of $p(x)=x^3+ax^2+bx+c=0$ is $2a^3-9ab-27c=0$, if the roots are three consecutive terms of an arithmetic series. **4**

c) Solve the differential equation $\frac{dy}{dx}=2y$ for y given that when $x=1$, $y=1$ **2**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

CORRECTED SOLNS. IGS TRIAL 2008

Question 1 **Trial HSC Examination- Mathematics Extension 2** **2008**

Part	Solution	Marks	Comment
(a)	$\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\frac{16}{9}-x^2}}$	2	1 for rearranging
	$= \frac{1}{3}sin^{-1}\frac{x}{4/3}+c$		1 for inv trig integral
	$= \frac{1}{3}sin^{-1}\frac{3x}{4}+c$		
(b)	$\int 5\cos x\ sin^2 x\ dx = \frac{5}{3}sin^3 x + c$	2	2 for solution 1 if simple error made
(c)	$\int_1^e x \ln x\ dx = \left[\frac{x^2}{2}ln x\right]_1^e - \int_1^e \frac{x^2}{2}\frac{1}{x}dx$	3	1 for breakup into parts
	$= \left[\frac{x^2}{2}ln x\right]_1^e - \int_1^e \frac{x}{2}dx$		
	$= \left[\frac{x^2}{2}ln x - \frac{x^2}{4}\right]_1^e$		1 for integral
	$= \left[\frac{x^2}{4}(2ln x - 1)\right]_1^e$		1 for final answer
	$= \frac{e^2}{4}(2-1) - \frac{1}{4}(-1)$		
	$= \frac{e^2}{4} + \frac{1}{4}$		
(d)	Let $\frac{A}{x+1} + \frac{B}{x-1} = \frac{1}{x^2-1}$	4	
	$A(x-1) + B(x+1) = 1$		1 value of B
	When $x = 1$ $2B = 1$ $\rightarrow B = \frac{1}{2}$		1 value of A
	When $x = -1$ $-2A = 1$ $\rightarrow A = -\frac{1}{2}$		
	$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2}\int_2^3\left(\frac{-1}{x+1} + \frac{1}{x-1}\right)dx$		
	$= \frac{1}{2}\left[ln(x-1) - ln(x+1)\right]_2^3$		1 integral
	$= \frac{1}{2}\left[ln\left(\frac{x-1}{x+1}\right)\right]_2^3$		
	$= \frac{1}{2}\left[ln\frac{1}{2} - ln\frac{1}{3}\right]_2^3$		1 for answer
	$= \frac{1}{2}ln\frac{3}{2}$		

Question 1 **Trial HSC Examination- Mathematics Extension 2** **2008**

Part **Solution** **Marks** **Comment**

4

(e) If $t = \tan\frac{\theta}{2}$

$$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$$

$$2\cos^2\frac{\theta}{2}dt = d\theta$$

$$\cos^2\frac{\theta}{2} = \frac{1}{1+t^2}$$

$$d\theta = \frac{2}{1+t^2}dt$$

1 for $d\theta$

Limits $\theta = \frac{\pi}{2} \longrightarrow t = 1$

$\theta = 0 \longrightarrow t = 0$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$

1 for correct statement of integral including limits

$$= \int_0^1 \frac{1+t^2}{\left(2+2t^2+1-t^2\right)} \cdot \frac{2}{1+t^2}dt$$

$$= \int_0^1 \frac{2}{\left(3+t^2\right)}dt$$

1 for completing integral

$$= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$$

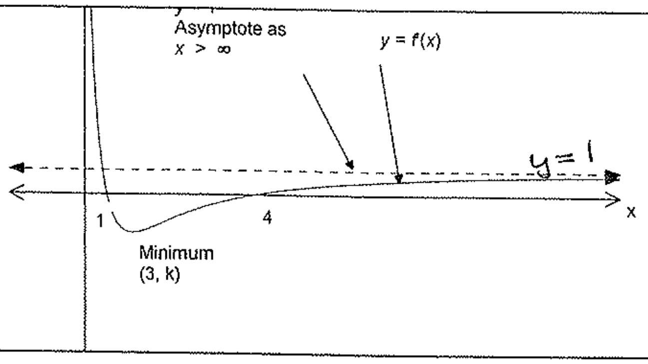
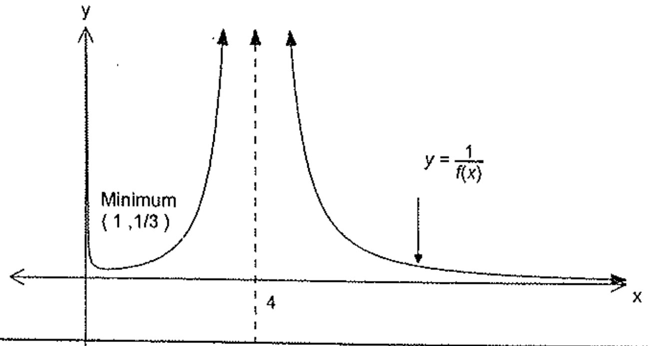
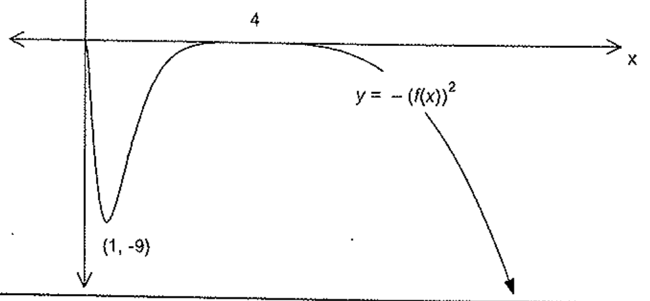
$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6}$$

1 for result

$$= \frac{\pi}{3\sqrt{3}}$$

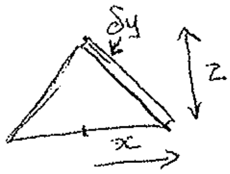
Question 2	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(a) i)	$\frac{a}{b}=\frac{3+4i}{2-2i}\times\frac{2+2i}{2+2i}$ $=\frac{6+6i+8i-8}{4+4}$ $=\frac{-2+14i}{8}$ $=-\frac{1}{4}+\frac{7}{4}i$	2	1 for multiplying by conjugate. 1 for correct answer
ii)	Let $\sqrt{A}=x+iy$ $\therefore A=x^2-y^2+2xyi$ $\therefore 3+4i=x^2-y^2+2xyi$ $\therefore 3=x^2-y^2$ ....(1) $\therefore 4=2xy$ ....(2) $(1)^2+(2)^2$ $x^4+2x^2y^2+y^4=25$ $(x^2+y^2)^2=25$ $x^2+y^2=5$ ....(3) (1) + (3) $2x^2=8$ $x=\pm 2$ $y=\pm 1$ $\sqrt{A}=\pm(2+i)$	3	1 for squaring and equating real and imaginary 1 for eliminating y (or x) 1 for final solution
iii)	$A-\overline{B}=3+4i-(2+2i)$ $=1+2i$	1	1 for answer
(b) i)	$1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ $\cos\theta=\frac{1}{2}$ and $\sin\theta=\frac{\sqrt{3}}{2}$ $\therefore \theta=\frac{\pi}{3}$ $\therefore 1+\sqrt{3}i=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$	2	1 value of θ 1 for result
ii)	$\left(1+\sqrt{3}i\right)^6=2^6\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^6$ $=64(\cos 2\pi+i\sin 2\pi)$ by De Moivres Theorem $=64(1+0i)$ $=64$ Which is ~~totally~~ real.	2	1 Use of De Moivre Thm 1 for answer

Question 2	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(c) i)	$OA = 2iOC$ $= 2i(p+iq)$ $= -2q+2pi$	1	1 for answer
ii)	$OB = OC + OA$ $= (p+iq)+(-2q+2pi)$ $= (p-2q)+(2p+q)i$	1	1 for answer
iii)	$BC = -OA$ $= 2q-2pi$	1	1 for answer
iv)	$AC = AB + BC$ $= OC - OA$ $= (p+iq)-(-2q+2pi)$ $= (p+2q)+(q-2p)i$	2	1 for sum of vectors 1 for answer

Question 3	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(a) i)		2	1 for basic shape 1 for asymptote
ii)		2	1 for basic shape 1 for discontinuity
iii)		2	1 for shape 1 for below axis
(b)	$x^2+2xy-y^2=17$ $2x+2y+2x\frac{dy}{dx}-2y\frac{dy}{dx}=0$ $\frac{dy}{dx}(2x-2y)=-(2x+2y)$ $\frac{dy}{dx}=\frac{-(x+y)}{(x-y)}$ $=\frac{x+y}{y-x}$ At (3,2) Gradient of tangent $=\frac{3+2}{2-3}=-5$	2	1 for implicit differentiation 1 for derivative

Question 3	Trial HSC Examination- Mathematics Extension 2	2008	
Part	Solution	Marks	Comment
(c) i)	For $x^3-3x^2-2x+4=0$ $x=\alpha,\ \beta$ and γ Let $X=x^2$ $\sqrt{X}=x$ $X\sqrt{X}-3X-2\sqrt{X}+4=0$ $\sqrt{X}(X-2)=3X-4$ Squaring $X\left(X^2-4X+4\right)=9X^2-24X+16$ $X^3-4X^2+4X=9X^2-24X+16$ $\therefore$ Required polynomial is $x^3-13x^2+28x-16=0$	2	Any method okay. 1 mark for partial solution or complete solution with simple error.
ii)	As above has roots $\alpha^2,\ \beta^2$ and γ^2 $\alpha^2+\beta^2+\gamma^2=\dfrac{-b}{a}=13$	1	1 for answer
iii)	As $\alpha,\ \beta$ and γ are roots of $x^3-3x^2-2x+4=0$ Then $\alpha^3-3\alpha^2-2\alpha+4=0$ $\beta^3-3\beta^2-2\beta+4=0$ $\gamma^3-3\gamma^2-2\gamma+4=0$ Adding $\left(\alpha^3+\beta^3+\gamma^3\right)-3\left(\alpha^2+\beta^2+\gamma^2\right)-2(\alpha+\beta+\gamma)+12=0$ $\left(\alpha^3+\beta^3+\gamma^3\right)-3(13)-2(3)+12=0$ $\left(\alpha^3+\beta^3+\gamma^3\right)=33$	2	Any method okay. 1 mark for partial solution or complete solution with simple error.
(d)	$P(x)=x^3+3x^2-24x+k$ $P'(x)=3x^2+6x-24$ $=3(x-2)(x+4)$ If $P'(x)=0$ $x=2,$ $x=-4$ If $x=2$ is a double zero, $P(2)=(2)^3+3(2)^2-24(2)+k=0$ $k=28$ $P(-4)=(-4)^3+3(-4)^2-24(-4)+k=0$ $k=-80$	2	1 possible zeros 1 values of k

Question 4	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(a) i)	$I_n = \int (\ln x)^n\,dx = x(\ln x)^n - \int x.n(\ln x)^{n-1}.\frac{1}{x}$ $= x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$ $= x(\ln x)^n - nI_{n-1}$	3	1 for use of Int by parts 1 for simplifying 1 for result in terms of I_n
ii)	Consider $\int (\ln x)^3\,dx = I_3$ $\therefore I_3 = x(\ln x)^3 - 3I_2$ Now $I_2 = x(\ln x)^2 - 2I_1$ and $I_1 = x(\ln x) - 1I_0$ $= x(\ln x) - x$ $\therefore I_3 = x(\ln x)^3 - 3\left(x(\ln x)^2 - 2\left(x(\ln x) - x\right)\right)$ $= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x$ $\therefore \int_1^{e^4} (\ln x)^3 dx = \left(e^4.64 - 3e^4.16 + 6e^4.4 - 6e^4\right) - (-6)$ $= 34e^4 + 6$	4	1 for I_3 1 for I_2 1 full expression including I_1 1 sub and evaluate
(b)	$y = 6x - x^2 - 8$ $y = 0 \quad x = 2,4$ For shells about Y axis $V = 2\pi \int_a^b xy\,dx$ About $x = 1$ $V = 2\pi \int_a^b (x-1)\,y\,dx$ $= 2\pi \int_2^4 (x-1)(6x - x^2 - 8)\,dx$ $= 2\pi \int_2^4 (7x^2 - x^3 - 14x + 8)\,dx$ $= 2\pi \left[\frac{7x^3}{3} - \frac{x^4}{4} - 7x^2 + 8x\right]_2^4$ $= 2\pi \left[\left(\frac{448}{3} - 64 - 112 + 32\right) - \left(\frac{56}{3} - 4 - 28 + 16\right)\right]$ $= \frac{16\pi}{3}$ units^3	4	4 marks for full solution 3 marks if simple error made 2 marks if major error or 2 simple errors 1 mark if start made using correct formula or from scratch with correct method.

Question 4	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(c)	 $V_{slice} \approx Ah\delta y$ $\approx \frac{1}{2} 2x \cdot z \cdot \delta y$ $\approx xz\delta y$	4	1 dimensions of slice
	$V_{solid} = \int_{-10}^{10} xz \, dy$ $\frac{x^2}{15^2} + \frac{y^2}{10^2} = 1$ $\therefore x = \sqrt{15^2\left(1 - \frac{y^2}{10^2}\right)}$ and $\frac{z^2}{15^2} + \frac{y^2}{10^2} = 1$ $\therefore z = \sqrt{15^2\left(1 - \frac{y^2}{10^2}\right)}$ $\therefore V_{solid} = 450 \int_0^{10} \left(1 - \frac{y^2}{10^2}\right) dy$ $= 450 \left[y - \frac{y^3}{300} \right]_0^{10}$ $= 3000$ cms		1 $x = f(y)$, $z = f(y)$ 1 correct integrand and limits 1 correct exact volume

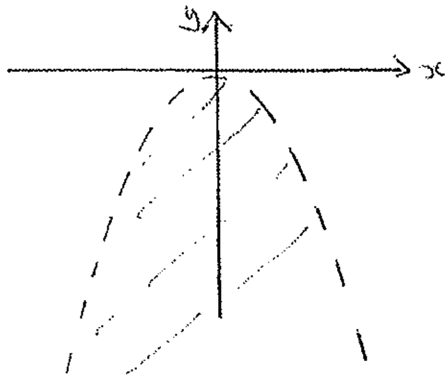
<table>
<tr><td colspan="2">Question 5</td><td>Trial HSC Examination- Mathematics Extension 2</td><td colspan="2">2008</td></tr>
<tr><td>Part</td><td colspan="2">Solution</td><td>Marks</td><td>Comment</td></tr>
<tr><td>(a) i)</td><td colspan="2">Solving simultaneously
$y = mx + a$ ....(1)
$b^2x^2 + a^2y^2 = a^2b^2$ .....(2)
sub (1) into (2)
$b^2x^2 + a^2(mx+a)^2 = a^2b^2$
$b^2x^2 + a^2m^2x^2 + 2a^3mx + a^4 = a^2b^2$
$(b^2 + a^2m^2)x^2 + 2a^3mx + a^4 - a^2b^2 = 0$
If $x = x_1$ and x_2 are the roots then
$x_1 + x_2 = \frac{-2a^3m}{b^2 + a^2m^2}$
$\therefore x_2 = \frac{-2a^3m}{b^2 + a^2m^2} - x_1$</td><td>3</td><td>1 for sub into equation

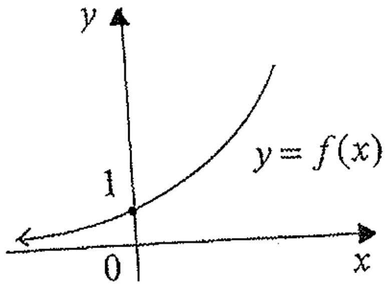
1 for simplify

1 for expression for x_2</td></tr>
<tr><td>ii)</td><td colspan="2">For a tangent $x_1 = x_2 = x$
$x_1 + x_2 = 2x = \frac{-2a^3m}{b^2 + a^2m^2}$
$\therefore x = \frac{-a^3m}{b^2 + a^2m^2}$</td><td>1</td><td>1 for answer</td></tr>
<tr><td>(b) i)</td><td colspan="2">$x = 2at$ and $y = at^2$
Grad of tangent $= \frac{dy}{dx} = t$
Grad of normal $= = \frac{-1}{t}$
At $t = p$ $m = \frac{-1}{p}$ [point $(2ap, ap^2)$]
Equation of normal
$y - ap^2 = \frac{-1}{p}(x - 2ap)$
$py - ap^3 = -x + 2ap$
$x + py - ap^3 - 2ap = 0$</td><td>2</td><td>1 for gradient of normal

1 for equation of normal</td></tr>
<tr><td>ii)</td><td colspan="2">Normal passes through (x_1, y_1) then
$x_1 + py_1 - ap^3 - 2ap = 0$
To find intersection with the parabola, this equation must be solved for p.
As the equation is a cubic in p, there can be from 1 to 3 values for p.
$\therefore$ Up to three normals can be drawn</td><td>2</td><td>2 for any reasonable explanation</td></tr>
</table>

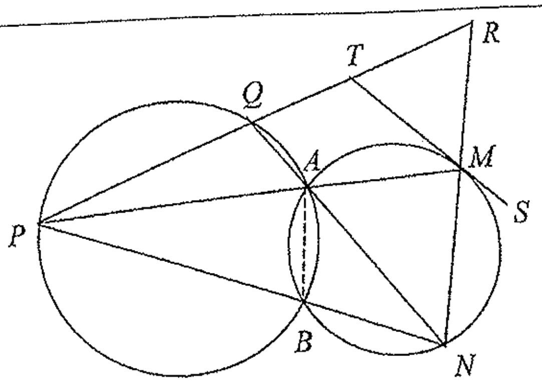
Question 5	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(c) i)	$z=\cos\theta+i\sin\theta=c+is$ $z^4=(c+is)^4$ $=c^4+4c^3(is)+6c^2(-s^2)+4c(-is^3)+s^4$ $=c^4-6c^2s^2+s^4+i(4c^3s-4cs^3)$ By De Moivres Thm $z^4=\cos 4\theta+i\sin 4\theta$ Equating real parts $\cos 4\theta=c^4-6c^2(1-c^2)+(1-c^2)^2$ $=c^4-6c^2+6c^4+1-2c^2+c^4$ $=8c^4-8c^2+1$ $=8\cos^4\theta-8\cos^2+1$	3	1 for expanding 1 for De Moivre 1 for solution
(ii)	$\left(z+\frac{1}{z}\right)^4=(2\cos\theta)^4$ $=16\cos^4\theta$ and $\left(z+\frac{1}{z}\right)^4=z^4+4z^3\frac{1}{z}+6z^2\frac{1}{z^2}+4z\frac{1}{z^3}+\frac{1}{z^4}$ $=z^4+\frac{1}{z^4}+4\left(z^2+\frac{1}{z^2}\right)+6$ $16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$ $\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$	2	1 for expansion 1 for expression
iii)	$\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta=\int_0^{\frac{\pi}{2}}\left(\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}\right)d\theta$ $=\left[\frac{1}{32}\sin 4\theta+\frac{1}{4}\sin 2\theta+\frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$ $=\left[\left(\frac{3}{8}\cdot\frac{\pi}{2}\right)-(0)\right]$ $=\frac{3\pi}{16}$	2	1 for integral 1 for evaluating

Question 6	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(a)	$\left[\text{Re}(z)\right]^2 + \text{Im}\, z < 0$ $x^2 + y < 0$ $y < -x^2$ 	2	1 for correct Cartesian equation 1 mark for correct region
b) i)	i. Consider the function $g(x) = e^{-x} + x - 1$. $g(0) = 0$ and $g'(x) = -e^{-x} + 1 \Rightarrow g'(0) = 0$ Also $g''(x) = e^{-x} > 0$ for all x $\therefore g(x)$ has a minimum value of 0 when $x = 0$. $\therefore e^{-x} + x - 1 \geq 0$ for all x, with equality only if $x = 0$. For $x \neq 0$, $f'(x) = \frac{d}{dx}\left(\frac{e^x - 1}{x}\right)$ $= \frac{e^x . x - (e^x - 1) . 1}{x^2}$ $= \frac{1 + xe^x - e^x}{x^2}$ $= \frac{e^x}{x^2}\left(e^{-x} + x - 1\right)$ > 0 Hence $f(x)$ is an increasing function for $x \neq 0$.	3	1 shows by differentiation that $e^{-x} + x + 1$ has min value of 0 when x=0 1 finds $f'(x)$ 1 rearranges $f'(x)$ as a product of $e^{-x} + x + 1$ and deduces $f'(x)$>0
ii)	ii. Let $h(x) = e^x$. Then $h'(x) = e^x$. $\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x - e^0}{x - 0}$ $= h'(0)$ $= 1$ $= f(0)$ $\therefore f(x)$ is continuous at $x = 0$.	2	1 Expresses the limiting value of $f(x)$ as $x \to 0$ as the derivative of e^x at x=0 1 Evaluates this derivative to show limiting value is 1

Question 6	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
iii)	iii. 	1	Correct shape curve with y intercept of 1 and asymptote
(c) i)	i. $\frac{dy}{dx} = e^x = 1$ at $x = 0$ Hence tangent AC has equation $y = x + 1$. Let $D(k, 0)$ $\therefore C(k, k+1)$. Also $B(k, e^k)$. Then $Area\ AODC < \int_0^k e^x\,dx < Area\ AODB$ $\frac{1}{2}k(k+2) < \left[e^x\right]_0^k < \frac{1}{2}k(1+e^k)$ $\therefore \frac{1}{2}k(k+2) < e^k - 1 < \frac{1}{2}k(1+e^k)$ For $k=1$, $1\cdot5 < e - 1 < 0\cdot5 + \frac{1}{2}e$ Hence $2\cdot5 < e$ and $\frac{1}{2}e < 1\cdot5$ $\therefore 2\cdot5 < e < 3$	4	1 for equation of tangent AC and coords of C 1 for lower bound for $e^k - 1$ using area AODC 1 for upper bound using area AODB 1 for using k=1 and rearranging
ii)	ii. Area of ΔABC is bisected if $(e^k - 1) - \frac{1}{2}k(k+2) = \frac{1}{2}k(1+e^k) - (e^k - 1)$ $(4-k)e^k - k^2 - 3k - 4 = 0$ Let $f(k) = (4-k)e^k - k^2 - 3k - 4$ Then $f(2) \approx 0\cdot78 > 0$, $f(3) \approx -1\cdot9 < 0$ and $f(k)$ is continuous. Hence $f(k) = 0$, and the area is bisected, for some k such that $2 < k < 3$ $f(k) = (4-k)e^k - k^2 - 3k - 4$ $f'(k) = \left\{-e^k + (4-k)e^k\right\} - 2k - 3$ $= (3-k)e^k - 2k - 3$ Taking $k_0 = 2\cdot7$, $k_1 = 2\cdot7 - \frac{f(2\cdot7)}{f'(2\cdot7)}$ $\therefore k_1 \approx 2\cdot7 - \frac{-0\cdot04635}{3\cdot9361}$ $\approx 2\cdot688$ Hence second approximation is $2\cdot7$ (to one decimal place).	3	1 for equation if triangle area bisected 1 for f(k)=0 and establishes existence of root k 2<k<3 1 for using Newton's method for 2nd approximation

Question 7	Trial HSC Examination- Mathematics Extension 2	2008	

Part	Solution	Marks	Comment
(a) i)	$z^5 = 1$ By De Moivres Thm $\cos 5\theta + i\sin 5\theta = 1$ Equating real and imaginary $\cos 5\theta = 1 \quad \sin 5\theta = 0$ $\therefore \; 5\theta = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi$ $\theta = 0,\ \frac{2\pi}{5},\ \frac{4\pi}{5},\ \frac{6\pi}{5},\ \frac{8\pi}{5}$ $z_1 = cis 0 = 1$ $z_2 = cis\frac{2\pi}{5}$ $z_3 = cis\frac{4\pi}{5}$ $z_4 = cis\frac{6\pi}{5} = cis\frac{-4\pi}{5} = \overline{z}_3$ $z_5 = cis\frac{8\pi}{5} = cis\frac{-2\pi}{5} = \overline{z}_2$	3	1 values of θ 2 for values of z_1 - z_5
ii)	$z^5 - 1 = (z - z_1)(z - z_2)(z - z_5)(z - z_3)(z - z_4)$ $= (z - z_1)(z^2 - (z_2 + z_5)z + z_2 z_5)(z^2 - (z_3 + z_4)z + z_3 z_4) =$ $= (z - z_1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$	2	1 factors 1 in quadratics
iii)	$z^5 - 1 = 0$ $(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$ Roots are $z_2,\ z_3,\ z_4,\ z_5$ from above.	1	
iv)	Sum of roots of $z^5 - 1 = 0$ is zero. $\therefore z_1 + z_2 + z_3 + z_4 + z_5 = 0$ $z_1 + z_2 + \overline{z}_2 + z_3 + \overline{z}_3 = 0$ $1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$ $2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$ $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \frac{-1}{2}$	2	1 for conjugates 1 for answer

(b)	 		
i)	$\angle RMA = \angle ABN$ *(exterior angle of cyclic quad. ABNM is equal to interior opposite angle)* Similarly $\angle ABN = \angle AQP$ in cyclic quadrilateral *ABPQ*. Hence quadrilateral *QAMR* is cyclic. *(exterior angle AQP is equal to interior opposite angle RMA)*	3	2 marks for using properties of cyclic quads to deduce equal angles 1 for applying test to QAMR
	ii. Produce *TM* to *S*. Then $\angle TMR = \angle SMN$ *(vertically opposite angles are equal)* $\angle SMN = \angle MAN$ *(angle between tangent and chord drawn to point of contact is equal to angle subtended by that chord in the alternate segment)* $\angle MAN = \angle PAQ$ *(vertically opposite angles are equal)* $\angle PAQ = \angle TRM$ *(exterior angle of cyclic quad. QAMR is equal to interior opposite angle)* Hence in ΔTMR, $\angle TMR = \angle TRM$ and hence $TM = TR$ *(sides opposite equal angles are equal)*	4	1 alt seg theorem 1 vert opp angles 1 equality of angles 1 deduces TMR has equal angles so equal sides

Question 8	Trial HSC Examination- Mathematics Extension 2	2008

Part	Solution	Marks	Comment
(a)	$\sin(x+y)-\sin(x-y)=\sin x\cos y+\cos x\sin y-(\sin x\cos y-\cos x\sin y)$ $=2\cos x\sin y$	1	1 for answer
a) ii)	If $n=1$ LHS $=\cos x$ RHS $=\dfrac{\sin\left(\frac{3x}{2}\right)-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ Using i) above RHS $=\dfrac{2\cos x\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ $=\cos x=$ LHS $\therefore$ true for $n=1$ Assume true for $n=k$ i.e $\cos x+\cos 2x+\cos 3x\ldots\ldots+\cos kx=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ When $n=k+1$ $\cos x+\cos 2x+\cos 3x\ldots\ldots+\cos kx+\cos(k+1)x=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}+\cos(k+1)x$ $=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\cos(k+1)x.2\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ $=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\sin\left((k+1)x+\frac{x}{2}\right)-\sin\left((k+1)x-\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ using i) above $=\dfrac{\sin\left(k+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)+\sin\left(k+\frac{3}{2}\right)x-\sin\left(k+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}$ $=\dfrac{\sin\left((k+1)+\frac{1}{2}\right)x-\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$ $\therefore$ True for $n=k+1$ $\therefore$ Since true for $n=1$, by induction is true for all positive integral values of $k\geq 1$	4	1 for n =1 case 1 for using i) 1 for simplifying 1 for stating k+1 case

Question 8	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(a) ii)	$\cos 2x + \cos 4x + \cos 6x ...... + \cos 16x = \cos(2x) + \cos 2(2x) + \cos 3(2x) ...... + \cos 8(2x)$ $= \dfrac{\sin\left(8+\frac{1}{2}\right)2x - \sin\left(\frac{2x}{2}\right)}{2\sin\left(\frac{2x}{2}\right)}$ $= \dfrac{\sin 17x - \sin x}{2\sin x}$ $= \dfrac{\sin(9+8)x - \sin(9-8)x}{2\sin x}$ $= \dfrac{2\cos 9x \sin 8x}{2\sin x}$ Using i) above $= \dfrac{2\cos 9x . 2\sin 4x \cos 4x}{2\sin x}$ Using double angle on $\sin 8x$ $= \dfrac{4\cos 9x . 2\sin 2x \cos 2x \cos 4x}{2\sin x}$ Using double angle on $\sin 4x$ $= \dfrac{8\cos 9x . 2\sin x \cos x \cos 2x \cos 4x}{2\sin x}$ Using double angle on $\sin 2x$ $= \dfrac{8\cos 9x . \cancel{2\sin x} \cos x \cos 2x \cos 4x}{\cancel{2\sin x}}$ $= 8\cos 9x \cos 4x \cos 2x \cos x$	4	1 for sub into expression 1 for breaking up 17x 2 for completing simplification
(b)	Let roots be $\alpha - d$, α and $\alpha + d$ $\therefore$ Sum of the roots $= (\alpha - d) + \alpha + (\alpha + d) = -a$ $\therefore 3\alpha = -a$ $\alpha = \dfrac{-a}{3}$ $\therefore$ Sum of the roots 2 at a time $= (\alpha - d)\alpha + (\alpha - d)(\alpha + d) + (\alpha + d)\alpha = b$ $\alpha^2 - \alpha d + \alpha^2 - d^2 + \alpha^2 + \alpha d = b$ $3\alpha^2 - d^2 = b$ $d^2 = 3\alpha^2 - b$ $d^2 = 3\left(\dfrac{-a}{3}\right)^2 - b = \dfrac{a^2}{3} - b$ $\therefore$ Product of the roots $= \alpha(\alpha - d)(\alpha + d) = c$ $\alpha^3 - \alpha d^2 = c$ $\left(\dfrac{-a}{3}\right)^3 - \left(\dfrac{-a}{3}\right)\left(\dfrac{a^2}{3} - b\right) = c$ $\dfrac{-a^3}{27} + \dfrac{a^3}{9} - \dfrac{ab}{3} = c$ $\dfrac{2a^3}{27} - \dfrac{ab}{3} = c$ $\therefore 2a^3 - 9ab - 27c = 0$	4	1 each for expressions for sums & products = 3 marks 1 for substitution and simplifying

Question 8	Trial HSC Examination- Mathematics Extension 2	2008	
Part	**Solution**	**Marks**	**Comment**
(c)	$\frac{dy}{dx}=2y$ $\therefore \frac{dx}{dy}=\frac{1}{2y}$ $\therefore x=\frac{1}{2}\ln y+c$ When $x=1, y=1$ $\therefore c=1$ $\therefore x=\frac{1}{2}\ln y+1$ $\frac{1}{2}\ln y=x-1$ $\ln y=2x-2$ $y=e^{2x-2}$	2	1 for expression for x 1 for result