International Grammar School

Year 12
Mathematics Extension 2
HSC Trial Examination
2011

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

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Total Marks – 120

Attempt Questions 1 - 8

All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) **Marks**

a) Evaluate:

i) By completing the square, find $\int \frac{2}{x^2+4x+13}dx$ **2**

ii) Use integration by parts to evaluate $\int 3xe^x\,dx$. **2**

iii) Evaluate $\int_0^1 xe^{-x^2}\,dx$ **2**

b) Use the substitution $t = tan\frac{\theta}{2}$ to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta+2\sin\theta+3}d\theta$$

Answer correct to 3 significant figures.

c) i) Find real numbers *a, b* and *c* such that:

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)} = \frac{ax+b}{x^2+1}+\frac{c}{x+2}$$ **3**

ii) Hence find $\int \frac{7x+4}{\left(x^2+1\right)(x+2)}dx$ **2**

End of Question 1.

Question 2 (15 marks) Use a separate writing booklet **Marks**

a) Given $A = 3 - 4i$ and $B = 5 + 3i$, express the following in the form $x + iy$, where x and y are real numbers.

i) $B - A$ **1**

ii) $\overline{AB}$ **2**

iii) $\dfrac{A}{B}$ **2**

iv) $\sqrt{A}$ **2**

b) If $z = 1 - \sqrt{3}\, i$,

i) Express z in mod-arg form. **1**

ii) Show that z^6 is an integer. **2**

c) On the Argand diagram, sketch the region where the inequalities $2 \le |\, z \,| \le 5$ and $\arg \dfrac{\pi}{6} < \arg \dfrac{2\pi}{3}$ hold simultaneously. **3**

d) The points A and B are drawn on an Argand Diagram and are represented by the lines p and q respectively. **2**

Copy this diagram into your answer booklet.

On this diagram plot the points $C(-q)$ and $D(p-q)$.

End of Question 2.

Question3 (15 marks) Use a separate writing booklet **Marks**

a) i) Sketch the curve $f(x) = (x+1)(x-2)(x+3)$ showing the intercepts with the coordinate axes. **2`**

ii) On the same diagram, sketch the graph of $y = x - f(x)$ **2**

iii) The area bounded by $y = f(x)$, the x – axis and the ordinates $x = -3$ and $x = -1$ is rotated about the y – axis. **3**

Use cylindrical shells to find the volume of the solid of revolution formed.

b) A particle of unit mass moves in a straight line against a resistance equal to $v + v^2$ where v is its velocity. Initially the particle is at the origin and is travelling with velocity q where $q > 0$.

i) Show that v is related to displacement by the formula **2**
$x = -\ln(1+v) + c$

ii) Show that the time t which has elapsed when the particle is travelling with velocity v is given by: **3**

$$t = \ln \frac{q(1+v)}{v(1+q)}$$

iii) Find v as a function of t. **2**

iv) Find the value of v as $t \to \infty$. **1**

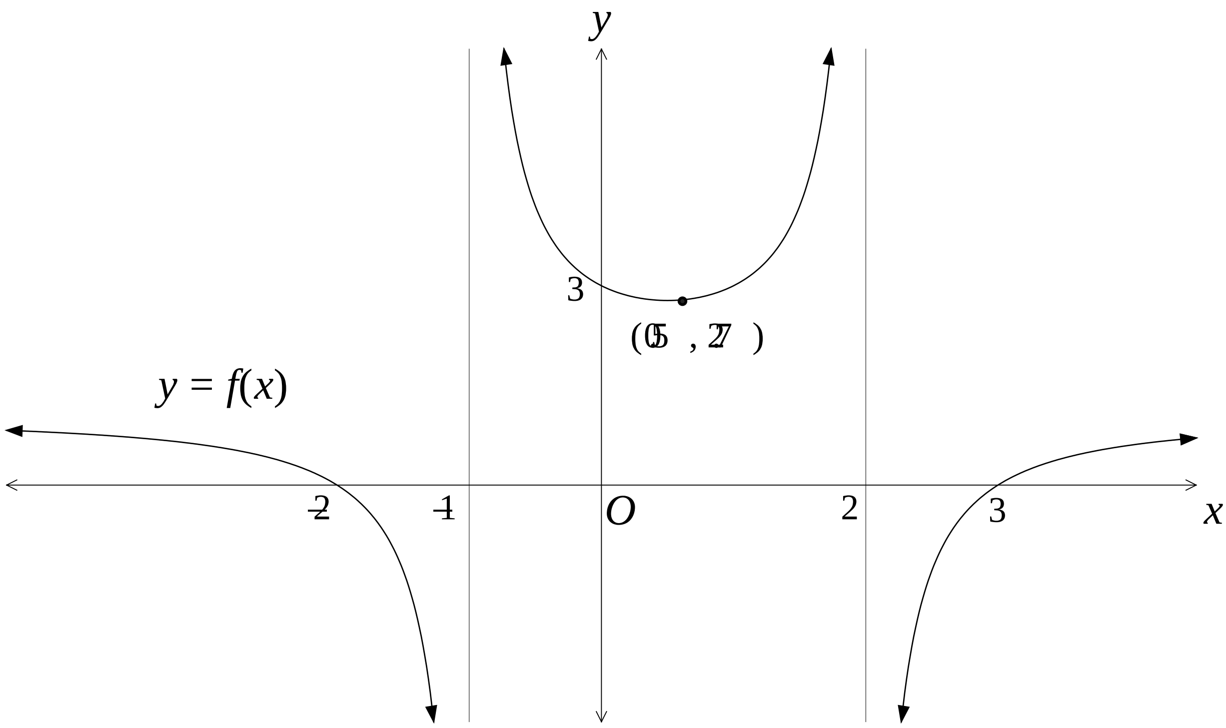
End of Question 3.

Question 4 (15 marks) Use a separate writing booklet **Marks**

a) The equation $x^4 + 2x^3 - 7x^2 - 20x - 12 = 0$ has a double root. Find this root and hence solve this equation. **3**

b) The equation $x^3 - 4x^2 + 2x - 7 = 0$ has roots α , β *and* γ .Find an equation which has roots $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ *and* $\dfrac{1}{\gamma}$. **2**

c) The graph of $y = f(x)$ is shown below.



On separate axes draw sketches of the following, showing any critical features.

i) $y = \dfrac{1}{f(x)}$

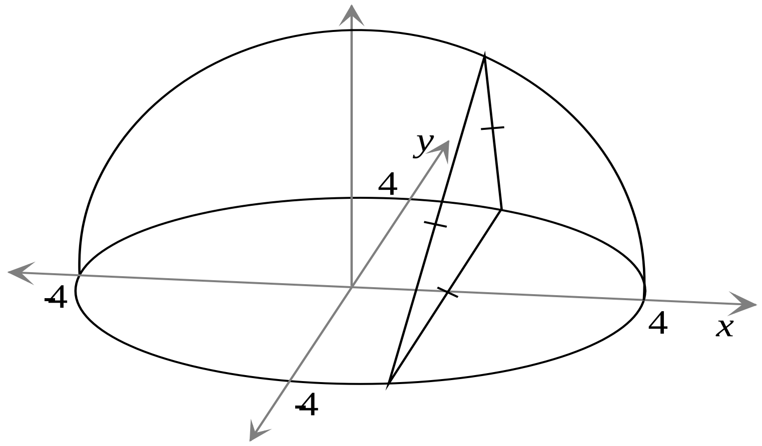
ii) $y = f'(x)$

iii) $y = \pm\sqrt{f(x)}$

Question 4 continues on the next page

Question 4 continued.

d) **4**



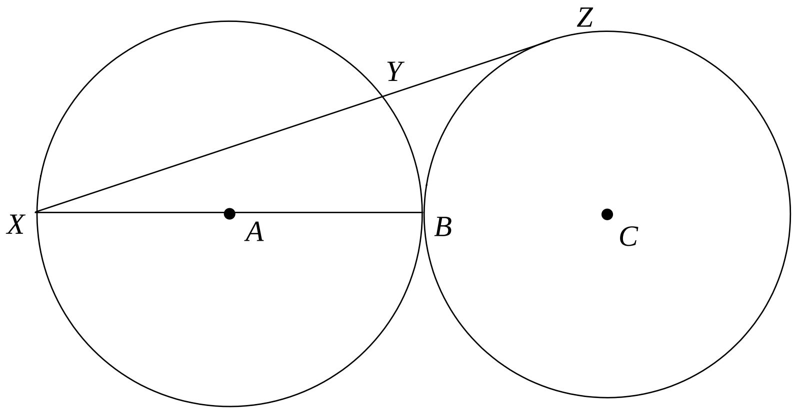
The diagram above shows a solid which has the circle $x^2 + y^2 = 16$ as its base. The cross-section perpendicular to the x axis is an equilateral triangle. Calculate the volume of the solid.

End of Question 4.

Question 5 (15 marks) Use a separate writing booklet **Marks**

a) The cubic $y = x^3$ is rotated about the *y* axis $\{x : 0 \le x \le 2\}$ to form a solid. Calculate the volume of this solid using the method of slicing. **3**

b) **3**



Two equal circles touch externally at *B*. *XB* is a diameter of one circle. *XZ* is the tangent from *X* to the other circle and cuts the first circle at *Y.*
Prove that $2XZ = 3XY$.

c) Use the principle of Mathematical induction to prove that: **3**

$$\frac{d}{dx}\left[x^2+1\right]^n = 2xn\left[x^2+1\right]^{n-1} \text{ for } n \ge 1,\ n \in \mathbb{Z}.$$

Note： You should not use the function of a function rule (chain rule) as part of your proof.

d) i) Derive the reduction formula **2**

$$\int x^m (\ln x)^n \, dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} \, dx$$

ii) Hence or otherwise, find $\int_1^e x^3 (\ln x)^3 \, dx$ **4**

End of Question 5.

Question 6 (15 marks) Use a separate writing booklet **Marks**

a) A hyperbola has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

i) Verify that the point P$\left(a\ sec\theta, b\ tan\theta\right)$ lies on the hyperbola. **2**

ii) The normal to the hyperbola at P cuts the *x*-axis at M and N is the foot of the perpendicular from P to the *x*-axis. Prove that the equation of the normal at P is: **3**

$ax\ sin\ \theta + by = \left(a^2 + b^2\right) tan\ \theta$.

iii) Show that $OM = e^2 ON$, where *O* is the origin and *e* is the eccentricity of the hyperbola. **3**

iv) Prove that $SM = e \times SP$, where S is the focus of the hyperbola. **3**

b) Find the equation of the tangent to the curve $x^2 y + 2x - 2xy = 0$ at the point (1, 2). **2**

c) Given that $3 + i$ is a root of $P(z) = z^3 + az^2 + bz + 10$, where *a* and *b* are real numbers, factorise $P(z)$ over real numbers. **2**

End of Question 6.

Question 7 (15 marks) Use a separate writing booklet **Marks**

a) A local council consists of 6 independents and 5 others aligned to political parties. A committee of 5 members is to be chosen at random.

i) How many committees of 5 can be chosen? **1**

ii) How many of these committees will have a majority of independents? **2**

b) The equation $x^3 - 3x^2 + ax + 8 = 0$ has roots that are in arithmetic sequence. Find the value of a and hence solve the equation. **4**

c) i) Differentiate $\sin^{-1} x - \sqrt{1-x^2}$ **2**

ii) Hence show that $\int_0^a \sqrt{\frac{1+x}{1-x}}dx = \sin^{-1} a + 1 - \sqrt{1-a^2}$ for $0 < a < 1$. **1**

d) Given that $sin^{-1}x, cos^{-1}\mathrm{x}$ and $sin^{-1}(1-x)$ are acute:

i) Show that: **3**

$$\sin(sin^{-1}x - cos^{-1}x) = 2x^2 - 1.$$

ii) Solve: **2**

$$sin^{-1}x - cos^{-1}x = sin^{-1}(1-x)$$

End of Question 7

Question 8 (15 marks) Use a separate writing booklet **Marks**

a) i) Write expressions for *sin* $(\alpha - \beta)$ and *cos* $(\alpha - \beta)$ and hence show that: **3**

$$tan\,(\alpha - \beta) = \frac{tan\,\alpha - tan\,\beta}{1 + tan\,\alpha\; tan\,\beta}$$

ii) Hence write an expression for $tan\left(\alpha - \frac{\pi}{3}\right)$ in terms of $tan\,\alpha$. **1**

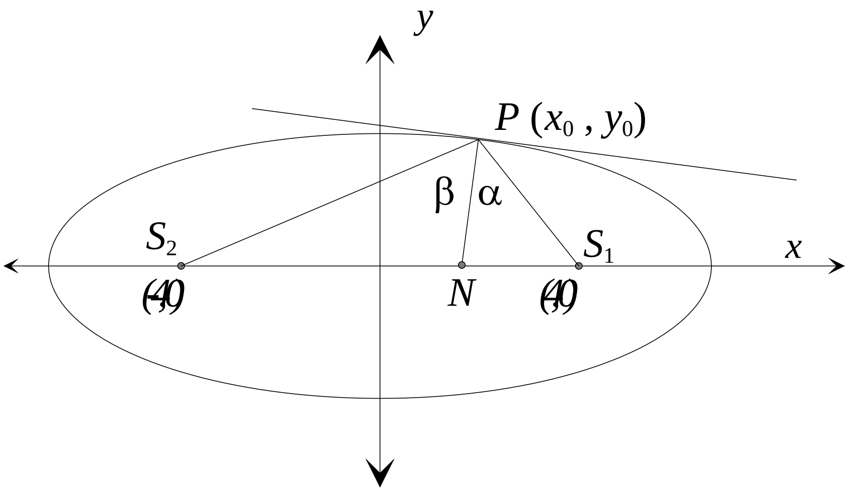
b) Given $f(x) = x - \log_e(1 + x^2)$

i) Show that $f'(x) \geq 0$ for all values of x. **3**

ii) Hence deduce that $e^x > 1 + x^2$ for all positive values of x. **3**

c) i) Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(x_0\ ,\ y_0)$ has equation: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$. **2**

ii)



3

In the diagram above, the line *PN* is the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at $P(x_0\ ,\ y_0)$ and S_1 and S_2 are the foci of the ellipse. $\angle NPS_1 = \alpha$ and $\angle NPS_2 = \beta$. Show that $\alpha = \beta$.

End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

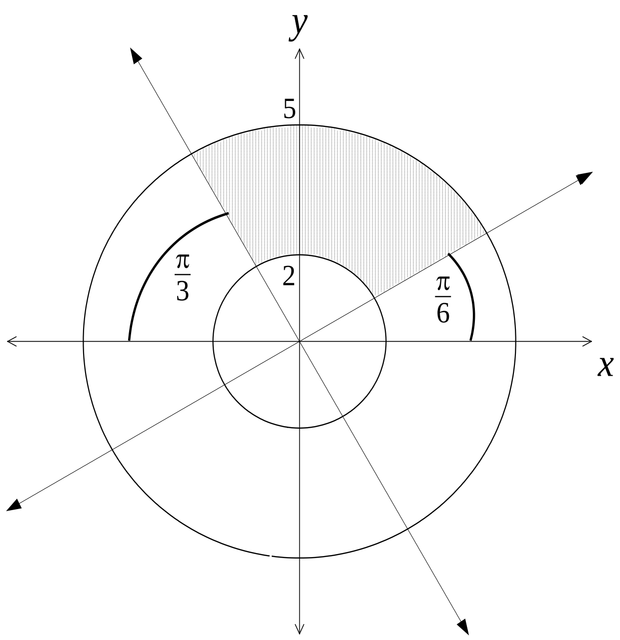
$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1	Trial HSC Examination - Mathematics Extension 2	2011
	Solution	**Criteria**
1(a) (i)	$\int \frac{2}{x^2+4x+13}dx = 2\int \frac{dx}{(x+2)^2+3^2}$ $= \frac{2}{3}\tan^{-1}\frac{(x+2)}{3}+c$	2 Marks: Correct answer. 1 Mark: Correctly completes the square
1(a) (ii)	$\int 3xe^x dx = 3\int x\frac{d}{dx}(e^x)dx$ $= 3(xe^x - \int e^x dx)$ $= 3xe^x - 3e^x + c$	2 Marks: Correct answer. 1 Mark: Set up of the integration by parts.
1(a) iii)	$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}dx$ $= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ $= -\frac{1}{2}(e^{-1}-e^0)$ $= \frac{1}{2}(1-\frac{1}{e})$ $= \frac{e-1}{2e}$	2 Marks: Correct answer. 1 Mark: Integrates correctly
1(b)	$t = \tan\frac{\theta}{2}$ $dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$ $dt = \frac{1}{2}(1+t^2)d\theta$ $d\theta = \frac{2}{1+t^2}dt$ When $\theta = 0$ then $t = 0$ and when $\theta = \frac{\pi}{2}$ then $t = 1$ $\cos\theta + 2\sin\theta + 3 = \frac{1-t^2+2(2t)+3(1+t^2)}{1+t^2}$ $= \frac{2(t^2+2t+2)}{1+t^2}$ $= \frac{2[1+(t+1)^2]}{1+t^2}$	4 Marks: Correct answer. 3 Marks: Correctly determines the primitive function 2 Marks: Correctly expresses the integral in terms of t 1 Mark: Correctly finds $d\theta$ in terms of dt and

Question 1	Trial HSC Examination - Mathematics Extension 2		2011
	$\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + 2\sin\theta + 3} d\theta = \int_0^1 \frac{1+t^2}{2[1+(t+1)^2]} \times \frac{2}{1+t^2} dt$ $= \int_0^1 \frac{1}{1+(t+1)^2} dt$ $= \left[\tan^{-1}(t+1)\right]_0^1$ $= \tan^{-1} 2 - \frac{\pi}{4}$ $= 0.322$		determines the new limits.
1(c) (i)	$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$ $7x+4 = (ax+b)(x+2) + c(x^2+1)$ Let $x=-2$ and $x=0$ $-10 = 5c$ $\quad$ $4 = b(0+2) - 2(0^2+1)$ $c=-2$ $\quad$ $b=3$ Equating the coefficients of x^2 $0 = a-2$ $a=2$ $\therefore a=2,\ b=3$ and $c=-2$		3 Marks: Correct answer. 2 Marks: Calculates two of the variables 1 Mark: Makes some progress in finding *a*,*b* or *c*.
1(c) (ii)	$\int \frac{7x+4}{\left(x^2+1\right)(x+2)} dx = \int \left(\frac{2x+3}{\left(x^2+1\right)} - \frac{2}{(x+2)}\right) dx$ $= \int \left(\frac{2x}{\left(x^2+1\right)} + \frac{3}{\left(x^2+1\right)} - \frac{2}{(x+2)}\right) dx$ $= \ln(x^2+1) + 3\tan^{-1} x - 2\ln\lvert x+2 \rvert + c$		2 Marks: Correct answer. 1 Mark: Correctly finds one of the integrals.
		/15	

Question 2	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	i. $B - A = (5+3i) - (3-4i)$ $= 5 + 3i - 3 + 4i$ $= 2 + 7i$	**1**	
	ii. $\overline{AB} = \overline{(3-4i)(5+3i)}$ $= \overline{15 + 9i - 20i + 12}$	**1**	Expansion
	$= \overline{27 - 11i}$ $= 27 + 11i$	**1**	Conjugate
	iii. $\dfrac{A}{B} = \dfrac{3-4i}{5+3i}$ $= \dfrac{3-4i}{5+3i} \times \dfrac{5-3i}{5-3i}$ $= \dfrac{15 - 9i - 20i - 12}{25+9}$	**1**	Realising denominator
	$= \dfrac{3-29i}{34} = \dfrac{3}{34} - \dfrac{29}{34}i$	**1**	Answer
	iv. Let $\sqrt{A} = x + iy$ (a and b real) $\therefore A = x^2 - y^2 + 2xyi$ $\therefore x^2 - y^2 = 3$ --------------------(1) $2xy = -4$ $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$ $= 3^2 + 4^2$ $= 25$ $x^2 + y^2 = 5$ -------------------- (2)	**1**	Any fair method
	(1) + (2) $2x^2 = 8 \rightarrow x = \pm 2$ $(2) - (1)$ $2y^2 = 2 \rightarrow y = \pm 1$ Since $2xy = -4$ $\sqrt{A} = \pm(2 - i)$	**1**	Answer

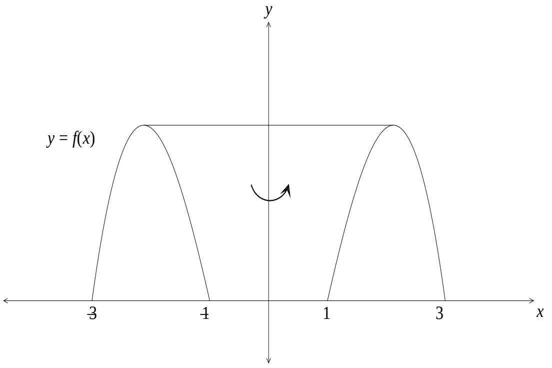
Question 2	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
b)	i. $r=\sqrt{1+3}=2$ $\tan\theta=\frac{-\sqrt{3}}{1}=-\sqrt{3}$ $\therefore \theta=-\frac{\pi}{3}$ $1-\sqrt{3}\,i=2\,cis\left(-\frac{\pi}{3}\right)$	**1**	Mod-Arg Form
	ii. $z=1-\sqrt{3}\,i=2\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]$ $\therefore z^6=64\left[\cos\left(-\frac{6\pi}{3}\right)+i\sin\left(-\frac{6\pi}{3}\right)\right]$ $=64\,cis\,(-2\pi)$ $=64$	**1** **1**	Use of De Moivre's Solution
c)		**1** **1** **1**	Circles Rays Correct Region

Question 2	**Trial HSC Examination - Mathematics Extension 2**		**2011**
Part	**Solution**	**Marks**	**Comment**
d)	D C	**2**	1 each for correctly plotting C and D
		/15	

Question 3	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	i. $f(x)=(x+1)(x-2)(x+3)$		
		1	Intercepts
		1	Correct shape
	-6		
	-6		
	ii. $y=x-f(x)$		
		2	Deduct a mark for a major feature missing or incorrect,

Question 3	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**

iii.



$$f(x) = (x+1)(x-2)(x+3)$$
$$= (x^3 + 2x^2 - 5x - 6)$$

By cylindrical shells

$$V = \int_a^b 2\pi xy\ dx$$

$$V = \int_1^3 2\pi x\left(x^3 + 2x^2 - 5x - 6\right)dx$$ **1**

$$= 2\pi \int_1^3 \left(x^4 + 2x^3 - 5x^2 - 6x\right)dx$$

$$= 2\pi\left[\frac{x^5}{5} + \frac{x^4}{2} - \frac{5x^3}{3} - 3x^2\right]_1^3$$ **1**

$$= 2\pi\left[\left(17\frac{1}{10}\right) - \left(-3\frac{29}{30}\right)\right]$$

$$= \frac{632\pi}{15}$$ **1**

b) i.

$$\ddot{x} = -\left(v + v^2\right)$$

$$v\frac{dv}{dx} = -\left(v + v^2\right)$$

$$\frac{dv}{dx} = \frac{-\left(v + v^2\right)}{v}$$

$$\frac{dx}{dv} = \frac{-1}{1+v}$$ **1**

$$x = -\int \frac{1}{1+v}\ dv$$

$$x = -\ell n\left(1+v\right) + c$$ **1**

Question 3	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**

ii.

$$\frac{dv}{dt} = -\left(v + v^2\right)$$

$$\frac{dt}{dv} = -\frac{1}{\left(v + v^2\right)}$$

$$t = -\int \frac{dv}{\left(v + v^2\right)}$$

$$t = -\int \frac{dv}{v(1+v)} \qquad \frac{1}{v(1+v)} = \frac{A}{v} + \frac{B}{1+v}$$

$$t = -\int \left(\frac{1}{v} - \frac{1}{1+v}\right) dv \qquad 1 = A(1 + v) + B(v)$$

$$t = \int \left(\frac{1}{(1+v)} - \frac{1}{v}\right) dv \qquad v = 0 \quad A = 1$$

$$t = \ell n\,(1+v) - \ell n\,(v) + c \qquad 1 = 1 + v + Bv$$

1 — Expression for t

$$t = \ell n \left(\frac{1+v}{v}\right) + c \qquad Bv = -v \quad \therefore \; B = -1$$

When $t = 0$, $v = q$

$$0 = \ell n \left(\frac{1+q}{q}\right) + c$$

$$c = -\ell n \left(\frac{1+q}{q}\right)$$

$$c = \ell n \left(\frac{q}{1+q}\right)$$

1 — *Value of* c

$$\therefore \; t = \ell n \left(\frac{1+v}{v}\right) + \ell n \left(\frac{q}{1+q}\right)$$

$$t = \ell n \left(\frac{q\,(1+v)}{v\,(1+q)}\right)$$

1 — Solution

iii. Now $e^{t} = \dfrac{q\,(1+v)}{v\,(1+q)}$

$$v\,(1+q)\,e^{t} = q + qv$$

1

$$v\,e^{t} + qve^{t} = q + qv$$

$$v\,e^{t} + qve^{t} - qv = q$$

$$v\left(e^{t} + qe^{t} - q\right) = q$$

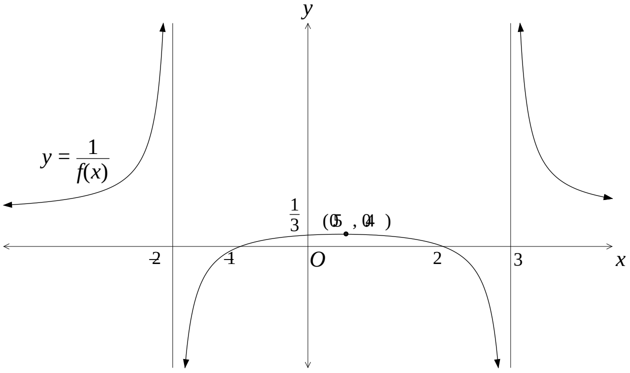
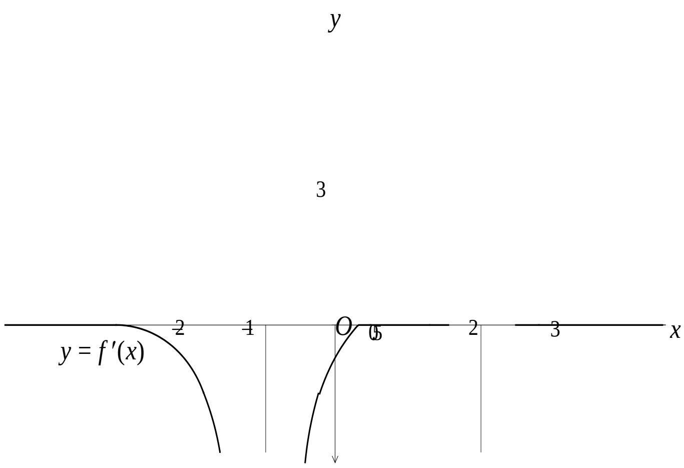
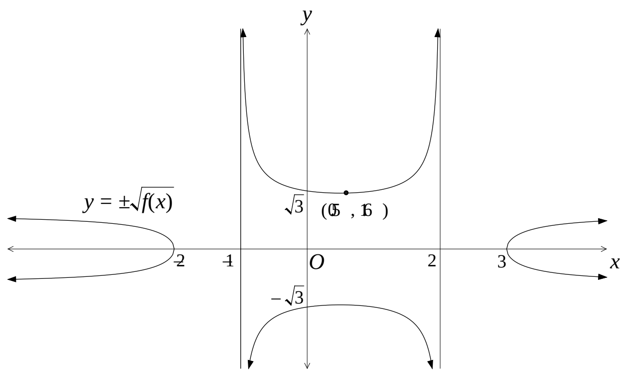
1

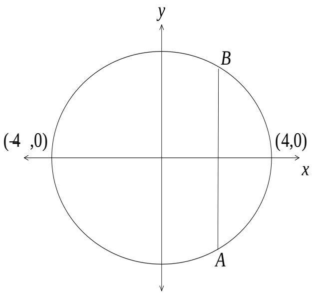
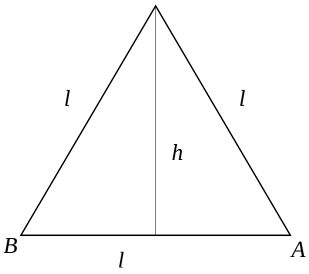
$$v = \frac{q}{e^{t} + qe^{t} - q}$$

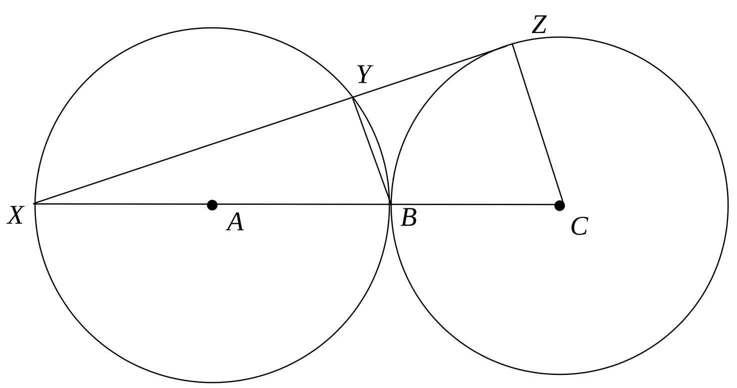
1 — Value

iv. *As* $t \to \infty, v \to 0$

Question 4	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	i. $x^4 + 2x^3 - 7x^2 - 20x - 12 = 0$ If $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12$ Double root then $f'(x) = 4x^3 + 6x^2 - 14x - 20 = 0$ Test roots of $f'(x)$ Since $f'(-2) = f(-2) = 0$, there is a double root at $x = -2$.	**1**	Double root
	Therefore $f(x)$ is divisible by $(x+2)^2$ i.e. $x^2 + 4x + 4$ By division, $f(x) = (x^2 + 4x + 4)(x^2 - 2x - 3)$ $= (x+2)^2 (x-3)(x+1)$	**1**	Division
	i.e. Solutions $x = -2, -2, 3, -1$	**1**	Solution
b)	$x^3 - 4x^2 + 2x - 7 = 0$ For roots of $\frac{1}{\alpha}, \frac{1}{\beta}$ *and* $\frac{1}{\gamma}$ $\quad x = \frac{1}{X}$ $\therefore \left(\frac{1}{X}\right)^3 - 4\left(\frac{1}{X}\right)^2 + 2\left(\frac{1}{X}\right) - 7 = 0$ $\frac{1}{X^3} - \frac{4}{X^2} + \frac{2}{X} - 7 = 0$ $1 - 4X + 2X^2 - 7X^3 = 0$	**1**	Substitution
	ie. Equation is $7x^3 - 2x^2 + 4x - 1 = 0$	**1**	Equation

Question 4	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
c)	i. $y = \dfrac{1}{f(x)}$ 	**2 Each**	Deduct a mark for a major feature missing or incorrect, e.g. asymptotes not correct
	ii. $y = f'(x)$ 	**2**	
	iii. $y = \pm\sqrt{f(x)}$ 	**2**	

Question 4	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
d)	 		
	$x^2+y^2=16$ $y=\sqrt{16-x^2}$ $\therefore l=2\sqrt{16-x^2}$		
	$sin\,60=\frac{h}{l}$ $h=l\,sin\,60$ $h=\frac{\sqrt{3}}{2}l$		
	$\therefore h=\sqrt{3}\sqrt{16-x^2}$	**1**	Expression for *h*
	$A(x)=\frac{1}{2}bh$ $=\frac{1}{2}\left(2\sqrt{16-x^2}\right)\left(\sqrt{3}\sqrt{16-x^2}\right)$ $=\sqrt{3}\left(16-x^2\right)$	**1**	Area
	$V=\int_{-4}^{4}\sqrt{3}\left(16-x^2\right)dx$	**1**	Integral
	$=\sqrt{3}\left[16x-\frac{x^3}{3}\right]_{-4}^{4}$ $=\frac{256\sqrt{3}}{3}$	**1**	Answer
		/15	

Question 5	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
5(a)	Area of the slice is a circle radius is x and height δy $A = \pi x^2$ $= \pi(y^{\frac{1}{3}})^2$ $= \pi y^{\frac{2}{3}}$ $\delta V = \delta A.\delta y$ $V = \lim_{\delta y \to 0} \sum_{y=0}^{8} \pi y^{\frac{2}{3}} \delta y$ $= \int_0^8 \pi y^{\frac{2}{3}} dy$ $= \pi \left[\frac{3}{5} y^{\frac{5}{3}} \right]_0^8$ $= \frac{3\pi}{5} \times 8^{\frac{5}{3}}$ $= \frac{96\pi}{5}$ cubic units		3 Marks: Correct answer. 2 Marks: Correct integral for the volume of the solid. 1Mark: Correct expression for the volume of the solid.
5(b)	 Construction: Join *BY*, produce *XB* to *C*, join *CZ*. Proof: $\angle XYB = 90^0$ (angle in a semicircle is a right angle) $\angle XZC = 90^0$ (angle between tangent and radius is a right angle) $BY \parallel CZ$ (corresponding angles are equal) $\Delta XYB \parallel\mid \Delta XZC$ (equiangular) $\frac{XY}{XZ} = \frac{XB}{XC}$ (corresponding sides of similar triangles) However $\frac{XB}{XC} = \frac{2}{3}$ ($BC = \frac{1}{2} XB$) $\frac{XY}{XZ} = \frac{2}{3}$ $\therefore 2XZ = 3XY$		3 Marks: Correct answer. 2 Marks: Makes significant progress towards the proof. 1 Mark: States a relevant circle theorem property or equivalent statement.

Question 5	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
c)	Prove $\frac{d}{dx}\left[x^2+1\right]^n = 2xn\left[x^2+1\right]^{n-1}$ Test $n = 1$ $\frac{d}{dx}\left[x^2+1\right]^1 = \frac{d}{dx}\left[x^2+1\right]$ $= 2x$ $= 2x(1)\left[x^2+1\right]^{1-1}$ $\therefore$ True for $n=1$	1	
	Assume true for $n = k$ i.e. Assume $\frac{d}{dx}\left[x^2+1\right]^k = 2xk\left[x^2+1\right]^{k-1}$ Consider $n = k + 1$ Want to show $\frac{d}{dx}\left[x^2+1\right]^{k+1} = 2x(k+1)\left[x^2+1\right]^k$ LHS = $\frac{d}{dx}\left[x^2+1\right]^{k+1} = \frac{d}{dx}\left(x^2+1\right)\left[x^2+1\right]^k$ $=\left(x^2+1\right).2kx\left(x^2+1\right)^{k-1}+\left(x^2+1\right)^k.2x$	1	
	$=2kx\left(x^2+1\right)^k+2x\left(x^2+1\right)^k$ $=2x(k+1)\left(x^2+1\right)^k$ $\therefore$ True for $n=k+1$ if true for $n=k$ But true for $n = 1$ $\therefore$ true for $n = 1 + 1 = 2$ etc Hence by Mathematical induction, true for all $n\ge 1$	1	
d)	i. $\int x^m\,(\ln x)^n\,dx$ $\quad u=(\ln x)^n \quad v'=x^m$ $u'=\frac{n(\ln x)^{n-1}}{x} \quad v=\frac{x^{m+1}}{m+1}$	1	
	$=uv-\int vu'\,dx$ $=\frac{(\ln x)^n\,x^{m+1}}{m+1}-\int\frac{x^{m+1}\,n\,(\ln x)^{n-1}}{x\,(m+1)}\,dx$ $=\frac{(\ln x)^n\,x^{m+1}}{m+1}-\frac{n}{m+1}\int\left[x^m\,(\ln x)^{n-1}\right]dx$	1	

Question 5	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
d)	ii.		
	$\int_1^e x^3\,(\ln x)^3\,dx = \left[\frac{x^4(\ln x)^3}{4}\right]_1^e - \frac{3}{4}\left(\int_1^e x^3\,(\ln x)^2\right)dx$		
	$= \left[\frac{x^4(\ln x)^3}{4}\right]_1^e - \frac{3}{4}\left\{\left[\frac{x^4(\ln x)^2}{4}\right]_1^e - \frac{2}{4}\int_1^e\left(x^3\,(\ln x)^1\right)dx\right\}$	**1**	
	$= \left[\frac{x^4(\ln x)^3}{4} - \frac{3x^4(\ln x)^2}{16}\right]_1^e + \frac{3}{8}\left\{\int_1^e\left(x^3\,(\ln x)^1\right)dx\right\}$		
	$= \left[\frac{x^4(\ln x)^3}{4} - \frac{3x^4(\ln x)^2}{16}\right]_1^e + \frac{3}{8}\left\{\left[\frac{x^4(\ln x)^1}{4}\right]_1^e - \frac{1}{4}\int_1^e\left(x^3\,(\ln x)^0\right.\right.$	**1**	
	$= \left[\frac{x^4(\ln x)^3}{4} - \frac{3x^4(\ln x)^2}{16} + \frac{3x^4(\ln x)^1}{32}\right]_1^e - \frac{3}{32}\int_1^e x^3 dx$		
	$= \left[\frac{x^4(\ln x)^3}{4} - \frac{3x^4(\ln x)^2}{16} + \frac{3x^4(\ln x)^1}{32} - \frac{3x^4}{128}\right]_1^e$	**1**	
	$= \left[\frac{e^4}{4} - \frac{3e^4}{16} + \frac{3e^4}{32} - \frac{3e^4}{128}\right] + \left[\frac{3}{128}\right]$	**1**	
	$= \frac{17e^4 + 3}{128}$		
		/15	

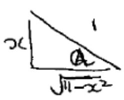
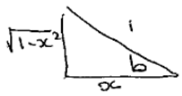
Question 6	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	i. $\frac{(a\sec\theta)^2}{a^2}-\frac{(b\tan\theta)^2}{b^2}=1$	**1**	
	$LHS=\frac{a^2\sec^2\theta}{a^2}-\frac{b^2\tan^2\theta}{b^2}$		
	$=\sec^2\theta-\tan^2\theta$		
	$=1+\tan^2\theta-\tan^2\theta$	**1**	
	$=1$		
	$=RHS$		
	Therefore P lies on the hyperbola		
	ii. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$		
	$\therefore\ \frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$		
	$\therefore\ \frac{dy}{dx}=\frac{b^2x}{a^2y}$		
	At $(a\sec\theta, b\tan\theta)$		
	$\frac{dy}{dx}=\frac{b^2(a\sec\theta)}{a^2(b\tan\theta)}=\frac{b\sec\theta}{a\tan\theta}$	**1**	
	$\therefore\ m=\frac{b}{a}\operatorname{cosec}\theta$		
	Gradient of Normal = $-\frac{a}{b}\sin\theta$	**1**	
	Equation:		
	$y-b\tan\theta=-\frac{a}{b}\sin\theta(x-a\sec\theta)$		
	$by-b^2\tan\theta=-ax\sin\theta+a^2\sec\theta\sin\theta$		
	$ax\sin\theta+by=b^2\tan\theta+a^2\tan\theta$	**1**	
	$ax\sin\theta+by=(a^2+b^2)\tan\theta \quad -------(1)$		

Question 6	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	iii. Coordinates of N are $(a\sec\theta, 0)$ For M, sub $y = 0$ into (1) $ax\sin\theta = (a^2+b^2)\tan\theta$ $\therefore x = \dfrac{(a^2+b^2)\tan\theta}{a\sin\theta}$ $\therefore x = \dfrac{(a^2+b^2)\sec\theta}{a}$ Coordinates of M are $\left(\dfrac{(a^2+b^2)\sec\theta}{a}, 0\right)$ $\therefore OM = \dfrac{(a^2+b^2)\sec\theta}{a}$ Now $e^2\ ON = e^2(a\sec\theta)$ Also for a hyperbola $b^2 = a^2(e^2-1)$ $\dfrac{b^2}{a^2}+1 = e^2$ $\dfrac{a^2+b^2}{a^2} = e^2$ $\therefore e^2\ ON = \left(\dfrac{a^2+b^2}{a^2}\right)a\sec\theta$ $= \dfrac{(a^2+b^2)\sec\theta}{a}$ $= OM$	 **1** **1** **1**	

Question 6	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
	iii. $SM = ae - \frac{a^2+b^2}{a}\sec\theta$ $= ae - \frac{a^2 e^2}{a}\sec\theta$ $= ae(1-e\sec\theta)$ $b^2 = a^2(e^2-1)$ $b^2 = a^2e^2 - a^2$ $a^2+b^2 = a^2e^2$	**1**	
	Now $SP = \sqrt{(ae - a\sec\theta)^2 + (0 - b\tan\theta)^2}$ $= \sqrt{a^2e^2 - 2ea^2\sec\theta + a^2\sec^2\theta + a^2(e^2-1)\tan^2\theta}$ $= a\sqrt{e^2 - 2e\sec\theta + \sec^2\theta + (e^2-1)(\sec^2\theta - 1)}$ $= a\sqrt{e^2 - 2e\sec\theta + \sec^2\theta + e^2\sec^2\theta - e^2 - \sec^2\theta + 1}$ $= a\sqrt{e^2\sec^2\theta - 2e\sec\theta + 1}$ $= a\sqrt{(1-e\sec\theta)^2}$ $= a(1-e\sec\theta)$	**1**	
	$\therefore e\,SP = ae(1-e\sec\theta)$ $= SM$	**1**	
b)	$x^2y + 2x - 2xy = 0$ $2xy + x^2\frac{dy}{dx} + 2 - 2y - 2x\frac{dy}{dx} = 0$ $\frac{dy}{dx}(x^2 - 2x) = 2y - 2xy - 2$ $\frac{dy}{dx} = \frac{2y - 2xy - 2}{x^2 - 2x}$	1	Implicit Differentiation
	At (1, 2) $\frac{dy}{dx} = \frac{2(2) - 2(1)(2) - 2}{1^2 - 2(1)}$ $= \frac{-2}{-1} = 2$ $y - 2 = 2(x-1)$ $y - 2 = 2x - 2$ $y = 2x$	1	Equation
c)	All the coefficients of $P(z)$ are real. Then any complex roots occur in conjugate pairs. Since $3+i$ is a root then $3-i$ is a root	1	3 – i with correct explanation

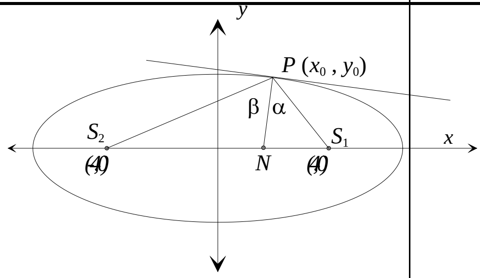
Question 6	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
	Roots are $3+i$, $3-i$ and α $(3+i)(3-i)\alpha = -\frac{10}{1}$ $(9-i^2)\alpha = -10$ $10\alpha = -10$ $\alpha = -1$ $P(z) = (z--1)[z-(3+i)][z-(3-i)]$ $= (z+1)(z^2-6z+10)$	1	Correct answer
		/15	

Question 7	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)i)	From 11 people, total number of committees of 5 $= {}^{11}C_5$ $= 462$	**1**	Correct
ii)	Independent majority on a committee of 5, chosen from 6 independents (I) and 5 politically aligned (PA). 5 (I) 0 (PA) Number = ${}^6C_5 \times {}^5C_0 = 6$ 4 (I) 1 (PA) Number = ${}^6C_4 \times {}^5C_1 = 15 \times 5 = 75$ 3 (I) 2 (PA) Number = ${}^6C_3 \times {}^5C_2 = 20 \times 10 = 200$ Total number of committees with Independent majority $= (6 + 75 + 200)$ $= 281$	**1** **1**	1 mark some progress showing understanding
b)	$x^3 - 3x^2 + ax + 8 = 0$ Let Roots be $\alpha - \beta, \alpha, \alpha + \beta$ Sum (1 at Time): $\alpha - \beta + \alpha + \alpha + \beta = \frac{-b}{a}$ $3\alpha = 3$ $\alpha = 1$ Product : $(\alpha - \beta) \times \alpha \times (\alpha + \beta) = \frac{-d}{a}$ $\alpha^3 - \alpha\beta^2 = -8$ $1 - \beta^2 = -8$ $\beta^2 = 9$ $\beta = \pm 3$	**1** **1**	
b)	Sum (2 at a time) = $\alpha(\alpha - \beta) + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = \frac{c}{a}$ $3\alpha^2 - \beta^2 = a$ $3(1)^2 - 3^2 = a$ $a = -6$ Therefore roots are $\alpha - \beta, \alpha, \alpha + \beta$ i.e. $-2, 1, 4$	**1** **1**	

Question 7	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
c)i)	$y=\sin^{-1}x-\sqrt{1-x^2}$ $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$ $=\frac{1}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$ $=\frac{1+x}{\sqrt{1-x^2}}$ $=\frac{1+x}{\sqrt{(1+x)(1-x)}}$ $=\frac{\sqrt{1+x}}{\sqrt{1-x}}$ Result defined for $-1\le x\le 1$	**1** **1**	1 Mark Correctly differentiates the function 2 Marks: Correct answer.
ii)	$\int_0^a \sqrt{\frac{1+x}{1-x}}dx=\left[\sin^{-1}x-\sqrt{1-x^2}\right]_0^a$ $=(\sin^{-1}a-\sqrt{1-a^2})-(\sin^{-1}0-\sqrt{1})$ $=\sin^{-1}a-\sqrt{1-a^2}+1$ $=\sin^{-1}a+1-\sqrt{1-a^2}$	**1**	1 Mark: Correct answer
d)i)	i) $\sin(\sin^{-1}x-\cos^{-1}x)=2x^2-1$. let $a=\sin^{-1}x$ $\quad b=\cos^{-1}x$ $\sin a = x$. $\quad \cos b = x$.   $\cos a=\sqrt{1-x^2}$. $\quad \sin b=\sqrt{1-x^2}$ LHS: $=\sin(a-b)$ $=\sin a\cos b-\cos a\sin b$ $=x.x-\sqrt{1-x^2}.\sqrt{1-x^2}$ $=x^2-(1-x^2)$ $=2x^2-1$ $=$ RHS.	**2** **1**	1 mark each for setting up cos a and sin b. (2 marks). 1 mark for expansion
ii)	ii) $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(1-x)$. ie $\sin(\sin^{-1}x-\cos^{-1}x)=1-x$ ie $2x^2-1=1-x$ from i) $2x^2+x-2=0$ $x=\frac{-1\pm\sqrt{1+16}}{4}$ $=\frac{-1\pm\sqrt{17}}{4}$	**2**	1 mark 1 mark.
		/15	

Question 8	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
a)	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ $\tan(\alpha-\beta)=\dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$ $=\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$ Divide everything by $\cos\alpha\cos\beta$ $=\dfrac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$ $=\dfrac{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}{1+\dfrac{\sin\alpha}{\cos\alpha}\dfrac{\sin\beta}{\cos\beta}}$ $=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ ii. $\tan\left(\alpha-\dfrac{\pi}{3}\right)=\dfrac{\tan\alpha-\tan\dfrac{\pi}{3}}{1+\tan\alpha\tan\dfrac{\pi}{3}}=\dfrac{\tan\alpha-\sqrt{3}}{1+\sqrt{3}\tan\alpha}$	1 1 1 1	1 for the two expressions. Any fair proof Final result
b)i)	$f(x)=x-\log_e(1+x^2)$, $1+x^2>0$ i) $f'(x)=1-\dfrac{2x}{1+x^2}$ $=\dfrac{1+x^2-2x}{1+x^2}$ $=\dfrac{(x-1)^2}{1+x^2}$ since $(x-1)^2\geq 0\ \forall x$ and $1+x^2>0\ \forall x$ $f'(x)\geq 0\ \forall x$.	3	1 mark correct differentiation. 2 marks correct explanation; 1 mark reasonable attempt.
ii)	ii) deduce $e^x>1+x^2$ $f'(x)\geq 0$ means $f(x)$ monotonic increasing for $x=0$, $f(0)=0$ so for $x\geq 0$, $f(x)\geq 0$. so for $x>0$, $x-\log_e(1+x^2)>0$ ie $x>\log_e(1+x^2)$ ie $e^x>1+x^2$, $x>0$.	3	1 mark. 1 mark. 1 mark.

Question 8	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
c)	i. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$ $\frac{2y}{b^2}\frac{dy}{dx}=-\frac{2x}{a^2}$ $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$ At (x_0, y_0) Equation $y-y_1=m(x-x_1)$ $m=-\frac{b^2x_0}{a^2y_0}$ $y-y_0=-\frac{b^2x_0}{a^2y_0}(x-x_0)$ $a^2yy_0-a^2{y_0}^2=-b^2{x_0}^2+b^2x\,x_0$ $b^2x\,x_0+a^2yy_0=a^2{y_0}^2+b^2x^2$ Divide every thing by a^2b^2 $\frac{x\,x_0}{a^2}+\frac{yy_0}{b^2}=\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}$ But $\frac{{x_0}^2}{a^2}+\frac{{y_0}^2}{b^2}=1$ $\therefore\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$	**1** **1**	Gradient

Question 8	Trial HSC Examination - Mathematics Extension 2		2011
Part	**Solution**	**Marks**	**Comment**
ii)	Ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$ or $9x^2+25y^2=225$  Equation of tangent is $\frac{xx_0}{25}+\frac{yy_0}{9}=1$ Differentiate implicitly. $\frac{x_0}{25}+\frac{dy}{dx}\frac{y_0}{9}=0$ $\frac{dy}{dx}\frac{y_0}{9}=-\frac{x_0}{25}$ $\frac{dy}{dx}=-\frac{9x_0}{25y_0}$ Gradient of normal $=\frac{25y_0}{9x_0}$ Gradient of $PS_1=\frac{y_0}{x_0-4}$ Gradient of $PS_2=\frac{y_0}{x_0+4}$	**1**	1 for individual gradients
	$tan\alpha=\left\|\frac{\frac{25y_0}{9x_0}-\frac{y_0}{x_0-4}}{1+\frac{25y_0}{9x_0}\cdot\frac{y_0}{x_0-4}}\right\|=\left\|\frac{\frac{25x_0y_0-100y_0-9x_0y_0}{9x_0(x_0-4)}}{\frac{9x_0^2-36x_0+25y_0^2}{9x_0(x_0-4)}}\right\|$ $=\left\|\frac{16x_0y_0-100y_0}{9x_0^2+25y_0^2-36x_0}\right\|$ $=\left\|\frac{4y_0(4x_0-25)}{225-36x_0}\right\|$ since P lies on ellipse $9x_0^2+25y_0^2=225$ $=\left\|\frac{4y_0(4x_0-25)}{9(25-4x_0)}\right\|$ $=\left\|\frac{4y_0}{9}\right\|$	**1**	1 for expressions for one angle or tan of angle
	$tan\beta=\left\|\frac{\frac{25y_0}{9x_0}-\frac{y_0}{x_0+4}}{1+\frac{25y_0}{9x_0}\cdot\frac{y_0}{x_0+4}}\right\|=\left\|\frac{\frac{25x_0y_0+100y_0-9x_0y_0}{9x_0(x_0+4)}}{\frac{9x_0^2+36x_0+25y_0^2}{9x_0(x_0+4)}}\right\|$ $=\left\|\frac{16x_0y_0+100y_0}{9x_0^2+25y_0^2+36x_0}\right\|$ $=\left\|\frac{4y_0(4x_0+25)}{9(25+4x_0)}\right\|$ since P lies on ellipse $9x_0^2+25y_0^2=225$ $=\left\|\frac{4y_0}{9}\right\|=tan\alpha$ $\therefore\ \alpha=\beta$	**1**	1 for second angle and equality
		/15	