International Grammar School

Year 12
Mathematics Extension 2
HSC Trial Examination
2014

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this paper
- In questions 11 – 16, show all relevant reasoning and/or calculations

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Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Question 1 – 10.

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1 An object moving in a circular path of radius 4 metres travels 48 metres in 3 seconds.
The angular speed of the object is:

(A) 3 rad/s

(B) 4 rad/s

(C) 12 rad/s

(D) 16 rad/s

2 What is the gradient of the tangent to the circle $x^2 - 2x + y^2 = 9$ at the point $(0, -3)$?

(A) $-\frac{1}{3}$

(B) $-\frac{11}{6}$

(C) $\frac{1}{3}$

(D) $\frac{1}{6}$

3 The number of ways that 6 items can be divided between 3 people so that each person receives 2 items is:

(A) 6

(B) 27

(C) 90

(D) 360

4 Which of the following is the expression for $\int \sin^3 x dx$?

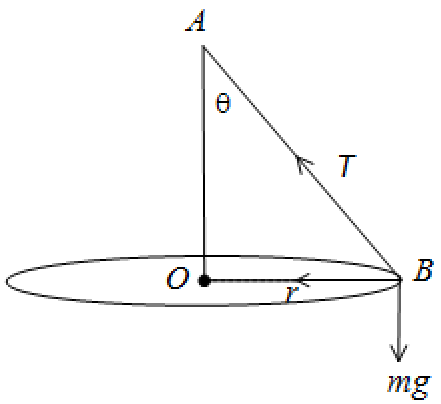
(A) $\frac{1}{3}\cos^3 x - \cos x + C$

(B) $\frac{1}{3}\cos^3 x + \cos x + C$

(C) $\frac{1}{3}\sin^3 x - \sin x + C$

(D) $\frac{1}{3}\sin^3 x + \sin x + C$

5 A particle at B is attached to a string AB that is fixed at A. The particle rotates in a horizontal circle with a radius of r. Let T be the tension in the string and $\angle BOA = \theta$.



Which of the following statements is correct?

(A) $T\cos\theta - mg = ma$

(B) $T\sin\theta = mr\omega^2$

(C) $T = -mg$

(D) $T - mg = ma$

6 The polynomial equation $3x^3 - 2x^2 + x - 7 = 0$ has roots α, β and γ.

Which polynomial equation has roots $\frac{2}{\alpha}, \frac{2}{\beta}$ and $\frac{2}{\gamma}$?

(A) $3x^3 - 4x^2 + 4x - 56 = 0$

(B) $7x^3 - 2x^2 + 8x - 24 = 0$

(C) $x^3 - 2x^2 - 27x - 49 = 0$

(D) $24x^3 - 8x^2 + 2x - 7 = 0$

7 The point $P\left(cp, \frac{c}{p}\right)$ lies on the rectangular hyperbola $xy = c^2$. The equation of the normal to the hyperbola at P is:

(A) $x + pqy = c(p + q)$

(B) $x + p^2y = 2cp$

(C) $px - \frac{1}{p}y = cp^2\left(1 - \frac{1}{p^2}\right)$

(D) $py - c = p^3(x - cp)$

8 What is the eccentricity for the hyperbola $\frac{y^2}{225} - \frac{x^2}{64} = 1$?

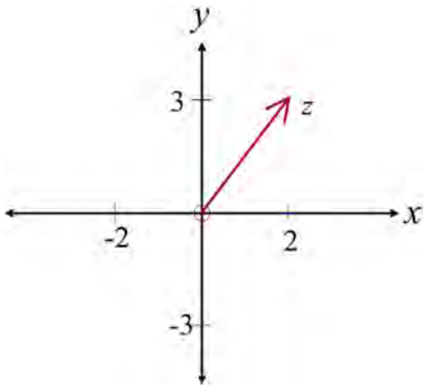
(A) $\frac{8}{17}$

(B) $\frac{15}{17}$

(C) $\frac{17}{15}$

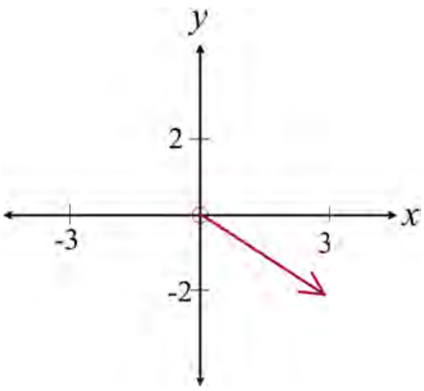
(D) $\frac{17}{8}$

9 The Argand diagram below shows the complex number z.

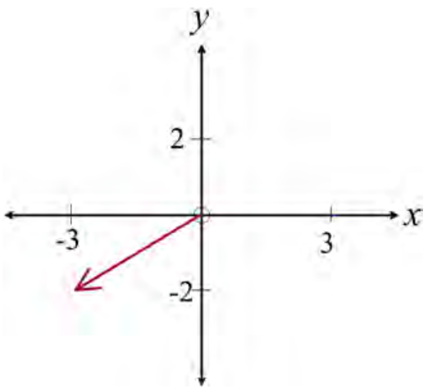


Which Argand diagram best represents $i\bar{z}$?

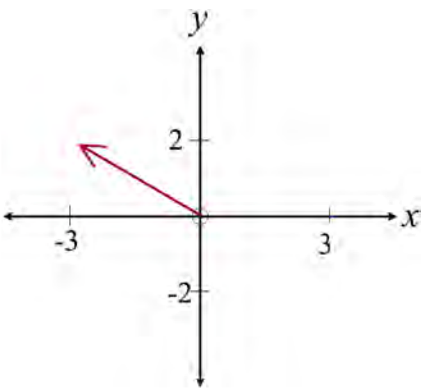
(A)



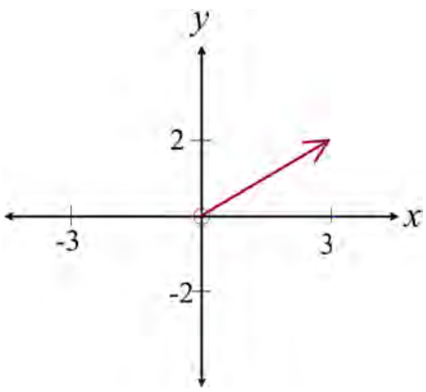
(B)



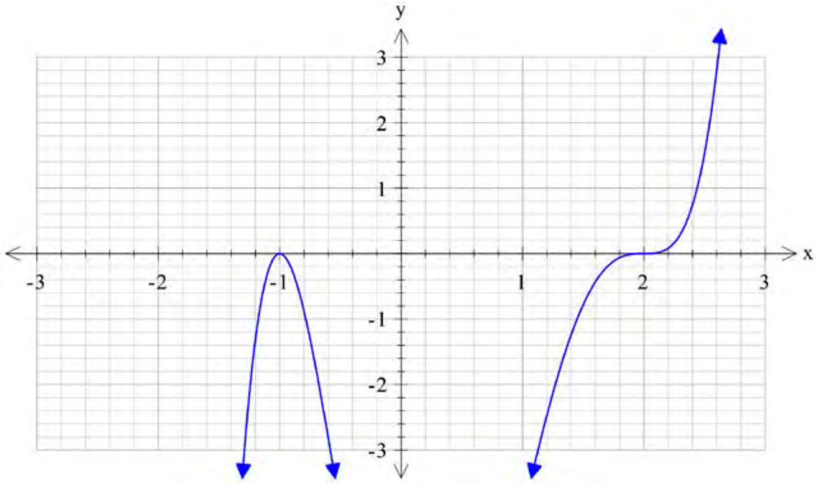
(C)



(D)

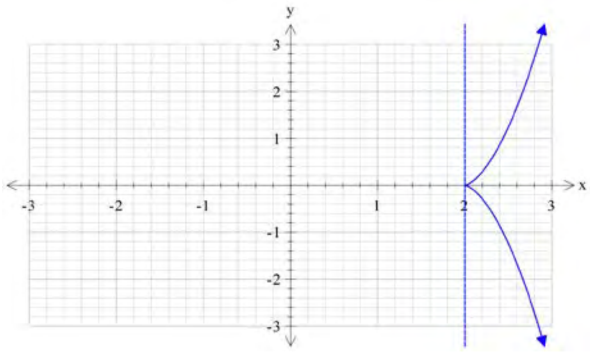


10 The graph of $y = f(x)$ is drawn below.

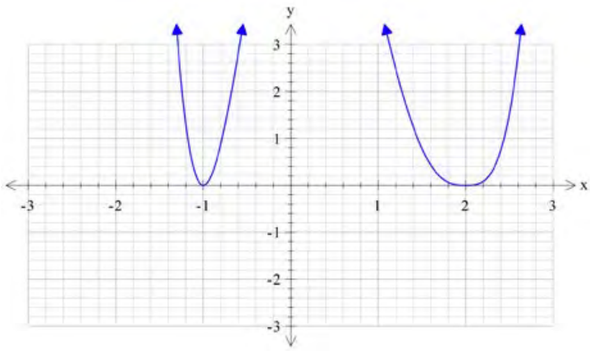


Which of the following graphs represents the graph of $y = [f(x)]^2$?

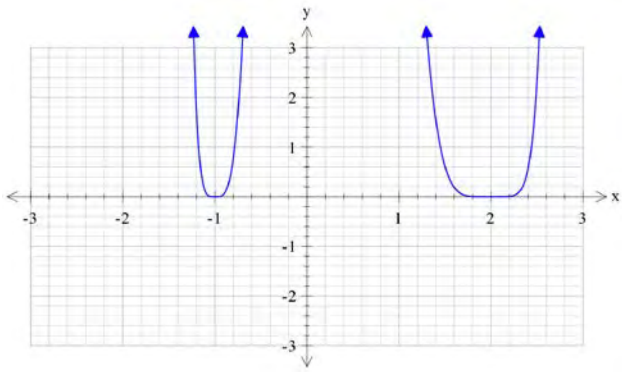
(A)



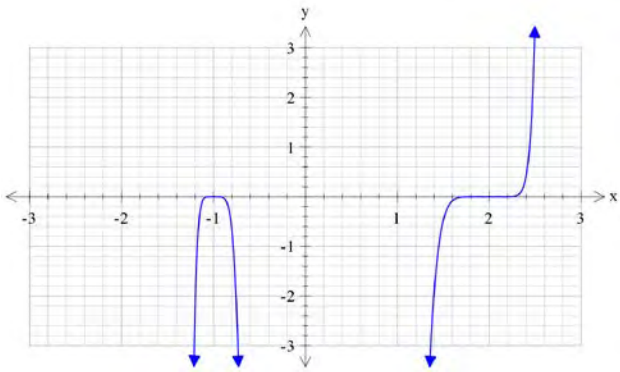
(B)



(C)



(D)



Section II

90 Marks
Attempt Questions 11 - 16.
Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 11 - 16, your responses should include relevant mathematics reasoning and/ calculations.

Question 11 (15 MARKS) Use a SEPARATE writing booklet **Marks**

(a) Let $w = \sqrt{3} + i$ and $z = 3 - \sqrt{3}i$.

(i) Find wz **1**

(ii) Express w in modulus-argument form. **2**

(iii) Write w^4 in simplest Cartesian form. **2**

(b) (i) Find the values of A, B, C and D such that: **2**

$$\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

(ii) Hence find $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx.$ **2**

(c) Find all solutions to the equation $x^4 - 2x^3 + x^2 - 8x - 12 = 0$, given that $x = 2i$ is a root of the equation. **3**

(d) Find $\int \frac{x\, dx}{x^2 - 3x + 4}$ **3**

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Using calculus, show that $x \geq \ln(1+x)$ for $x \geq -1$. **3**

(b) Consider the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(i) Show that the point $P(4\cos\theta, 3\sin\theta)$ lies on the ellipse. **1**

(ii) Calculate the eccentricity of the ellipse and hence find the foci and the directrices of the ellipse. **3**

(iii) Find the equation of the tangent at $P(4\cos\theta, 3\sin\theta)$. **2**

(iv) Find the equation of the normal at $P(4\cos\theta, 3\sin\theta)$. **2**

(v) Show that the tangent at P cuts the positive directrix at **2**

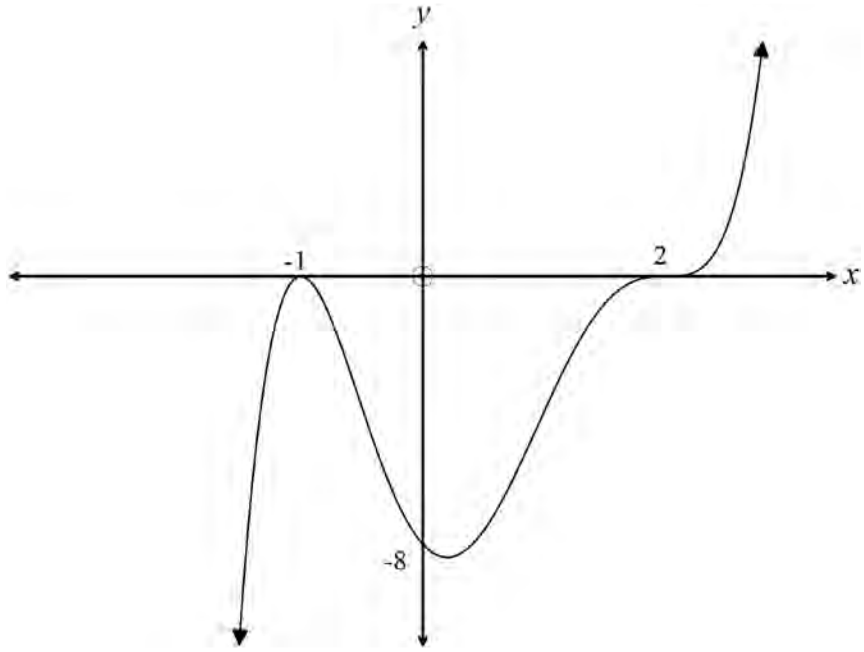
$M\left(\frac{16\sqrt{7}}{7}, \frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}\right)$.

(vi) Hence show that $\angle PSM = 90°$, if S is the positive focus. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Express $\dfrac{(1+i)^2}{\left(1-i\sqrt{3}\right)^2}$ in the form $r\ cis\ \theta$. **4**

(b) The graph of $y = f(x)$ is shown below.



Sketch the following curves on separate half page diagrams.

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \dfrac{d}{dx}[f(x)]$ **2**

(iv) $y^2 = f(x)$ **2**

(c) Prove that $cis\ (\alpha + \beta) = cis\ \alpha\ cis\ \beta$ **2**

(d) Find the Cartesian equation of the following curve and sketch it on an Argand Diagram. **2**

$$|z + 3 + 2i| = |z - 2 + i|$$

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Let $I_n = \int x(\ln x)^n dx$ for $n = 0,\ 1,\ 2,\ 3,...$ **3**

Show that $I_n = \frac{x^2}{2}(\ln x)^n - \frac{n}{2}I_{n-1}$ for $n \geq 1$

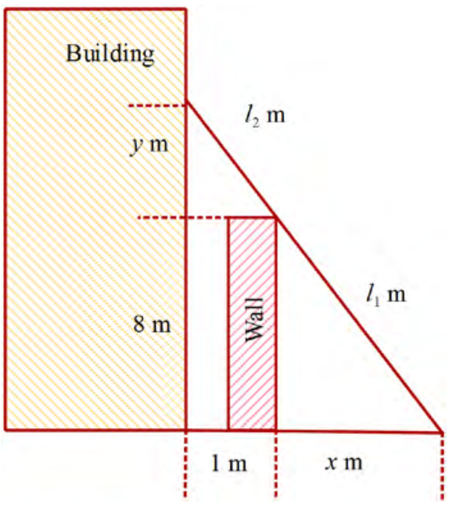
(ii) Hence, or otherwise, find $\int x(\ln x)^2 dx$. **1**

(b) If $T_1 = 8$, $T_2 = 20$ and $T_n = 4T_{n-1} - 4T_{n-2}$ for $n \geq 3$ prove by mathematical induction that: **3**

$$T_n = (n+3)2^n \text{ for } n \geq 1$$

(c) Using the method of cylindrical shells, find the volume of the solid of revolution formed when the area bounded by $y = \sin x$, the x-axis, between $x = 0$ and $x = \pi$, is rotated about the y-axis. **3**

(d)



A ladder reaches from the ground, over a wall 8 metres high, to the side of a building 1 metre behind the wall.

(i) By using similar triangles, show that $y = \frac{8}{x}$ and hence the length l of the ladder, where $l = l_1 + l_2$, is given by: **2**

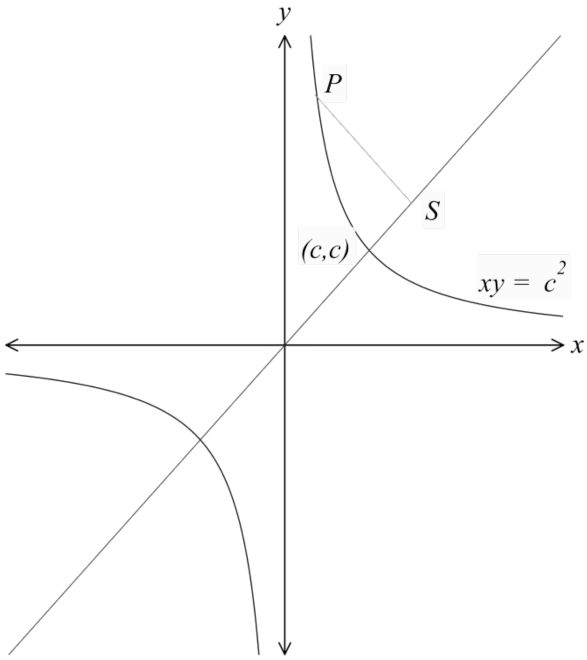
$$l = \sqrt{x^2 + 64} + \sqrt{1 + \frac{64}{x^2}}.$$

(ii) Hence find the length of the shortest ladder which will satisfy the conditions described above. **3**

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use the substitution $u = e^x$ to find $\int \frac{e^x + e^{2x}}{1+e^{2x}} dx$. **4**

(b) The point $P\left(cp, \frac{c}{p}\right)$ with $p > 0$ lies on the rectangular hyperbola $xy = c^2$ with focus S. The point T divides the interval PS in the ratio 1:2.



(i) Show that the coordinates of T are: **2**

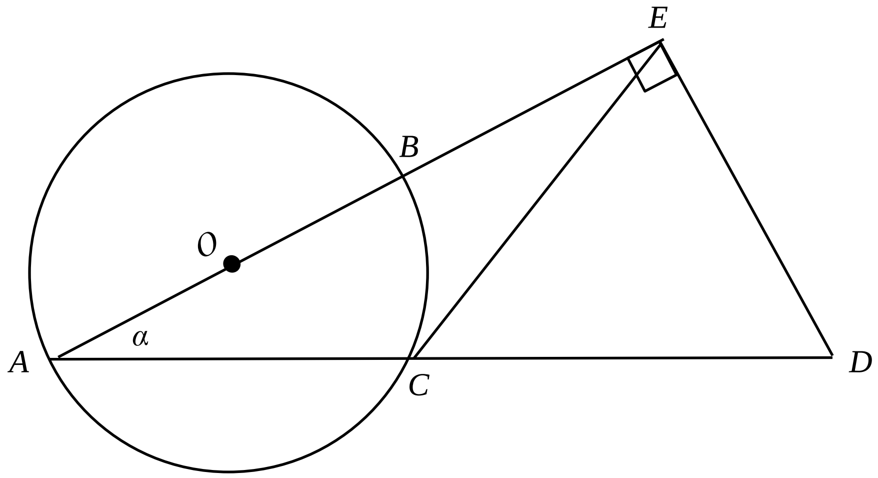
$$T\left(\frac{2cp + c\sqrt{2}}{3}, \frac{\frac{2c}{p} + c\sqrt{2}}{3}\right)$$

(ii) Show that the Cartesian equation of the locus of T can be written as: **2**

$4c^2 = \left(3x - c\sqrt{2}\right)\left(3y - c\sqrt{2}\right)$.

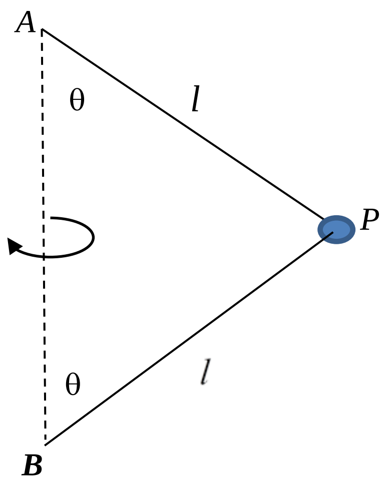
HINT: find an expression for $2p$ in terms of x, and $\frac{2}{p}$ in terms of y.

(c)



The diameter AB of a circle centre O is produced to E. EC is a tangent touching the circle at C, and the perpendicular to AE at E meets AC produced at D. **3**
$\angle BAC = \alpha$
Show that ΔCDE is isosceles.

(d) A particle P of mass 5kg is attached by two chains, each of length 3 m, to two fixed points A and B, which lie on a vertical plane.
P revolves with constant angular velocity ω about AB. AP makes an angle of θ with the vertical. The tension in AP is T_1 and the tension in BP is T_2 where $T_1 \geq 0$ and $T_2 \geq 0$.



(i) Resolve the forces on P in the horizontal and vertical directions **2**

(ii) If the object is rotating in a circle of radius *1.5m* at *12m/s*, find the tension in both parts of the string. (Use $g = 10\ m/s^2$) **2**

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the first derivative of $y = \ln\left(\dfrac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right)$ **2**

(b) (i) The displacement (from a fixed point) of a body moving in a straight line is given by x, and its velocity is v. **1**

Show that $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = v\dfrac{dv}{dx}$.

(ii) A particle of mass one kg is moving in a straight line. It is initially at the origin and is travelling with velocity $\sqrt{3}\,ms^{-1}$. The particle is moving against a resisting force $v+v^3$, where v is the velocity.

A Briefly explain why the acceleration of the particle is given by $\dfrac{dv}{dt} = -\left(v+v^3\right)$. **1**

B Show that the displacement x of the particle from the origin is given by $x = \tan^{-1}\left(\dfrac{\sqrt{3}-v}{1+v\sqrt{3}}\right)$. **4**

C Show that the time t which has elapsed when the particle is travelling with velocity V is given by $t = \dfrac{1}{2}\log_e\left[\dfrac{3\left(1+V^2\right)}{4V^2}\right]$ **4**

D Find V^2 as a function of t. **2**

E Hence find the limiting position of the particle as $t \to \infty$. **1**

End of Examination

2014 Year 12 Trial Examination **Student Name: . . . .**

Mathematics Extension 2

Section I Multiple-Choice Answer Sheet

1	A ⬭	B ⬭	C ⬭	D ⬭
2	A ⬭	B ⬭	C ⬭	D ⬭
3	A ⬭	B ⬭	C ⬭	D ⬭
4	A ⬭	B ⬭	C ⬭	D ⬭
5	A ⬭	B ⬭	C ⬭	D ⬭
6	A ⬭	B ⬭	C ⬭	D ⬭
7	A ⬭	B ⬭	C ⬭	D ⬭
8	A ⬭	B ⬭	C ⬭	D ⬭
9	A ⬭	B ⬭	C ⬭	D ⬭
10	A ⬭	B ⬭	C ⬭	D ⬭

MULTIPLE CHOICE		
1	$v = \frac{48}{3} = 16m/s$ $v = rw$ $16 = 4w$ $w = \frac{16}{4} = 4\,rads\,/s$	**B**
2	$x^2 - 2x + y^2 = 9$ Differentiate w.r.t x $2x - 2 + 2y\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{1-x}{y}$ At $(0,-3)$, $\frac{dy}{dx} = -\frac{1}{3}$	**A** 1
3	First person can select 2 out of 6 i.e. $\binom{6}{2}$ ways. Second person can select 2 out of 4 i.e. $\binom{4}{2}$ ways. Final person can chose remaining 2 in 1 way. Number of ways $= \binom{6}{2} \times \binom{4}{2} \times 1 = 15 \times 6 \times 1 = 90$ ways.	**C**
4	$\int \sin^3 x dx = \int \sin x(\sin^2 x)dx$ $= \int \sin x(1-\cos^2 x)dx$ $= \int \sin x dx - \int \sin x \cos^2 x dx$ $= -\cos x + \frac{1}{3}\cos^3 x + C = \frac{1}{3}\cos^3 x - \cos x + C$	1 Mark: A
5	Resolving the forces vertically and horizontally at P $T\cos\theta - mg = 0$ $T\sin\theta = mr\omega^2$ Statement (B) is correct.	1 Mark: B
6	$x = \alpha, \beta, \gamma$ so $y = \frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$. $\therefore y = \frac{2}{x}$ and hence $x = \frac{2}{y}$ $3\left(\frac{2}{y}\right)^3 - 2\left(\frac{2}{y}\right)^2 + \left(\frac{2}{y}\right) - 7 = 0$ $\left(\frac{24}{y^3}\right) - \left(\frac{8}{y^2}\right) + \frac{2}{y} - 7 = 0$ $24 - 8y + 2y^2 - 7y^3 = 0$ Required equation is : $7x^3 - 2x^2 + 8x - 24 = 0$	**B**

7	$xy = c$ $y + x\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ $\frac{dy}{dx} = -\frac{c}{p} \div cp$ $= -\frac{c}{cp^2}$ $= -\frac{1}{p^2}$ $\therefore$ Gradient of Normal $= p^2$ $y - y_1 = m(x - x_1)$ $y - \frac{c}{p} = p^2(x - cp)$ $py - c = p^3(x - cp)$	**D**
8	$\frac{y^2}{225} - \frac{x^2}{64} = 1$, $a^2 = 64$ and $b^2 = 225$ $a^2 = b^2(e^2 - 1)$ $64 = 225 \times (e^2 - 1)$ $e = \sqrt{\frac{64}{225} + 1} = \sqrt{\frac{289}{225}} = \frac{17}{15}$	1 Mark: C
9	$z = 2 + 3i$ $i\bar{z} = i\overline{(2+3i)}$ $= i(2 - 3i)$ $= 3 + 2i$	**D**
10	Graph (C)	**C**

QUESTION 11

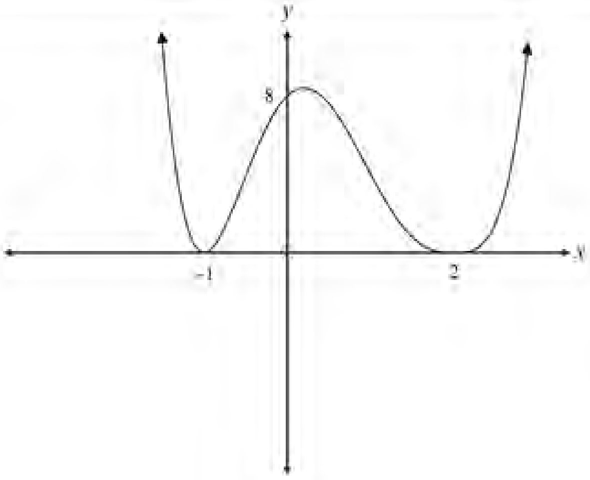
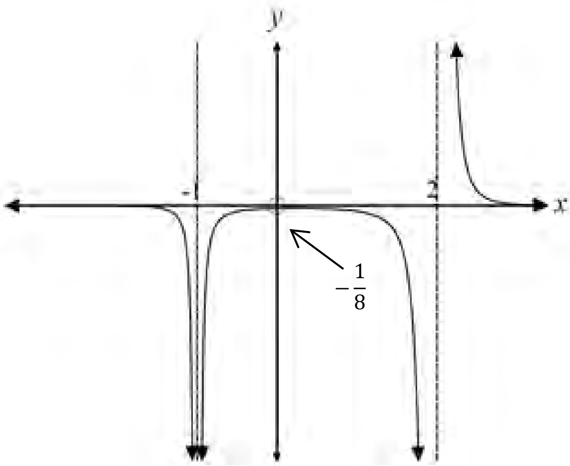
		Marks	Criteria
a)	$w = \sqrt{3} + i$ and $z = 3 - \sqrt{3}i$. (i) wz $= (\sqrt{3} + i)(3 - \sqrt{3}i)$ $= 3\sqrt{3} - 3i + 3i + \sqrt{3}$ $= 3\sqrt{3} + \sqrt{3}$ $= 4\sqrt{3}$	1	Correct Answer
	(ii) $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ $\tan\theta = \frac{1}{\sqrt{3}}, \quad \theta = \frac{\pi}{6}$ $\therefore w = 2\, cis\, \frac{\pi}{6}$	2	1 for correct r 1 for correct θ
	(iii) $w^4 = \left(2\, cis\, \frac{\pi}{6}\right)^4$ $= 2^4\, cis\, \frac{4\pi}{6}$ $= 16\, cis\, \frac{2\pi}{3}$ $= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= 16\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$ $= -8 + 8\sqrt{3}i$	2	1 – Evaluating Power 1 Answer in Cartesian form
b)	(i) $\frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$ $5x^3 - 3x^2 + 2x - 1 \equiv Ax(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2$ $\equiv Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$ $(A + C)x^3 = 5x^3 \quad \therefore A + C = 5$ $(B + D)x^2 = -3x^2 \quad \therefore B + D = -3$ $Ax = 2 \quad \therefore A = 2 \quad \therefore C = 3$ $B = -1 \quad \therefore D = -2$ Hence, $A = 2, B = -1, C = 3, D = -2$.	2	2 – Correct A, B, C and D 1 – 3 correct
	(ii) $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2}\, dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x - 2}{x^2 + 1}\right) dx$ $= \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3x}{x^2 + 1} - \frac{2}{x^2 + 1}\right) dx$ $= 2\ln x + \frac{1}{x} + \frac{3}{2}\ln(x^2 + 1) - 2\tan^{-1}x + c$	2	1 – Breakup of Integral 1 – Correct Answer
c)	$x^4 - 2x^3 + x^2 - 8x - 12$ Since $x = 2i$ is one root, $(x - 2i)(x + 2i)$ are factors as coeffs are real, so $(x^2 + 4)$ is a factor By division, $x^4 - 2x^3 + x^2 - 8x - 12 = (x^2 + 4)(x^2 - 2x - 3)$ $= (x + 2i)(x - 2i)(x - 3)(x + 1)$ $\therefore$ Solution is $x = \pm 2i, -1$ and 3	3	1 obtaining factor 1 – Correct division 1 all 4 roots

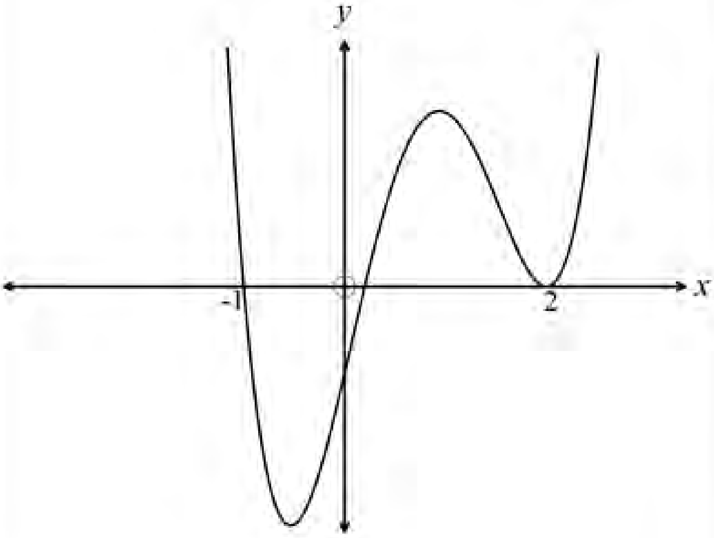
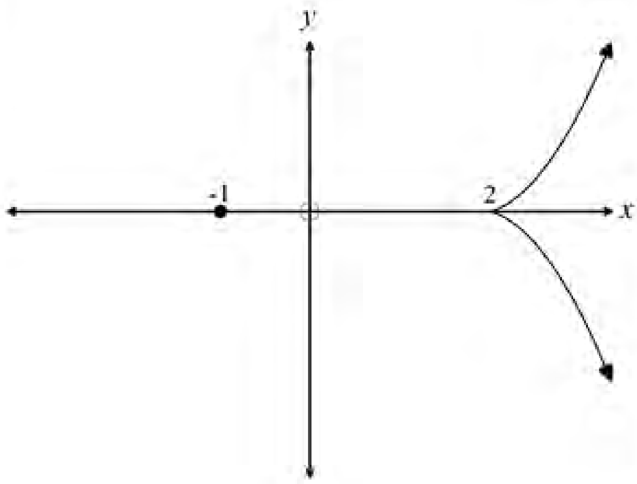
d)	$\int \frac{x}{x^2-3x+4}\,dx$ $= \frac{1}{2}\int \frac{2x-3}{x^2-3x+4} + \frac{3}{2}\int \frac{1}{x^2-3x+4}$ $= \frac{1}{2}\ln(x^2-3x+4) + \frac{3}{2}\int \frac{1}{\left(x-\frac{3}{2}\right)^2+\frac{7}{4}}$ $= \frac{1}{2}\ln(x^2-3x+4) + \frac{\frac{3}{2}}{\sqrt{\frac{7}{4}}}\, tan^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{\frac{7}{4}}}\right) + c$ $= = \frac{1}{2}\ln(x^2-3x+4) + \frac{3}{\sqrt{7}}\, tan^{-1}\left(\frac{2x-3}{\sqrt{7}}\right) + c$	1 1 1	Total marks - 3

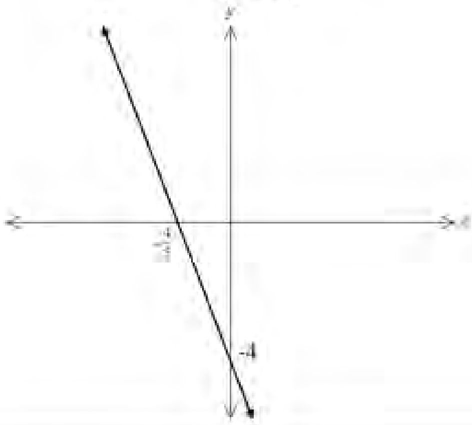
QUESTION 12

a)	Let $f(x)=x-\ln(1+x)$ and $f'(x)=1-\frac{1}{1+x}$ Minimum occurs if $f'(x)=0$ $1-\frac{1}{1+x}=0 \quad \frac{1+x}{1+x}-\frac{1}{1+x}=0$ $\therefore 1+x-1=0$ or $x=0 \quad (x\neq -1)$ Test $f''(x)=\frac{1}{(1+x)^2}$, $f''(0)=1>0$ Minima Therefore the least value of $f(x)$ is at $x=0$ $f(0)=0-\ln(1+0)=0$ hence $f(x)\geq 0$ $f(x)=x-\ln(1+x)\geq 0$ $\therefore x\geq \ln(1+x)$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 2 Marks: 1 Mark: Sets up the function and correctly uses calculus.	
b)	(i) $\frac{x^2}{16}+\frac{Y^2}{9}=1 \qquad P(4\cos\theta, 3\sin\theta)$ $\frac{(4\cos\theta)^2}{16}+\frac{(3\sin\theta)^2}{9}=1$ $\frac{16\cos^2\theta}{16}+\frac{9\sin^2\theta}{9}=1$ $\cos^2\theta+\sin^2\theta=1$ $1=1$ $\therefore$ P lies on the ellipse. (ii) $b^2=a^2(1-e^2)$ $3^2=4^2(1-e^2)$ $\frac{9}{16}=1-e^2$ $e^2=1-\frac{9}{16}$ $e^2=\frac{7}{16}$ $e=\frac{\sqrt{7}}{4}$ Foci $=(\pm ae,0)=\left(\pm 4\times\frac{\sqrt{7}}{4},0\right)=(\pm\sqrt{7},0)$ Directrices : $x=\pm\frac{a}{e}$ $x=\pm 4\div\frac{\sqrt{7}}{4}$ $x=\pm\frac{16}{\sqrt{7}}=\pm\frac{16\sqrt{7}}{7}$	**1** **3**	Working 1 – eccentricity 1 – foci 1 – directrices
	(iii) If $\frac{x^2}{16}+\frac{Y^2}{9}=1$ $\frac{2x}{16}+\frac{2y}{9}\frac{dy}{dx}=0$ $\frac{dy}{dx}=-\frac{2x}{16}\div\frac{2y}{9}$ $\frac{dy}{dx}=-\frac{2x}{16}\times\frac{9}{2y}$ $\frac{dy}{dx}=-\frac{9x}{16y}$ At $P(4\cos\theta, 3\sin\theta)$ $\quad \frac{dy}{dx}=-\frac{36\cos\theta}{48\sin\theta}=-\frac{3\cos\theta}{4\sin\theta}$ $y-y_1=m(x-x_1)$ $y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}(x-4\cos\theta)$ $4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$ $3x\cos\theta+4y\sin\theta=12$	**2**	1 – gradient 1 – Equation

	(iv) Normal $\frac{dy}{dx} = \frac{4\sin\theta}{3\cos\theta}$ $y - y_1 = m(x - x_1)$ $y - 3\sin\theta = \frac{4\sin\theta}{3\cos\theta}(x - 4\cos\theta)$ $3y\cos\theta - 9\sin\theta\cos\theta = 4x\sin\theta - 16\sin\theta\cos\theta$ $4x\sin\theta - 3y\cos\theta - 7\sin\theta\cos\theta = 0$	**2**	1 – substitution 1 - answer
	(v) $3x\cos\theta + 4y\sin\theta = 12$ $x = \frac{16\sqrt{7}}{7}$ $\frac{48\sqrt{7}}{7}\cos\theta + 4y\sin\theta = 12$ $4y\sin\theta = 12 - \frac{48\sqrt{7}}{7}\cos\theta$ $y = \frac{12 - \frac{48\sqrt{7}}{7}\cos\theta}{4\sin\theta}$ $y = \frac{84 - 48\sqrt{7}\cos\theta}{28\sin\theta} = \frac{21 - 12\sqrt{7}\cos\theta}{7\sin\theta}$ $Therefore\ M = \left(\frac{16\sqrt{7}}{7}, \frac{21 - 12\sqrt{7}\cos\theta}{7\sin\theta}\right)$	**2**	1 – substitution 1 – working
	(vi) Gradient PS $= \frac{3\sin\theta - 0}{4\cos\theta - \sqrt{7}} = \frac{3\sin\theta}{4\cos\theta - \sqrt{7}}$ Gradient MS $= \frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta} - 0}{\frac{16\sqrt{7}}{7} - \sqrt{7}} = \frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}}{\frac{16\sqrt{7}-7\sqrt{7}}{7}} = \frac{\frac{21-12\sqrt{7}\cos\theta}{7\sin\theta}}{\frac{9\sqrt{7}}{7}}$ $= \frac{21-12\sqrt{7}\cos\theta}{9\sqrt{7}\sin\theta}$ $m(PS).\,m(MS) = \frac{3\sin\theta}{4\cos\theta - \sqrt{7}} \cdot \frac{21-12\sqrt{7}\cos\theta}{9\sqrt{7}\sin\theta}$ $= \frac{7-4\sqrt{7}\cos\theta}{(4\cos\theta - \sqrt{7})\sqrt{7}}$ $= \frac{7-4\sqrt{7}\cos\theta}{4\sqrt{7}\cos\theta - 7}$ $= \frac{7-4\sqrt{7}\cos\theta}{-(7-4\sqrt{7}\cos\theta)}$ $= -1$ $\therefore MS \perp PS\ \ and\ \angle PSM = 90^\circ$	**2**	1 – Both Gradients 1 – proving perpendicular

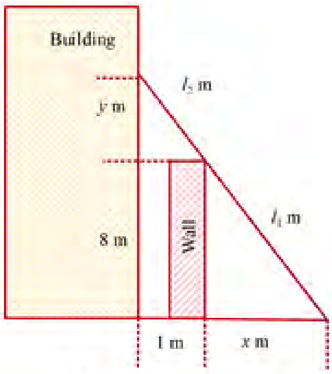
QUESTION 13			
a)	$\dfrac{(1+i)^2}{\left(1-i\sqrt{3}\right)^2}$		
	$(1+i)$: $r=\sqrt{2}, \theta=\frac{\pi}{4}$ $\quad 1+i=\sqrt{2}\,cis\,\frac{\pi}{4}$	1	2 marks for converting to mod-arg form
	$\left(1-i\sqrt{3}\right)$: $r=2, \theta=-\frac{\pi}{3}$ so $1-i\sqrt{3}=2\,cis\left(-\frac{\pi}{3}\right)$	1	
	$\dfrac{(1+i)^2}{\left(1-i\sqrt{3}\right)^2}=\dfrac{\left(\sqrt{2}\,cis\,\frac{\pi}{4}\right)^2}{\left(2\,cis\left(-\frac{\pi}{3}\right)\right)^2}$	1	
	$=\dfrac{2\,cis\,\frac{\pi}{2}}{4\,cis\left(-\frac{2\pi}{3}\right)}$ $=\frac{1}{2}\,cis\left(\frac{7\pi}{6}\right)$ $=\frac{1}{2}\,cis\left(\frac{-5\pi}{6}\right)$	1	2 marks for any valid method of simplifying.
b)	(i) 	1	Correct Graph
	(ii) 	2	1- Shape of Graph 1 – Accuracy of critical points

	(iii) 	2	1- Shape of Graph 1 – Accuracy of critical points
	(iv) 	2	1- Shape of Graph 1 – Accuracy of critical points
c)	$cis\ (\alpha+\ \beta) = cis\ \alpha\ cis\ \beta$ LHS $= cos(\alpha+\ \beta) +\ isin\ (\alpha+\ \beta)$ $= \cos\alpha\cos\beta - \sin\alpha\ sin\ \beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\ \beta)$ RHS $= cis\ \alpha\ cis\ \beta$ $= (\cos\alpha + isin\ \alpha)(\cos\beta + isin\ \beta)$ $= \cos\alpha\cos\beta + isin\ \beta\cos\alpha + isin\alpha\cos\beta - \sin\alpha\sin\beta$ $= \cos\alpha\cos\beta - \sin\alpha\ sin\ \beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\ \beta)$ $=$ LHS	1 1	LHS RHS

d)	$\lvert z+3+2i\rvert = \lvert z-2+i\rvert$ $(x+3)^2+(y+2)^2=(x-2)^2+(y+1)^2$ $x^2+6x+9+y^2+4y+4=x^2-4x+4+y^2+2y+1$ $10x+2y+8=0$ $5x+y+4=0$ $y=-5x-4$ 	1 1	1 – equation 1 - Graph

QUESTION 14

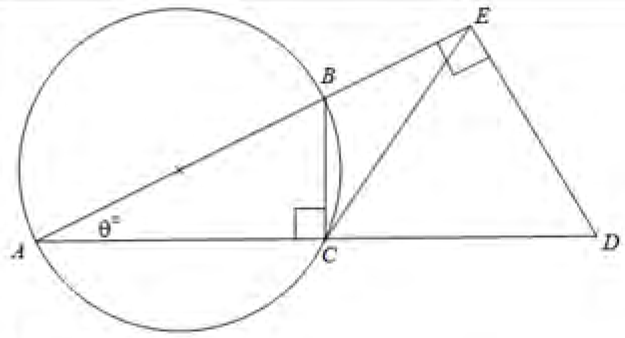
	Solution	Marking
a)	$I_n = \int x(\ln x)^n dx$ $= (\ln x)^n \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{n}{x} (\ln x)^{n-1} dx$ $= (\ln x)^n \frac{x^2}{2} - \frac{n}{2} \int x(\ln x)^{n-1} dx$ $= (\ln x)^n \frac{x^2}{2} - \frac{n}{2} I_{n-1}$ for $n \geq 1$	3 Marks: Correct answer. 2 Marks: Makes significant progress towards the solution. 1 Mark: Sets up the integration and shows some understanding. by parts
	$I_2 = \int x(\ln x)^2 dx = \frac{x^2(\ln x)^2}{2} - \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	1 mark correct
b)	Step 1: To prove the statement true for $n = 1$ and $n = 2$ $T_1 = (1+3)2^1 = 8$ $T_2 = (2+3)2^2 = 20$ Result is true for $n = 1$ Result is true for $n = 2$ Step 2: Assume the result true for $n = k$, $n = k-1$ $T_k = (k+3)2^k$, $T_{k-1} = (k+2)2^{k-1}$ To prove the result is true for, $T_{k+1} = (n+4)2^{k+1}$ $T_{k+1} = 4T_k - 4T_{k-1}$ $= 4(k+3)2^k - 4(k+2)2^{k-1}$ Using assumption in 2 $= 4k2^k + 12 \times 2^k - 4k2^{k-1} - 8 \times 2^{k-1}$ $= 4k2^k + 12 \times 2^k - 2k2^k - 4 \times 2^k$ $= 2^{k+1}(2k + 6 - k - 2)$ $= (k+4)2^{k+1}$ Result is true for $n = k+1$ if true for $n = k$ Step 3: Result true by principle of mathematical induction.	3 Marks: Correct answer. 2 Marks: Proves the result true for $n = 1$ and attempts to use the result of $n = k$ to prove the result for $n = k+1$. 1 Mark: Proves the result true for $n = 1$ and $n = 2$.
c)	By shell Method $\lim \partial x \to 0 \quad 2\pi x \sin x \partial x$ $V = 2\pi \int_0^{\pi} x \sin x\, dx$ $let\ u = x$ $v' = \sin x$ $u' = 1$ $v = -\cos x$ $I = uv - \int vu'$ $= 2\pi[-x\cos x]_0^{\pi} - \int_0^{\pi} -\cos x\, dx$ $= 2\pi\, [\pi + (\sin x)_0^{\pi}]$ $= 2\pi^2$	1 Shell Method 1 Integration by parts 1 Answer

d) (i)	 $\frac{y}{8} = \frac{1}{x} \Rightarrow y = \frac{8}{x}$ $l_1{}^2 = x^2 + 64 \rightarrow l_1 = \sqrt{x^2+64}$ $l_2{}^2 = y^2 + 1 \rightarrow l_2 = \sqrt{y^2+1}$ $l = \sqrt{x^2+64} + \sqrt{y^2+1}$ $l = (x^2+64)^{\frac{1}{2}} + \left(\frac{64}{x^2}+1\right)^{\frac{1}{2}}$ $l = (x^2+64)^{\frac{1}{2}} + (1+64x^{-2})^{\frac{1}{2}}$	1 1	Equations for l_1 and l_2 Equation for l
(ii)	$l' = \frac{1}{2}(2x)(x^2+64)^{-\frac{1}{2}} + \frac{1}{2}(-128x^{-3})(1+64x^{-2})^{-\frac{1}{2}}$ $l' = \frac{2x}{2\sqrt{x^2+64}} - \frac{128}{2x^3\sqrt{1+\frac{64}{x^2}}}$ $l' = \frac{x}{\sqrt{x^2+64}} - \frac{64}{x^3\sqrt{\frac{x^2+64}{x^2}}}$ $l' = \frac{x}{\sqrt{x^2+64}} - \frac{64}{x^2\sqrt{x^2+64}}$	1	Correct derivative
	Stat pt: $l' = \frac{x}{\sqrt{x^2+64}} - \frac{64}{x^2\sqrt{x^2+64}} = 0$ $\frac{x}{\sqrt{x^2+64}} = \frac{64}{x^2\sqrt{x^2+64}}$		
	$x^3 = \frac{64\sqrt{x^2+64}}{\sqrt{x^2+64}}$ $x^3 = 64$ $x = 4$ When $x = 1, l' < 0$ When $x = 5, l' > 0$ $\therefore \textit{minimum when } x = 4 \quad \textit{hence } y = 2$ $l_1 = \sqrt{80} = 4\sqrt{5}, l_2 = \sqrt{5}$ $\therefore L = 4\sqrt{5} + \sqrt{5} = 5\sqrt{5} \approx 11.2 \textit{ metres}$	1 1	Value of x and test Minimum length

QUESTION 15

a)	$u = e^x$ or $du = e^x dx$ or $dx = \frac{1}{u}du$ $\int \frac{e^x + e^{2x}}{1+e^{2x}}dx = \int \frac{u+u^2}{1+u^2} \times \frac{1}{u}du$ $= \int \frac{1+u}{1+u^2}du$ $= \int \frac{1}{1+u^2}du + \int \frac{u}{1+u^2}du$ $= \tan^{-1} u + \frac{1}{2}\ln(u^2+1) + C$ $= \tan^{-1} e^x + \frac{1}{2}\ln(e^{2x}+1) + C$	4 Marks: Correct answer. 3 Marks: Separates and integrates one part correctly. 2 Marks: Correctly expresses the integral in terms of u 1 Mark: Correctly finds dx in terms of du
b) (i)	$P\left(cp, \frac{c}{p}\right)$, $S(c\sqrt{2}, c\sqrt{2})$ and $PT:TS = 1:2$ Coordinates of T $x = \frac{mx_2 + nx_1}{m+n} = \frac{1 \times c\sqrt{2} + 2 \times cp}{1+2} = \frac{c(\sqrt{2}+2p)}{3}$ $y = \frac{my_2 + ny_1}{m+n} = \frac{1 \times c\sqrt{2} + 2 \times \frac{c}{p}}{1+2} = \frac{c\left(\sqrt{2}+\frac{2}{p}\right)}{3}$	2 Marks: Correct answer. 1 Mark: Finds one of the coordinates or makes some progress towards the solution.
(ii)	To find the locus of T eliminate p from the above equations. $x = \frac{c(\sqrt{2}+2p)}{3}$ $3x = c(\sqrt{2}+2p)$ $\frac{3x}{c} - \sqrt{2} = 2p$ $y = \frac{c\left(\sqrt{2}+\frac{2}{p}\right)}{3}$ $\frac{3y}{c} - \sqrt{2} = \frac{2}{p}$ $\left(\frac{3x}{c} - \sqrt{2}\right) \times \left(\frac{3y}{c} - \sqrt{2}\right) = 2p \times \frac{2}{p} = 4$ $(3x - \sqrt{2}c)(3y - \sqrt{2}c) = 4c^2$	2 Marks: Correct answer. 1 Mark: Uses the coordinates of T and attempts to eliminate p.

c)



Let $\angle BAC = \theta$

$\angle ACB = 90^\circ$ (angle in a semi-circle)

$\angle BCD = 90^\circ$ (adjacent angles on a straight line)

$\angle BCE = \angle BAC$ (angle between a tangent and a chord equals the angle in the alternate segment)

$\therefore \angle BCE = \theta$

$\therefore \angle ECD = 90 - \theta$ (angle sum of $\angle BCD$)

$\therefore \angle EDC = 90 - \theta$ ((angle sum of ΔAED)

$\therefore \angle ECD = \angle EDC = 90 - \theta$

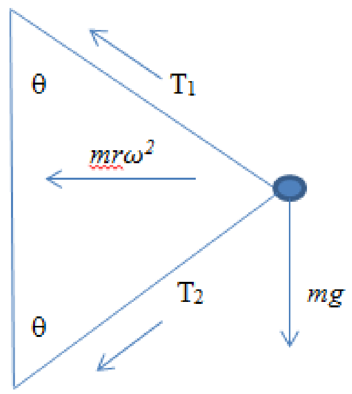
ΔECD is isosceles (base angles of the triangle are equal)

3 Marks: Correct answer.

2 Marks: Makes significant progress towards the solution.

1 Mark: Draws the diagram and applies a relevant circle theorem.

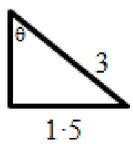
d)



$v = 12\ m/s$ $\qquad v = r\omega$

$m = 5$ kg $\qquad 12 = 1{\cdot}5\omega$

$r = 1{\cdot}5$ $\qquad \omega = 8$



$\sin\theta = \frac{1{\cdot}5}{3}$

$\theta = 30^\circ$

$$T_1 \cos\theta = mg + T_2 \cos\theta$$
$$T_1 \cos 30 = (5)(10) + T_2 \cos 30$$
$$T_1 . \frac{\sqrt{3}}{2} = 50 + T_2 . \frac{\sqrt{3}}{2}$$
$$T_1 \sqrt{3} = 100 + T_2\sqrt{3} \quad \dots\dots\dots (1)$$

(can be without numerical sub)

1 Vertical Equation

$$mr\omega^2 = T_1 \sin\theta + T_2 \sin\theta$$
$$5 \times 1 \cdot 5 \times 8^2 = T_1 \sin 30 + T_2 \sin 30$$
$$480 = \frac{T_1}{2} + \frac{T_2}{2}$$
$$960 = T_1 + T_2 \quad \dots\dots\dots (2)$$

(can be without numerical sub)

1 Horizontal Equation

		From (2), $$T_1 = 960 - T_2 \dots\dots\dots(3)$$ (3) in (1) $$(960 - T_2)\sqrt{3} = 100 + T_2\sqrt{3}$$ $$960\sqrt{3} - T_2\sqrt{3} = 100 + T_2\sqrt{3}$$ $$T_2 2\sqrt{3} = 960\sqrt{3} - 100$$ $$T_2 = \frac{960\sqrt{3} - 100}{2\sqrt{3}} = 451\ N$$	1	Value for T_2
		$$T_1 = 960 - T_2$$ $$T_1 = 960 - 451$$ $$T_1 = 509$$	1	Value for T_1

QUESTION 16		
a)	$y = \ln\left(\frac{\sqrt{x^2+1}}{\sqrt[3]{x^3+1}}\right)$ $y = \ln\sqrt{x^2+1} - \ln\sqrt[3]{x^3+1}$ $y = \ln(x^2+1)^{\frac{1}{2}} - \ln(x^3+1)^{\frac{1}{3}}$ $y = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}\ln(x^3+1)$ $\frac{dy}{dx} = \frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{3}\frac{3x^2}{x^3+1}$ $= \frac{x}{x^2+1} - \frac{x^2}{x^3+1}$	1 Use of logarithmic rules 1 Answer
b) (i)	$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dx}\times\frac{d}{dv}\left(\frac{1}{2}v^2\right)$ $= v\frac{dv}{dx}$	1 mark correctly establishes result
(ii) A	Resistance force acts against the direction of motion, $\therefore F = m\times a = 1\times a = -(v+v^3)$ $a = -(v+v^3)$ $\frac{dv}{dt} = -(v+v^3)$	1 mark correct explanation
B	$v\frac{dv}{dx} = -(v+v^3)$ $\frac{dv}{dx} = -(1+v^2)$ $\frac{dx}{dv} = \frac{-1}{1+v^2}$ $x = -\tan^{-1}v + c$ when $x = 0, v = \sqrt{3} \Rightarrow c = \tan^{-1}\sqrt{3}$ $\therefore x = \tan^{-1}\sqrt{3} - \tan^{-1}v$ $\tan x = \tan\left(\tan^{-1}\sqrt{3} - \tan^{-1}v\right)$ $= \frac{\tan\left(\tan^{-1}\sqrt{3}\right) - \tan\left(\tan^{-1}v\right)}{1+\tan\left(\tan^{-1}\sqrt{3}\right)\times\tan\left(\tan^{-1}v\right)}$ $= \frac{\sqrt{3}-v}{1+v\sqrt{3}}$ $\therefore x = \tan^{-1}\left(\frac{\sqrt{3}-v}{1+v\sqrt{3}}\right)$	1 mark some correct progress towards result 1 mark $\frac{dx}{dv} = -\frac{1}{1+v^2}$ 1 mark $x = \tan^{-1}\sqrt{3} - \tan^{-1}v$ 1 mark correctly establishes result

C	$\frac{dv}{dt}=-\left(v+v^3\right)$ $\frac{dt}{dv}=-\frac{1}{v+v^3}$ $=\frac{-1}{v\left(1+v^2\right)}$ $=\frac{v}{1+v^2}-\frac{1}{v}$ $t=\int_{\sqrt{3}}^{V}\frac{v}{1+v^2}-\frac{1}{v}\,dv$ $=\left[\frac{1}{2}\log_e\left(1+v^2\right)-\log_e v\right]_{\sqrt{3}}^{V}$ $=\frac{1}{2}\left[\log_e\left(1+v^2\right)-2\log_e v\right]_{\sqrt{3}}^{V}$ $=\frac{1}{2}\left[\log_e\frac{1+v^2}{v^2}\right]_{\sqrt{3}}^{V}$ $=\frac{1}{2}\left[\log_e\left(\frac{1+V^2}{V^2}\right)-\log_e\left(\frac{4}{3}\right)\right]$ $=\frac{1}{2}\log_e\left[\frac{3\left(1+V^2\right)}{4V^2}\right]$	1 mark $\frac{dt}{dv}=-1\frac{1}{v+v^3}$ 1 mark correct partial fractions 1 mark correct integration 1 mark correct result
D	$t=\frac{1}{2}\log_e\left[\frac{3\left(1+V^2\right)}{4V^2}\right]$ $e^{2t}=\frac{3\left(1+V^2\right)}{4V^2}$ $4V^2e^{2t}=3+3V^2$ $V^2\left(4e^{2t}-3\right)=3$ $V^2=\frac{3}{4e^{2t}-3}$	1 mark $e^{2t}=\frac{3\left(1+V^2\right)}{4V^2}$ 1 mark correct result
E	As $t\to\infty, v\to 0$ hence $x\to\tan^{-1}\sqrt{3}$	1 mark correct both parts