International Grammar School

Mathematics Extension 2
Trial Higher School Certificate Examination
2010

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

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Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

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Total Marks – 120
Attempt Questions 1-8
All Questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Begin a NEW writing booklet **Marks**

a) Evaluate $\displaystyle\int_{\frac{\pi}{16}}^{\frac{\pi}{12}} \sec 4x \ \tan 4x \ dx$ **2**

b) Find $\displaystyle\int x \ln x \ dx$ **2**

c) Find $\displaystyle\int \frac{9x^3 + 9x^2 + 5x + 4}{3x + 1} \ dx$ **3**

d) i. Find constants a, b and c such that **2**

$$\frac{3x^2 - 2x - 3}{(x^2 + 9)(x - 3)} = \frac{ax + b}{x^2 + 9} + \frac{c}{x - 3}$$

ii. Hence find $\displaystyle\int \frac{3x^2 - 2x - 3}{(x^2 + 9)(x - 3)} \ dx$. **2**

e) By making the substitution $t = \tan \dfrac{\theta}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin \theta + \cos \theta}$ **4**

End of Question 1

Question 2 (15 Marks) Begin a NEW booklet **Marks**

a) Given A = 3 + 4i and B = 1 – i, express the following in the form x + iy where x and y are real numbers.

i. AB **1**

ii. $\frac{A}{iB}$ **2**

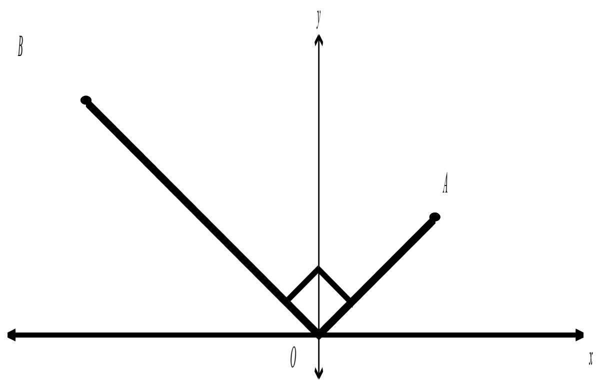
iii. $\sqrt{A}$ **3**

b) If $w = \sqrt{3} - i$,

i. Find the exact value of $|w|$ and arg w. **2**

ii. Find the exact value of w^5 in the form a + ib where a and b are real. **2**

c)



On the Argand diagram, OA represents the complex number $z_1 = x + iy$, $\angle AOB = \frac{\pi}{2}$ and the length of OB is twice that of OA.

i. Show that OB represents the complex number $-2y + 2ix$. **1**

ii. Given that AOBC is a rectangle, find the complex number represented by OC. **1**

iii. Find the complex number represented by BA. **1**

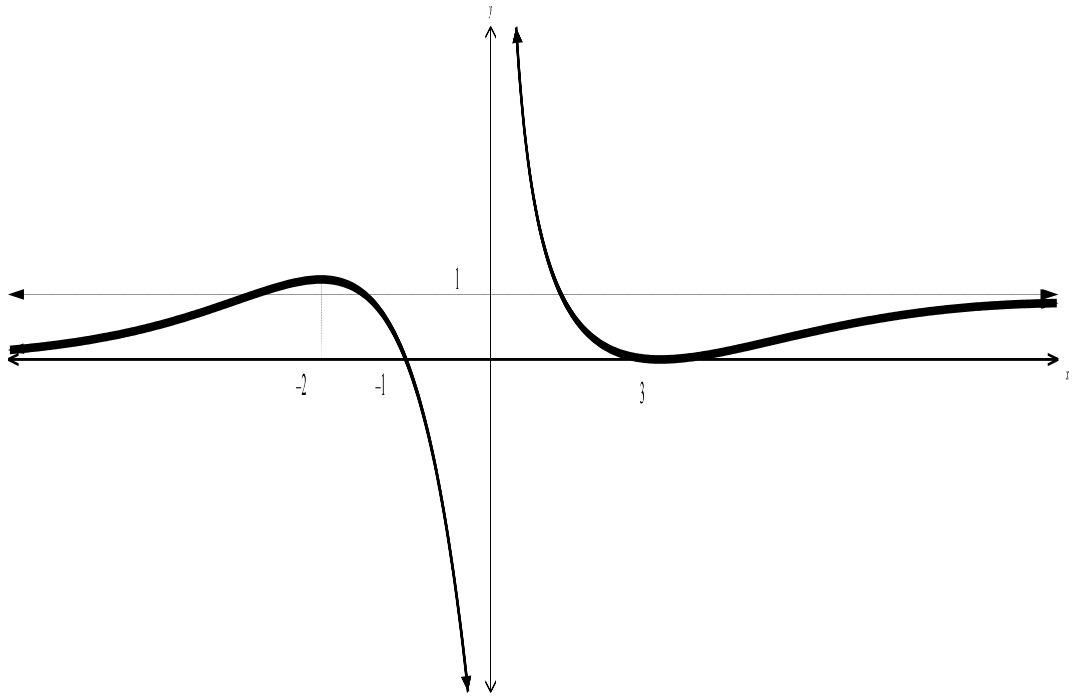
d) Sketch the region on an argand diagram where **2**

$$|z-1| \geq \sqrt{2} \text{ and } 0 \leq \arg(z+i) \leq \frac{\pi}{4} \text{ both hold.}$$

End of Question 2

Question 3 (15 Marks) Begin a NEW booklet **Marks**

a)



The diagram shows the graph of the function y = f(x) which has asymptotes, vertically at x = 0 and horizontally at y = 1 for $x \geq 0$ and at $y = 0$ for $x \leq 0$.

Draw separate sketches of the following showing any critical features.

i. $y = \dfrac{1}{f(x)}$ **2**

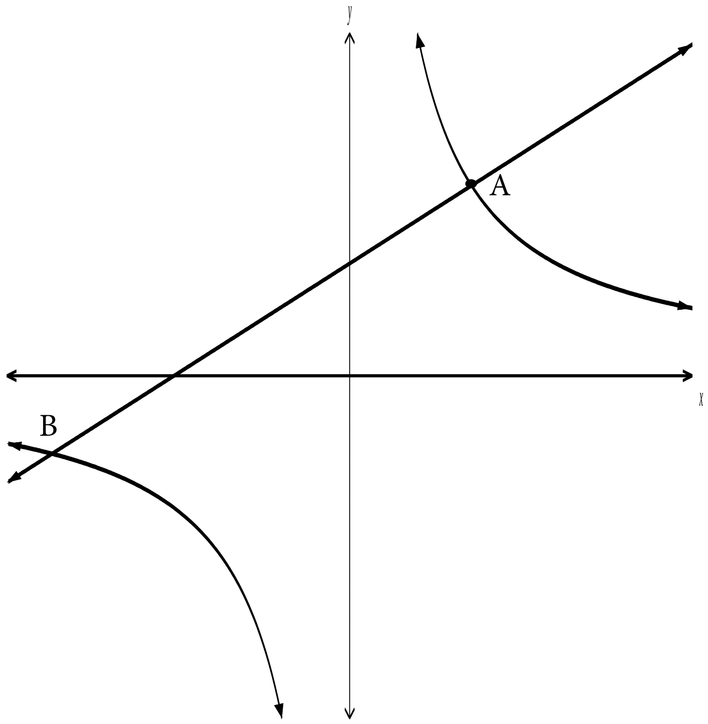
ii. $y = \left[\, f(x) \,\right]^2$ **2**

iii. $y = f'(x)$ **2**

Question 3 continues

Question 3 continued **Marks**

b)



The point A $\left(ca, \frac{c}{a}\right)$, where $a \neq \pm 1$ lies on the hyperbola $xy = c^2$. The normal through A meets the other branch of the curve at B.

i. Show that the equation of the normal through A is **2**

$$y = a^2 x + \frac{c}{a}(1 - a^4)$$

ii. Hence if B has coordinates $\left(cb, \frac{c}{b}\right)$, show that $b = \frac{-1}{a^3}$. **3**

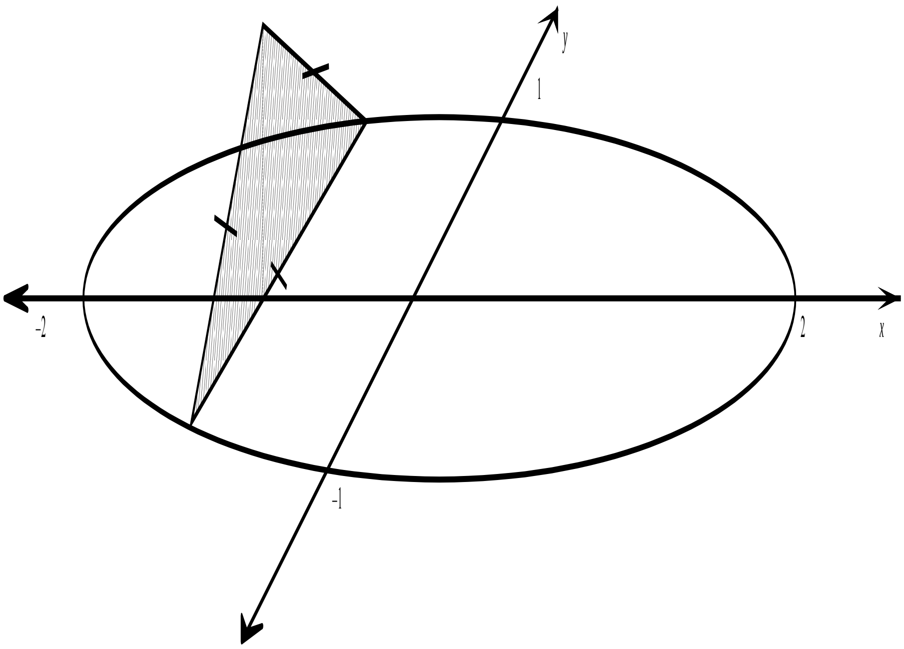
iii. If this hyperbola is rotated clockwise through 45°, show that the equation becomes **4**

$$x^2 - y^2 = 2c^2.$$

End of Question 3

Question 4 (15 Marks) Begin a NEW booklet **Marks**

a) A solid shape has as its base an ellipse in the XY plane as shown below. Sections taken perpendicular to the X-axis are equilateral triangles. The major and minor axes of the ellipse are 4 metres and 2 metres respectively.



i. Write down the equation of the ellipse. **1**

ii. Show that the area of the cross-section at x = k is given by **2**

$$A = \frac{\sqrt{3}}{4}(4 - k^2)$$

iii. By using the technique of slicing, find the volume of the solid. **2**

b) The region enclosed by the curve $y = 5x - x^2$, the x axis and the lines x = 1 and x = 3 is rotated about the y axis. By using the method of cylindrical shells, find the volume of the solid so produced. **4**

Question 4 continues

Question 4 continued **Marks**

c) The roots of the equation $x^3 - 3x^2 + 9 = 0$ are α, β and γ .

i. Determine the polynomial equation with roots α^2, β^2 and γ^2. **1**

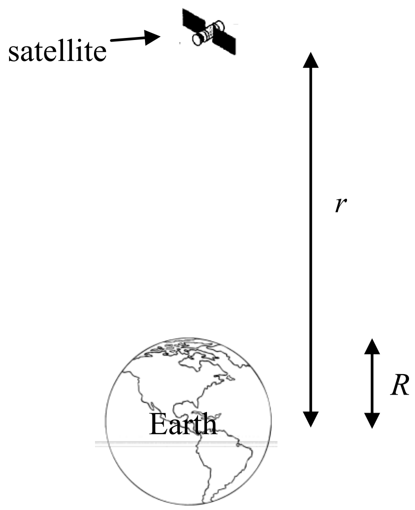
ii. Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence evaluate $\alpha^3 + \beta^3 + \gamma^3$. **3**

d) Given that the polynomial $P(x)$ has a double root at $x = \alpha$, show that the polynomial $P'(x)$ will have a single root at $x = \alpha$. **2**

End of Question 4

Question 5 (15 Marks) Begin a NEW booklet **Marks**

a)



The gravitational force between two objects of masses m and M placed at a distance r metres apart is proportional to their masses and inversely proportional to the square of their distance apart, ie $F = k\dfrac{mM}{r^2}$, $k > 0$.

A satellite is to be placed in orbit so that it will rotate about the earth once every 36 hours.

i. Show that $F = \dfrac{R^2 mg}{r^2}$ **2**

Taking $g = 9.8\ m/s^2$ and the earth's radius $R = 6400\ km$, find:

ii. The height of the satellite from the earth's surface **3**

iii. The linear velocity of the satellite. **1**

b) By taking logarithms of both sides and then differentiating implicitly, verify the rule for differentiating the quotient $y = \dfrac{u(x)}{v(x)}$ is given by **2**

$$\frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$$

Question 5 continues

Question 5 continued **Marks**

c) i. Show that the recurrence (reduction) formula for **4**

$$I_n = \int tan^n x dx \quad \text{is} \quad I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$

ii. Hence evaluate $\int_0^{\frac{\pi}{4}} tan^3 x \, dx$ **3**

End of Question 5

Question 6 (15 Marks) Begin a NEW booklet **Marks**

a) A solid of unit mass is dropped under gravity from rest at a height of H metres. Air resistance is proportional to the speed (V) of the mass. (acceleration under gravity = g)

i. Write the equation for the acceleration of the mass. (Use k as the constant of proportionality) **1**

ii. Show that the velocity (V) of the solid after t seconds is given by **3**

$$V = \frac{g}{k}(1 - e^{-kt})$$

iii. By using the fact that $\frac{d}{dx}\left(\frac{1}{2}V^2\right) = \ddot{x}$, show that **4**

$$x = \frac{g}{k^2}\left[\ln \frac{g}{g - kV} - \frac{kV}{g}\right].$$

b) Given z = cos θ + isin θ, and using De Moivres' Theorem

i. Find an expression for cos 4θ in terms of powers of cos θ. **3**
Hint: you may use the expansion:
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

ii. Determine the roots of the equation $z^4 = -1$. **2**

iii. Using the fact that $z^n + \frac{1}{z^n} = 2\cos n\theta$, find an expression for $\cos^4\theta$ in terms of cos nθ. **2**

End of Question 6

Question 7 (15 Marks) Begin a NEW booklet **Marks**

a) i. Prove that $\cos(k-1)\theta - 2\cos\theta\cos k\theta = -\cos(k+1)\theta$ **1**

ii. Hence, using mathematical induction, prove that if n is a positive integer then **4**

$$1 + \cos\theta + \cos 2\theta + \ldots\ldots\ldots\ \ldots + \cos(n-1)\theta = \frac{1 - \cos\theta - \cos n\theta + \cos(n-1)\theta}{2 - 2\cos\theta}$$

b) A mass of 20kg hangs from the end of a rope and is hauled up vertically from rest by winding up the rope. The pulling force on the rope starts at 250N and decreases uniformly at a rate of 10N for every metre wound up. **3**

Find the velocity of the mass when 10 metres have been wound up.

(Neglect the weight of the rope and take g = 10ms^{-2})

c) When a polynomial P(x) is divided by (x – 1) the remainder is 3 and when divided by (x – 2) the remainder is 5. Find the remainder when the polynomial is divided by (x – 1)(x – 2). **3**

d) How many different words can be formed from the letters A, A, B, C, D, E, E if the word must contain:

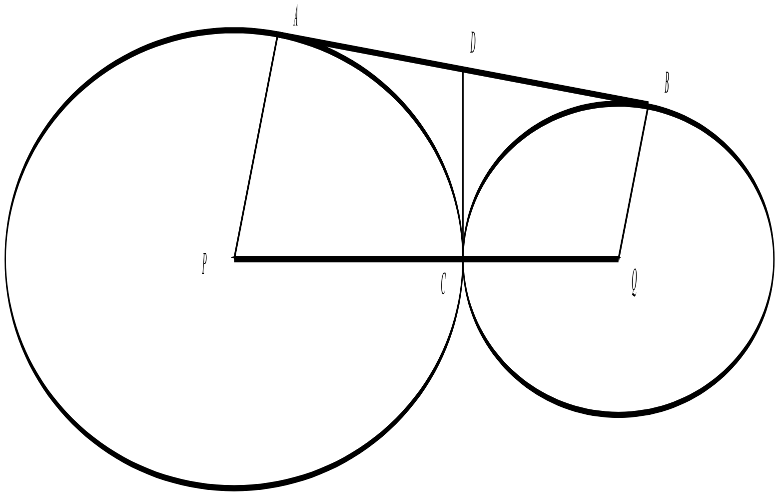
i. All seven letters (1 mark) **1**

ii. Exactly four letters **3**

End of Question 7

Question 8 (15 Marks) Begin a NEW booklet **Marks**

a)



In the diagram PCQ is a straight line joining the centres of the circles P and Q. AB and DC are common tangents.

i. Explain why PADC and CDBQ are cyclic quadrilaterals. **2**

ii. Show that $\Delta ADC \parallel\!| \Delta BQC$. **2**

iii. Show that PD || CB. **2**

b) Given $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

If $P = 1 + 2 \cos \theta + 2 \cos 2\theta + 2\cos 3\theta$

i. Prove that $P \sin \frac{\theta}{2} = \sin \frac{7\theta}{2}$. **3**

ii. Hence show that if $\theta = \frac{2\pi}{7}$ then **2**

$$P = 1 + 2 \cos \theta + 2 \cos 2\theta + 2 \cos 3\theta = 0$$

iii. By writing P in terms of $\cos \theta$, prove that $\cos \frac{2\pi}{7}$ is a root of the Polynomial equation **4**

$$8x^3 + 4x^2 - 4x - 1 = 0$$

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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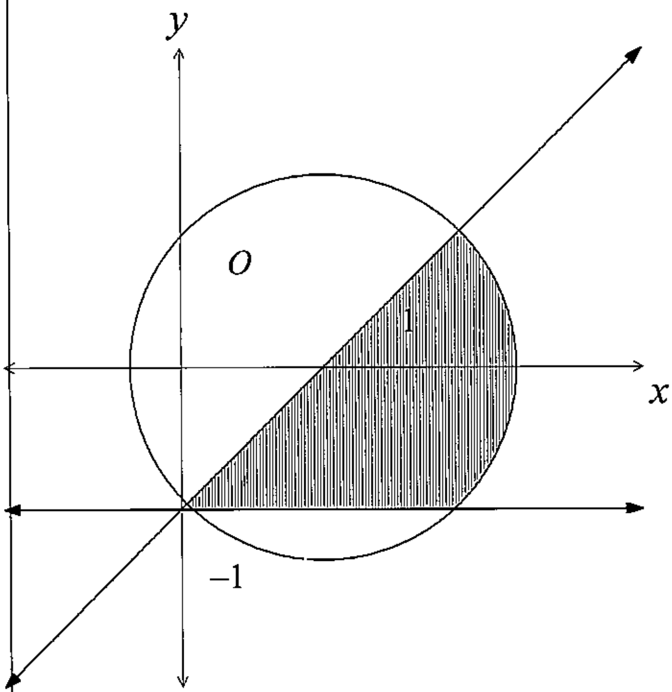
SOLUTIONS

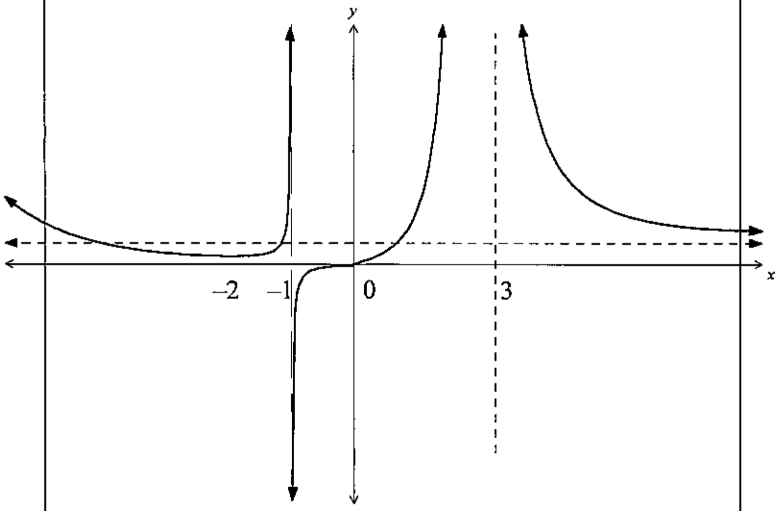
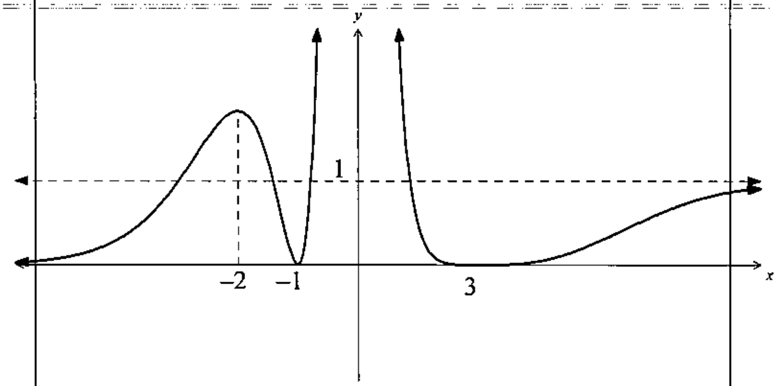
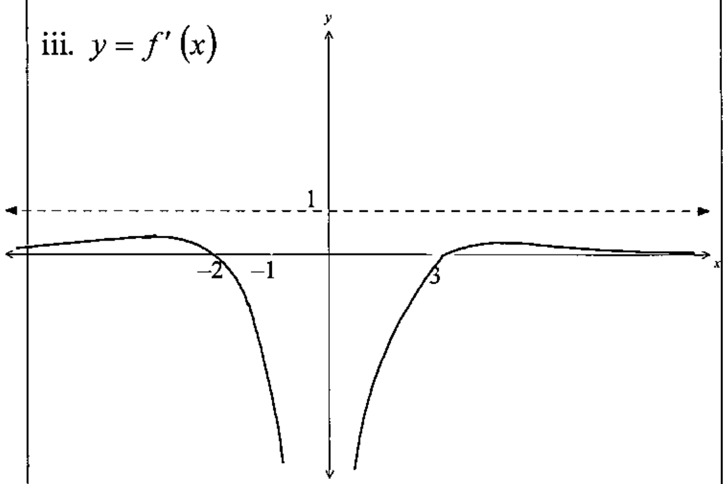
Question 1	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	$\int_{\frac{\pi}{16}}^{\frac{\pi}{12}} sec\,4x\,.tan\,4x\,dx = \left[\frac{sec\,4x}{4}\right]_{\frac{\pi}{16}}^{\frac{\pi}{12}}$	1	Use standard integral sheet
	$= \frac{1}{4}\left[sec\frac{\pi}{3} - sec\frac{\pi}{4}\right]$		
	$= \frac{1}{4}\left[2-\sqrt{2}\right]$	1	
	$= \frac{2-\sqrt{2}}{4}$		
b)	$\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2.\frac{1}{x}\,dx$	1	
	$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$		
	$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$	1	
c)	$\int \frac{9x^3+9x^2+5x+4}{3x+1}\,dx = \int\left(3x^2+2x+1+\frac{3}{3x+1}\right)dx$	2	By division
	$= x^3+x^2+x+ln\,(3x+1)+c$	1	
d)	i. $\frac{3x^2-2x-3}{(x^2+9)(x-3)} = \frac{ax+b}{x^2+9}+\frac{c}{x-3}$		
	$3x^2-2x-3=(ax+b)(x-3)+c(x^2+9)$	1	
	$x=3 \rightarrow 18=18c$		
	$\therefore c=1$		
	$x=0 \rightarrow -3=-3b+9$		Any fair method
	$\therefore b=4$		
	$x=1 \rightarrow -2=(a+4)(-2)+10$		
	$=-2a+2$		
	$\therefore a=2$		
	$\therefore a=2,\ b=4,\ c=1$	1	
	ii. $\int \frac{3x^2-2x-3}{(x^2+9)(x-3)}\,dx = \int\left(\frac{2x+4}{x^2+9}+\frac{1}{x-3}\right)dx$		
	$= \int\left(\frac{2x}{x^2+9}+\frac{4}{x^2+9}+\frac{1}{x-3}\right)dx$	1	
	$= \ln(x^2+9)+\frac{4}{3}\tan^{-1}\frac{x}{3}+\ln(x-3)+c$	1	Fully simplified not needed
	$= \ln(x^2+9)(x-3)+\frac{4}{3}\tan^{-1}\frac{x}{3}+c$		

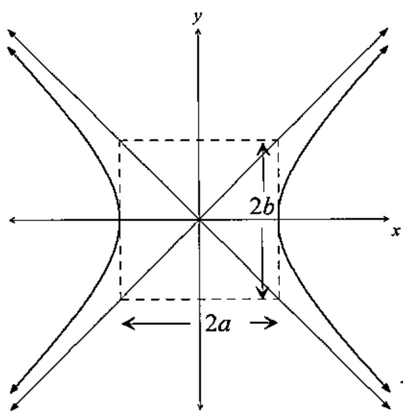
Question 1	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
e)	$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$ if $t=\tan\frac{\theta}{2}$ $\frac{dt}{d\theta}=\frac{1}{2}sec^2\frac{\theta}{2}$ $=\frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$ $=\frac{1}{2}\left(1+t^2\right)$ $\frac{d\theta}{dt}=\frac{2}{1+t^2}$ $d\theta=\frac{2\,dt}{1+t^2}$	1	
	$\theta=\frac{\pi}{2}, t=1$ $\theta=0, t=0$	1	
	$\int_0^1 \frac{\frac{2}{1+t^2}dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int_0^1\frac{2\,dt}{1+t^2+2t+1-t^2}$	1	
	$=\int_0^1 \frac{2\,dt}{2+2t}$ $=\left[\ln(1+t)\right]_0^1$ $=ln\,2-ln\,1$ $=ln\,2$	1	
		/15	

Question 2	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	i. $AB = (3+4i)(1-i)$ $= 3 - 3i + 4i + 4$ $= 7 + i$	1	
	ii. $\dfrac{A}{iB} = \dfrac{3+4i}{i(1-i)}$ $= \dfrac{3+4i}{1+i} \times \dfrac{1-i}{1-i}$ $= \dfrac{3-3i+4i+4}{2}$ $= \dfrac{7+i}{2} = \dfrac{7}{2} + \dfrac{1}{2}i$	1 1	
	iii. Let $\sqrt{A} = a + ib$ (a and b real) $\therefore A = a^2 - b^2 + 2abi$ $\therefore a^2 - b^2 = 3$, $2ab = 4$ $ab = 2$ $\therefore b = \dfrac{2}{a}$ $\therefore a^2 - \dfrac{4}{a^2} = 3$ $a^4 - 3a^2 - 4 = 0$ $(a^2-4)(a^2+1) = 0$ $\therefore a = \pm 2$ only real solution $\therefore$ b $= \pm 1$ $\therefore \sqrt{A} = \pm(2+i)$	1 1 1	Any fair method

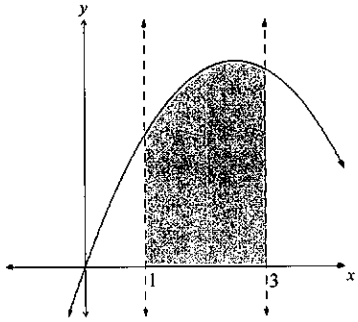
Question 2	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	i. $\lvert w\rvert = \sqrt{3+1} = 2$	1	
	$w = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$		
	If $Arg\ w = \theta$ $\quad cos\ \theta = \frac{\sqrt{3}}{2},\ sin\ \theta = -\frac{1}{2}$		
	$\therefore \theta = -\frac{\pi}{6}$		
	$\therefore\ Arg\ \ w = -\frac{\pi}{6}$	1	
	ii. $w = \sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$		
	$\therefore w^5 = 32\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$	1	
	$= 32\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$		
	$= -16\sqrt{3} - 16i$	1	
c)	i. $OB = iOA \times 2$ $= i(x+iy) \times 2$ $= -2y + 2ix$	1	
	ii. $OC = OB + OA$ $= (-2y + 2ix) + (x + iy)$ $= (x - 2y) + (2x + y)i$	1	
	iii. $BA = BO + OA$ $= -OB + OA$ $= -(-2y + 2ix) + (x + iy)$ $= (x + 2y) + (y - 2x)\,i$	1	0 if signs incorrect

d)		**2**	1 for region inside circle 1 for between lines
		/15	

Question 3	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	i. $y = \dfrac{1}{f(x)}$ 	2	2 marks each, deduct a mark for a major feature missing or incorrect, e.g. asymptotes not correct in (i)
	ii. $y = [f(x)]^2$ 	2	
	iii. $y = f'(x)$ 	2	

Question 3	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	i. $y = \frac{c^2}{x}$		
	$\frac{dy}{dx} = -\frac{c^2}{x^2}$, $at \quad x = ca \quad \frac{dy}{dx} = -\frac{1}{a^2}$		
	$\therefore$ Gradient of normal $= a^2$	1	
	$\therefore$ Equation of normal is $y - \frac{c}{a} = a^2(x - ca)$		
	$= a^2x - ca^3$		
	$y = a^2x + \frac{c}{a} - ca^3$		
	$y = a^2x + \frac{c}{a}(1 - a^4)$	1	
	ii. Solving $y = \frac{c^2}{x}$ with equation in (i)		
	$\frac{c^2}{x} = a^2x + \frac{c}{a}(1 - a^4)$		
	$\therefore a^2x^2 + \frac{c}{a}(1 - a^4)x - c^2 = 0$	1	
	Product of roots $= -\frac{c^2}{a^2}$	1	
	The roots are $x = cb \; and \; x = ca$		
	$\therefore c^2ab = -\frac{c^2}{a^2}$		
	$\therefore b = -\frac{1}{a^3}$	1	
	iii. Asymptotes become $y = \pm x$		
			
	Original Vertices $(\pm c, \pm c)$	1	Any logical reasoning
	Distance from O is $\sqrt{2}\,c$	1	
	$\therefore$ New vertices $= (\pm\sqrt{2}, 0)$		
	$\therefore a = \sqrt{2}\,c$	1	
	As it is rectangular $b = \sqrt{2}\,c$		
	$\therefore \frac{x^2}{2c^2} - \frac{y^2}{2c^2} = 1$		
	$\therefore x^2 - y^2 = 2c^2$	1	

Question 4	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	i. $a=2$, $b=1$ $\therefore$ Equation is $\frac{x^2}{4}+y^2=1$	1	
	ii. At $x=k$, $y^2=1-\frac{k^2}{4}$ $\therefore y=\pm\sqrt{\frac{4-k^2}{4}}$ $\therefore$ Length of side of triangle = $\sqrt{4-k^2}$	1	
	$\therefore$ Area = $\frac{1}{2}\sqrt{4-k^2}\,.\sqrt{4-k^2}\ \sin 60°$ $=\frac{1}{2}\left(4-k^2\right)\cdot\frac{\sqrt{3}}{2}$ $=\frac{\sqrt{3}}{4}\left(4-k^2\right)$	1	
	iii. Let slice thickness = δk $\therefore$ Volume of slice $\delta V=\frac{\sqrt{3}}{4}\left(4-k^2\right).\delta k$ $\therefore V=\int_{-2}^{2}\frac{\sqrt{3}}{4}\left(4-k^2\right)dk$	1	
	$=\frac{\sqrt{3}}{4}\left[4k-\frac{k^3}{3}\right]_{-2}^{2}$ $=\frac{\sqrt{3}}{4}\left[\left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)\right]$ $=\frac{\sqrt{3}}{4}\cdot\frac{32}{3}$ Volume = $\frac{8\sqrt{3}}{3}$ units ^{3}	1	

Question 4	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	 At $x = k$ $y = 5k - k^2$ Let thickness of shell be δk $\therefore$ Volume of shell		
	$\delta V = \pi\left(k^2 - (k - \delta k)^2\right)y$ $= \pi\left(2k\delta k - \delta k^2\right)y$	1	Can use formula
	As $\delta k \to 0$ $V = \int_1^3 2\pi . k . y\, dk$	1	
	$= 2\pi \int_1^3 k\left(5k - k^2\right) dk$	1	
	$= 2\pi\left[\frac{5}{3}k^3 - \frac{k^4}{4}\right]_1^3$		
	$= 2\pi\left[\left(45 - \frac{81}{4}\right) - \left(\frac{5}{3} - \frac{1}{4}\right)\right]$		
	Volume $= \frac{140\pi}{3}$ units ^{3}	1	

Question 4	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
c)	i. $X = x^2$ $\therefore x = \sqrt{X}$ $\therefore X\sqrt{X} - 3X + 9 = 0$ $X\left(\sqrt{X} - 3\right) = -9$ $\sqrt{X} - 3 = \frac{-9}{X}$ $\sqrt{X} = \frac{-9}{X} + 3$ $X = \frac{81}{X^2} - \frac{54}{X} + 9$ $X^3 = 81 - 54X + 9X^2$ Required equation is $x^3 - 9x^2 + 54x - 81 = 0$	1	Any method
	ii. from equation in (i) sum of roots is given by $\alpha^2 + \beta^2 + \chi^2 = \frac{-b}{a} = 9$	1	Any method
	Now, in original equation $\left.\begin{matrix}\alpha^3 - 3\alpha^2 + 9 = 0\\ \beta^3 - 3\beta^2 + 9 = 0\\ \chi^3 - 3\chi^2 + 9 = 0\end{matrix}\right\\}$ as $x = \alpha, \beta, \chi$ are roots Adding, $\alpha^3 + \beta^3 + \chi^3 - 3\left(\alpha^2 + \beta^2 + \chi^2\right) + 27 = 0$ $\alpha^3 + \beta^3 + \chi^3 - 3(9) + 27 = 0$ $\therefore \alpha^3 + \beta^3 + \chi^3 = 0$	1 1	
d)	Let $P(x) = (x-\alpha)^2 Q(x)$ $\therefore P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$ $= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$ $\therefore P'(x)$ has a single root at $x = \alpha$	1 1	
		/15	

Question 5	Trial HSC Examination - Mathematics Extension 2	2010

5(a)

Rotate around earth every 36 hours.

i) Period: $T = 36 \times 3600$

$= 129600$

$= 1.296 \times 10^5$ s.

$T = \frac{2\pi}{\omega}$

$\omega = \frac{2\pi}{T} = 4.84814 \times 10^{-5}$.

$F = k\frac{Mm}{r^2}$

at earth's surface, $r = R$, $F = mg$ — 1 mark.

$mg = \frac{kMm}{R^2}$

$kMm = R^2 mg$

so $F = \frac{R^2 mg}{r^2}$ — 1 mark.

ii) Now F is centripedal towards centre of earth

ie $F = mr\omega^2$

so $mr\omega^2 = \frac{R^2 mg}{r^2}$

$r^3 = \frac{R^2 mg}{m\omega^2} = \frac{R^2 g}{\omega^2}$

$= \frac{9.8 \times (6.4 \times 10^6)^2}{(4.8 \times 10^{-5})^2}$

$= 1.74 \times 10^{23}$ or 1.742×10^{23}

$\therefore r = 55827701.72$ m — 55851458 m

$\doteqdot 55828$ km — $= 55851$ km — 1 mark.

so height above earth's surface

$= 55851 - 6400 = 49451$ kms — 1 mark

or 55828 − 6400 = 49428 km

$\therefore$ satellite needs to be 49451 kms above earth's surface.

iii) $V = r\omega$ — 1 mark

$= 55851 \times 4.84813 \times 10^{-5}$

$= 2.7$ m/s

b)	Let $y = \frac{u(x)}{v(x)}$ $\therefore \ln y = \ln[u(x)] - \ln[v(x)]$ $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}$ $\frac{v(x)}{u(x)} \cdot \frac{dy}{dx} = \frac{u'(x)}{u(x)} - \frac{v'(x)}{v(x)}$ $\frac{dy}{dx} = \frac{u'(x)}{v(x)} - \frac{v'(x).u(x)}{[v(x)]^2}$ $\therefore \frac{dy}{dx} = \frac{v(x).u'(x) - u(x).v'(x)}{[v(x)]^2}$	1 1	
c)	i. $I_n = \int \tan^n x\, dx$ $= \int \tan^2 x . \tan^{n-2} x\, dx$ $= \int (\sec^2 x - 1) \tan^{n-2} x\, dx$ $= \int \sec^2 x \tan^{n-2} x\, dx - \int \tan^{n-2} x\, dx$ $= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$ ii. $\int_0^{\frac{\pi}{4}} \tan^3 x\, dx = \left[\frac{1}{2}\tan^2 x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, dx$ $= \frac{1}{2}[1-0] + [\ln(\cos x)]_0^{\frac{\pi}{4}}$ $= \frac{1}{2} + \left[\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right]$ $= \frac{1}{2} + \ln\frac{1}{\sqrt{2}}$	1 1 1 1 1 1 1	
		/15	

Question 6	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	i. $\ddot{x} = g - kv$	1	
	ii. $\ddot{x} = \frac{dv}{dt} = g - kv$ $\frac{dt}{dv} = \frac{1}{g - kv}$ $t = -\frac{1}{k}\ln\left(g - kv\right) + c$	1	
	When $t = 0$, $v = 0$ $\therefore c = \frac{1}{k}\ln g$ $\therefore t = \frac{1}{k}\ln g - \frac{1}{k}\ln\left(g - kv\right)$ $kt = \ln\left(\frac{g}{g - kv}\right)$	1	
	$e^{kt} = \frac{g}{g - kv}$ $ge^{kt} - kve^{kt} = g$ $ge^{kt} - g = kve^{kt}$ $g\left(e^{kt} - 1\right) = kve^{kt}$ $\therefore v = \frac{g}{k}\left(1 - e^{-kv}\right)$	1	

Question 6	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	iii. $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\cdot\frac{dv}{dx}$ $= v\frac{dv}{dx}$ $\therefore v\frac{dv}{dx} = g - kv$ $\frac{dv}{dx} = \frac{g-kv}{v}$ $\frac{dx}{dv} = \frac{v}{g-kv}$ $\frac{dx}{dv} = \frac{1}{k}\left[\frac{v}{\frac{g}{k}-v}\right] = \frac{1}{k}\left[\frac{v-\frac{g}{k}}{\frac{g}{k}-v} + \frac{\frac{g}{k}}{\frac{g}{k}-v}\right]$ $\frac{dx}{dv} = \frac{1}{k}\left[-1 + \frac{\frac{g}{k}}{\frac{g}{k}-v}\right]$ $\therefore x = \frac{1}{k}\left[-v - \frac{g}{k}\ln\left(\frac{g}{k}-v\right)\right] + c$ When $x = 0$, $v = 0$ $\therefore c = \frac{1}{k}\cdot\frac{g}{k}\ln\frac{g}{k}$ $\therefore x = \frac{1}{k}\left[\frac{g}{k}\ln\frac{g}{k} - v - \frac{g}{k}\ln\left(\frac{g}{k}-v\right)\right]$ $= \frac{1}{k}\left[\frac{g}{k}\ln\left(\frac{\frac{g}{k}}{\frac{g}{k}-v}\right) - v\right]$ $x = \frac{g}{k^2}\left[\ln\left(\frac{g}{g-kv}\right) - \frac{kv}{g}\right]$	1 1 1 1	

Question 6	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	i. $z^4 = \cos 4\theta + i\sin 4\theta$ also by expansion $= c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$ Where $c = \cos\theta$ and $s = \sin\theta$ $= c^4 + s^4 - 6c^2s^2 + 4i\left(c^3s - cs^3\right)$ Equating real parts $\cos 4\theta = \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta$ $= \cos^4\theta + (1-\cos^2\theta)^2 - 6\cos^2\theta(1-\cos^2\theta)$ $= \cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta$ $= 8\cos^4\theta - 8\cos^2\theta + 1$	1 1 1	
	ii. $\cos 4\theta = -1$ (Equating real parts) $\therefore 4\theta = \pi, -\pi, 3\pi, -3\pi$ $\theta = \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}$ $\therefore$ Roots are $cis\frac{\pi}{4}, cis\left(-\frac{\pi}{4}\right), cis\frac{3\pi}{4}, cis\left(-\frac{3\pi}{4}\right)$	1 1	
	iii. $z + \frac{1}{z} = 2\cos\theta$ $\left(z+\frac{1}{z}\right)^4 = 16\cos^4\theta$ LHS $= z^4 + 4z^3.\frac{1}{z} + 6z^2.\frac{1}{z^2} + 4z.\frac{1}{z^3} + \frac{1}{z^4}$ $= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ $\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ $\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$	1 1	
		/15	

Question 7	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)	i. $\cos\left[(k-1)\theta\right]-2\cos\theta\cos k\theta$		
	$=\cos k\theta\cos\theta+\sin k\theta\sin\theta-2\cos\theta\cos k\theta$		
	$=-\left(\cos k\theta\cos\theta-\sin k\theta\sin\theta\right)$		
	$=-\cos\left(k\theta+\theta\right)$		
	$=-\cos\left[(k+1)\theta\right]$	1	
	ii. When $n=1$		
	LHS = 1 RHS $=\dfrac{1-\cos\theta-\cos\theta+\cos 0}{2-2\cos\theta}$		
	$=\dfrac{2-2\cos\theta}{2-2\cos\theta}=1=LHS$	1	
	$\therefore$ True for $n=1$		
	Assume true for $n=k$.		
	i.e.		
	$1+\cos\theta+.....+\cos\left[(k-1)\theta\right]=\dfrac{1-\cos\theta-\cos k\theta+\cos\left[(k-1)\theta\right]}{2-2\cos\theta}$		
	When $n=k+1$		
	$1+\cos\theta+.......\cos\left[(k-1)\theta\right]+\cos k\theta$		
	$=\dfrac{1-\cos\theta-\cos k\theta+\cos\left[(k-1)\theta\right]}{2-2\cos\theta}+\cos k\theta$	1	
	$=\dfrac{1-\cos\theta-\cos k\theta+\cos\left[(k-1)\theta\right]+2\cos k\theta-2\cos\theta\cos k\theta}{2-2\cos\theta}$		
	$=\dfrac{1-\cos\theta+\cos k\theta-\cos\left[(k+1)\theta\right]}{2-2\cos\theta}$		
	$=\dfrac{1-\cos\theta-\cos\left[(k+1)\theta\right]+\cos\left[((k+1)-1)\theta\right]}{2-2\cos\theta}$	1	
	If true for $n=1$ then true for $n=1+1=2$ etc		
	$\therefore$ By induction true for all n positive integers.	1	

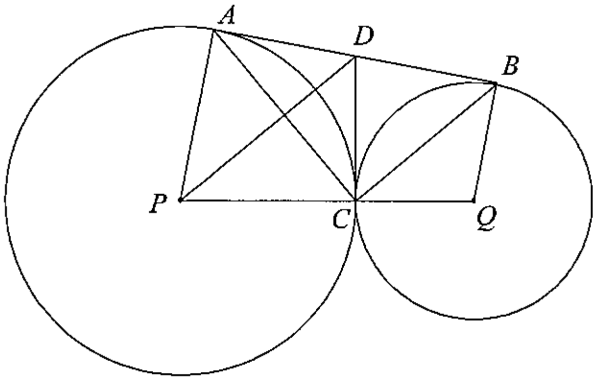
Question 7	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	Let Pulling force = F and distance pulled up be x metres. $\frac{dF}{dx} = -10$ $F = -10x + c$ When $x = 0$, $F = 250$ $\therefore F = 250 - 10x$ At the mass $20\ddot{x} = F - 20g$ $= 250 - 10x - 200$ $= 50 - 10x$ $\therefore \ddot{x} = \frac{5}{2} - \frac{x}{2}$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{5}{2} - \frac{x}{2}$ $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{4} + c$ When $x = 0$, $v = 0$, $\therefore c = 0$ $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{4}$ $v^2 = 5x - \frac{x^2}{2}$ When $x = 10$, $v^2 = 0$ $\therefore$ The mass is stationary.	 1 1 1	
c)	When dividing by $(x-1)(x-2)$ the remainder is in the form $ax + b$. $P(x) = (x-1)(x-2)\,Q(x) + ax + b$ $P(1) = a + b = 3$ $P(2) = 2a + b = 5$ $\therefore a = 2,\ b = 1$ $\therefore$ Remainder is $2x + 1$	 1 1 1	

7(d)
A A B C D E E $n=7$ $2 \times A$ $2 \times E$

i. using all 7 letters.
number $= \frac{7!}{2! \times 2!} = 1260$ — 1 mark correct answer.

ii) using 4 letters:
1) all different: ${}^5C_4 \times 4! = 120$
2) 2A's, no E's. ${}^3C_2 \times \frac{4!}{2!} = 36$
3) 2A's, 1 E ${}^3C_1 \times \frac{4!}{2!} = 36$
4) 2A's, 2E's $\frac{4!}{2! \times 2!} = 6.$
5) 2E's, no A ${}^3C_2 \times \frac{4!}{2!} = 36$
6) 2E's, 1 A ${}^3C_1 \times \frac{4!}{2!} = 36$

Total number = 270. — 3 marks correct answer

Question 8	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
a)			
	i. $\angle PAD = \angle DCP = 90°$ (Radius is perpendicular to tangent at point of contact) $\therefore PADC$ is cyclic (Opposite angles supplementary) Similar for $CDBQ$	2	
	ii. Let $\angle ADC = \theta$ $\therefore \angle BQC = \theta$ (Ext. angle of cyclic quadrilateral) $DA = DC$ (Equal Tangents) $\therefore \Delta ADC$ is isosceles $\therefore \angle DAC = \angle DCA = \left(90 - \frac{\theta}{2}\right)^\circ$ $BQ = CQ$ (Equal radii) $\therefore \Delta BQC$ is isosceles $\therefore \angle BCQ = \angle CBQ = \left(90 - \frac{\theta}{2}\right)^\circ$ $\therefore \Delta ADC \parallel\!\mid \Delta BQC$ (AAA)	2	Any correct proof
	iii. From above $\angle APC = 180 - \theta$ (opposite $\angle$ of $PADC$) $\angle PDC = \left(90 - \frac{\theta}{2}\right)^\circ$ (PD bisects $\angle APC$) $= \angle BCQ$ (from (ii)) $\therefore PD \parallel CB$ (corresponding $\angle$ equal)	2	

Question 8	Trial HSC Examination - Mathematics Extension 2		2010
Part	**Solution**	**Marks**	**Comment**
b)	i. $P\sin\frac{\theta}{2}$		
	$=(1+2\cos\theta+2\cos 2\theta+2\cos 3\theta)\sin\frac{\theta}{2}$		
	$=\sin\frac{\theta}{2}+2\cos\theta\sin\frac{\theta}{2}+2\cos 2\theta\sin\frac{\theta}{2}+2\cos 3\theta\sin\frac{\theta}{2}$	1	
	$\therefore =\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}-\sin\frac{3\theta}{2}+\sin\frac{7\theta}{2}-\sin\frac{5\theta}{2}$	1	
	$=\sin\frac{7\theta}{2}$	1	
	ii. From (i) $P\sin\frac{\theta}{2}=\sin\frac{7\theta}{2}$		
	When $\theta=\frac{2\pi}{7}$		
	$P\sin\frac{\pi}{7}=\sin\pi$ $=0$	1	
	As $\sin\frac{\pi}{7}\neq 0$ then $P=0$		
	i.e. $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta=0$	1	
	iii. $P=1+2\cos\theta+2(2\cos^2\theta-1)+2(4\cos^3\theta-3\cos\theta)$	1	
	$=8\cos^3\theta+4\cos^2\theta-4\cos\theta-1$	1	
	$P=8x^3+4x^2-4x-1$ when $x=\cos\theta$	1	
	From (ii) $P=0$ when $\theta=\frac{2\pi}{7}$		
	$\therefore x=\cos\frac{2\pi}{7}$ is a solution.	1	
		/15	