International Grammar School

Year 12

Mathematics Extension 2

HSC Trial Examination

2015

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this paper
- In questions 11 – 16, show all relevant reasoning and/or calculations

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Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks

Attempt Questions 1 – 10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

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1. Evaluate $\int \frac{dx}{x^2-4x+13}$

(A) $\frac{1}{3}\,tan^{-1}\left(\frac{x-2}{3}\right)$ + C

(B) $\frac{2}{3}\,tan^{-1}\left(\frac{x-2}{3}\right)$ + C

(C) $\frac{1}{3}\,tan^{-1}\left(\frac{2x-4}{3}\right)$ + C

(D) $\frac{2}{3}\,tan^{-1}\left(\frac{2x-4}{3}\right)$ + C

2. The foci of the hyperbola $\frac{y^2}{8} - \frac{x^2}{12} = 1$ are:

(A) $(\pm 2\sqrt{5}, 0))$

(B) $(\pm\sqrt{30}, 0))$

(C) $(0, \pm 2\sqrt{5}))$

(D) $(0, \pm\sqrt{30}))$

3. The gradient of the curve $xy - x^2 + 3 = 0$ at the point when $x = 1$ is:

(A) -4

(B) -1

(C) 1

(D) 4

4. The region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant is rotated about the y–axis. The volume of the solid of revolution formed can be found using:

(A) $V = \pi \int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}}\right) dy$

(B) $V = \pi \int_0^1 \left(y^{\frac{1}{2}} - y^{\frac{1}{3}}\right) dy$

(C) $V = \pi \int_0^1 \left(y^{\frac{2}{3}} - y\right) dy$

(D) $V = \pi \int_0^1 (x^4 - x^6)\ dx$

5. The five fifth roots of $1 + \sqrt{3}i$ are:

(A) $2^{\frac{1}{5}}\, cis\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

(B) $2^5\, cis\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$

(C) $2^{\frac{1}{5}}\, cis\left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

(D) $2^5\, cis\left(\frac{2k\pi}{5} + \frac{\pi}{30}\right), k = 0, 1, 2, 3, 4$

6. The locus of a complex number z is the line $4x - 3y - 12 = 0$

What is the minimum value of $|z|$?

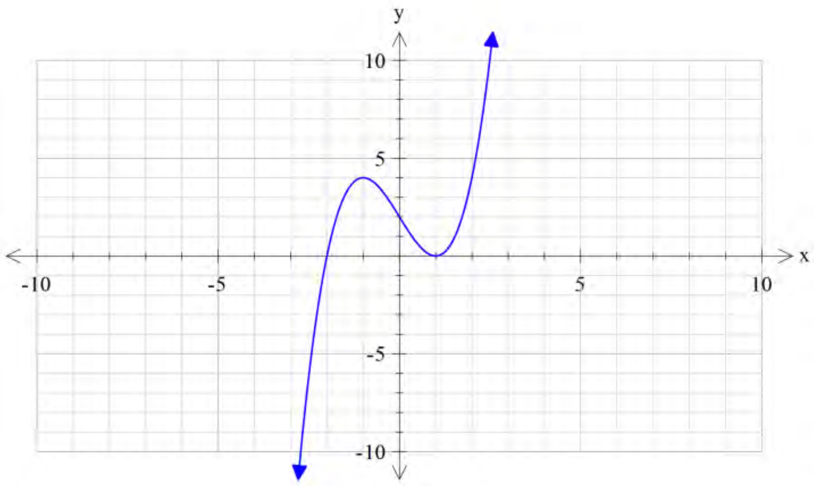
(A) $\frac{12}{5}$

(B) 3

(C) 4

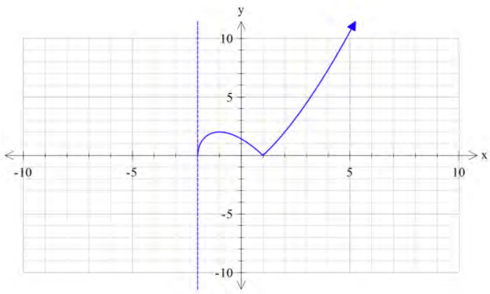
(D) 5

7. The diagram of $y = f(x)$ is drawn below.

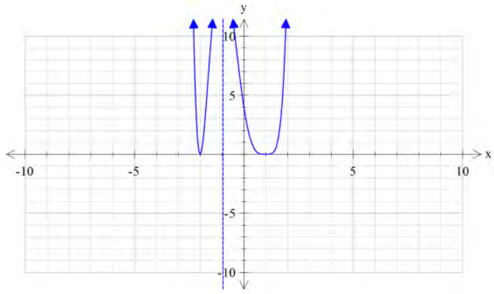


Which of the diagrams below best represents $y = \sqrt{f(x)}$

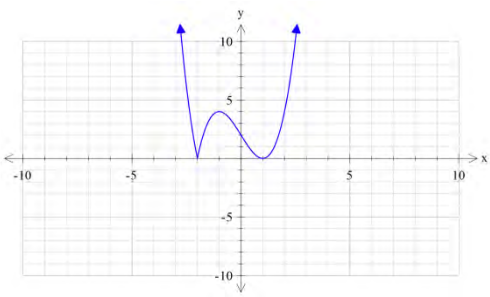
(A)



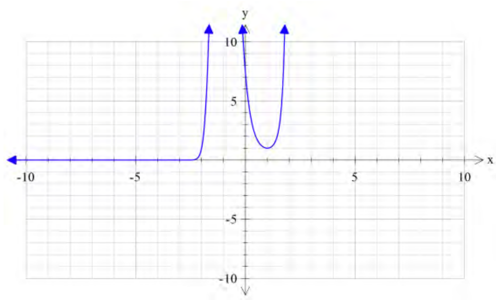
(B)



(C)



(D)



8. An object of mass 5kg is tied to a piece of rope, 3 metres in length, which has a breaking strain of 240N.

The rope is then swung in a horizontal circle.

What is the angular velocity of the object at the moment the rope breaks?

(A) 2

(B) 4

(C) 8

(D) 16

9. What is the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$?

(A) $-3i - 2$

(B) $3i - 2$

(C) $3i + 2$

(D) $2 - 3i$

10. Solve the inequality: $\frac{x+1}{x-3} \leq \frac{x+3}{x-2}$.

(A) $x < 2$ and $x > 3$

(B) $x < 2$ and $3 < x \leq 7$

(C) $2 < x < 3$

(D) $2 < x < 3$ and $x \geq 7$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$. Find the value of:

(i) $\bar{B}$ **1**

(ii) $\frac{A}{B}$ **2**

(iii) $\sqrt{B}$ **2**

(iv) The modulus and argument of A **2**

(v) A^4 **1**

(b) The roots of the polynomial equation $2x^3 - 3x^2 + 4x - 5 = 0$ are α, β and γ.
Find the polynomial equation which has roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$. **2**

(ii) $2\alpha, 2\beta$ and 2γ. **2**

(c) Find $\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} 3x \ \sin(x^2)\ dx$. **3**

(b) (i) Find the values of A, B, and C such that: **2**

$$\frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

(ii) Hence evaluate $\displaystyle\int \frac{4x^2-3x-4}{x^3+x^2-2x}\ .dx$ **2**

(c) Solve the equation $x^4 - 7x^3 + 17x^2 - x - 26 = 0$, given that $x = (3 - 2i)$ is a root of the equation. **3**

(d) (i) Show that the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$ is $x + t^2 y - 2ct = 0$. **2**

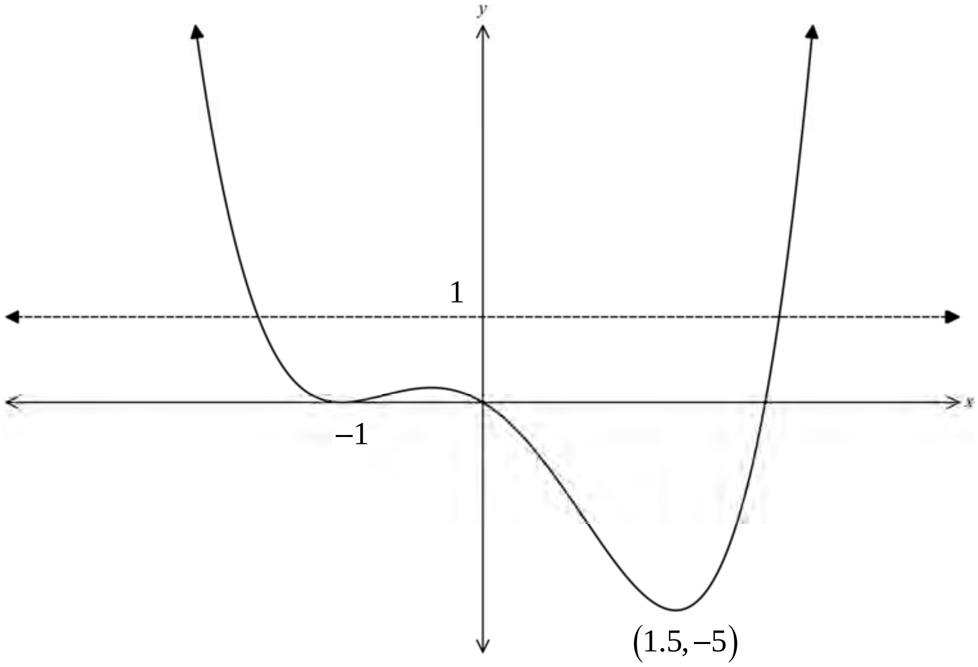
(ii) Find the coordinates of A and B where this tangent cuts the x and y axes respectively. **2**

(iii) Prove that the area of the triangle OAB is a constant, where O is the origin. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is shown below.



Draw separate sketches for each of the following:

(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) Show that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(x_1, \ y_1)$ is given by the equation: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$. **3**

Question 13 continues on the next page.

(c) A particle of unit mass is projected vertically upwards. The resistance to the motion is proportional to the square of the velocity. The velocity of projection is V.

(i) Show that the acceleration is given by: $\ddot{x} = -(g + kv^2)$. **1**

(ii) Show that the maximum height H reached is: **2**

$$H = \frac{1}{2k}\ln\left\{\frac{(g + kV^2)}{(g)}\right\}$$

(iii) Show that T, the time taken to reach H is: **2**

$$T = \frac{1}{\sqrt{kg}}\, tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)V$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Show that: $\dfrac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$. **3**

(b) A mass of 5kg, on the end of a string 0·5 metre long, is rotating in a conical pendulum with angular velocity 2π radians per second. Use $g = 10\text{m}/\text{s}^2$ and let θ be the angle that the string makes with the vertical.

(i) Draw a diagram showing all the forces acting on the mass. **1**

(ii) By resolving forces, find the tension in the string. **2**

(iii) Find θ, correct to the nearest degree. **1**

(c) A sequence is defined such that $u_1 = 1, u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. **4**

Prove by induction that $u_n < \left(\dfrac{7}{4}\right)^n$ for integers $n \geq 1$.

(d) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by $y = 3x^2 - x^3$ and the x axis around the y-axis. **4**

End of Question 14

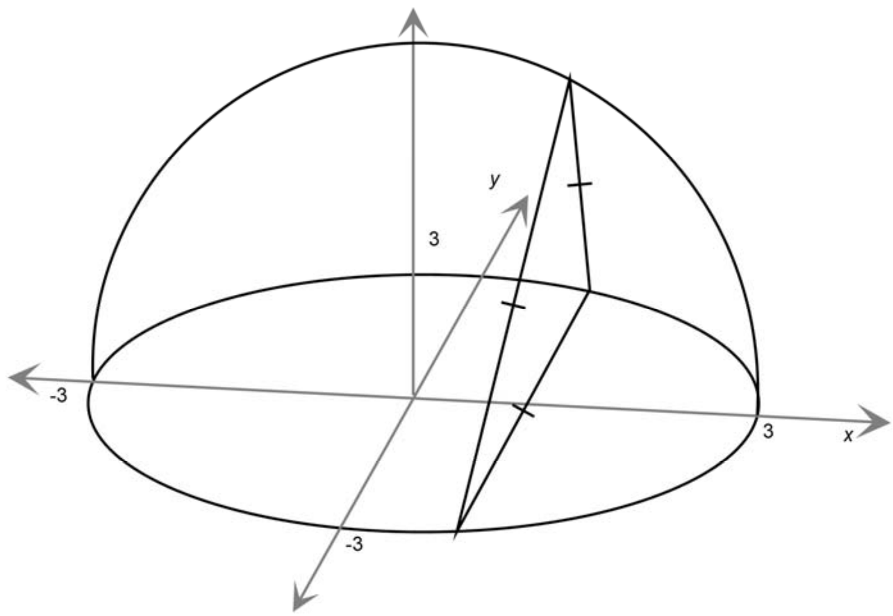
Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Derive the reduction formula: **2**

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2}\, dx$$

(ii) Use this reduction formula to evaluate $\int_0^1 x^5 e^{-x^2}\, dx$ **2**

(b)



The diagram above shows a solid which has the circle $x^2 + y^2 = 9$ as its base.

The cross-section perpendicular to the *x* axis is an equilateral triangle.

(i) Show that the area of a triangle is given by: **2**
$Area = \sqrt{3}\left(9 - x^2\right)$

(ii) Hence or otherwise find the volume of the solid. **2**

Question 15 continues on the next page.

(c) Given that $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, has a double root at $x = \alpha$, find the value of α. **3**

(d) If z represents the complex number $x + iy$, sketch the regions:

(i) $|arg\, z| < \frac{\pi}{4}$ **2**

(ii) $Im\,(z^2) = 4$ **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > b$.

(i) Show that the equation of the tangent at the point $P(asec\,\theta, btan\,\theta)$ has the equation $bxsec\,\theta - aytan\,\theta\ = ab$. **2**

(ii) Find the equation of the normal at P. **2**

(iii) Find the coordinates of the points A and B where the tangent and normal respectively cut the y-axis. **2**

(iv) Show that AB is the diameter of the circle that passes through the foci of the hyperbola. **3**

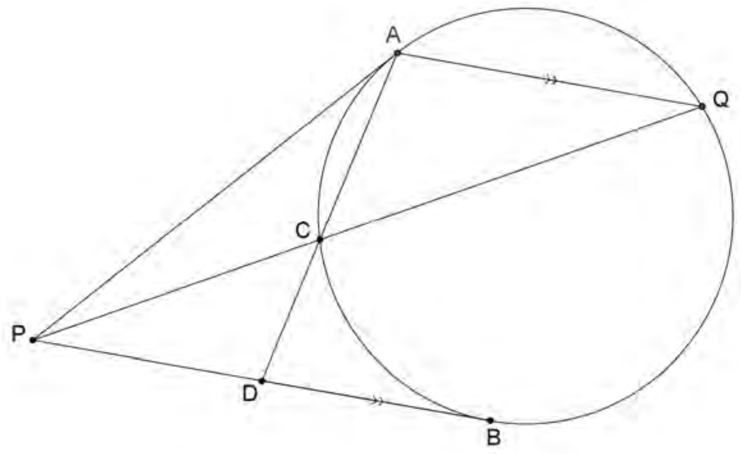
(b) Five letters are chosen from the letters of the word *CHRISTMAS*. **2**

These five letters are then placed alongside one another to form a five letter arrangement.

Find the number of distinct five letter arrangements which are possible, considering all choices.

Question 16 continues on the next page.

(c) In the diagram below, PA and PB are tangents to the circle. The chord AQ is parallel to the tangent PB. PCQ is a secant to the circle and chord AC produced meets PB at D.



i) Show that ΔCDP is similar to ΔPDA. **2**

ii) Show that $PD^2 = AD \times CD$ and hence, or otherwise, prove that AD bisects PB. **2**

End of Examination.

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

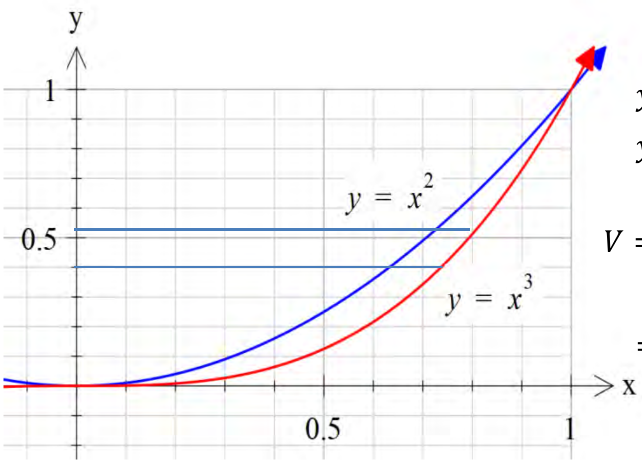
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2015 Extension 2 Trial solutions

Multiple Choice Worked Solutions

No	Working	Answer
1	$\int \frac{dx}{x^2-4x+13} = \int \frac{dx}{x^2-4x+4+9}$ $= \int \frac{dx}{(x-2)^2+9}$ $= \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$	**A**
2	$\frac{y^2}{8} - \frac{x^2}{12} = 1$ $a = 2\sqrt{2}, b = 2\sqrt{3}$ $b^2 = a^2(e^2-1)$ $(2\sqrt{3})^2 = (2\sqrt{2})^2(e^2-1)$ $12 = 8(e^2-1)$ $\frac{12}{8} = e^2 - 1$ $e^2 = \frac{20}{8} = \frac{10}{4}$ $e = \frac{\sqrt{10}}{2}$ Foci = $(0, \pm ae) = \left(0, \pm 2\sqrt{2}\left(\frac{\sqrt{10}}{2}\right)\right) = (0, \pm\sqrt{20}) = (0, \pm 2\sqrt{5})$	**C**
3	$xy - x^2 + 3 = 0$ $x\frac{dy}{dx} + y - 2x = 0$ $x\frac{dy}{dx} = 2x - y$ $\frac{dy}{dx} = \frac{2x-y}{x}$ when $x = 1$, $y - 1 + 3 = 0$ $y = -2$ At $(1, -2)$ $\frac{dy}{dx} = \frac{2(1)-(-2)}{1} = 2 + 2 = 4$	**D**
4	 $y = x^3 \rightarrow x = y^{\frac{1}{3}}$ $y = x^2 \rightarrow x = y^{\frac{1}{2}}$ $V = \pi \int_0^1 \left[\left(y^{\frac{1}{3}}\right)^2 - \left(y^{\frac{1}{2}}\right)^2\right] dy$ $= \pi \int_0^1 \left(y^{\frac{2}{3}} - y\right) dy$	**C**
5	$z^5 = 1 + \sqrt{3}i$ $R = \sqrt{1^2 + (\sqrt{3})^2} = 2$ Arg z: $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ $\therefore z^5 = 2\,cis\,\frac{\pi}{3}$ $z = 2^{\frac{1}{5}}\,cis\left(\frac{2k\pi}{5} + \frac{\pi}{15}\right), k = 0, 1, 2, 3, 4$	**A**

6	$\lvert z\rvert$ represents the length of the vector from the origin to z. Hence the minimum distance from the origin to z is the perpendicular distance from (0, 0) to 4x – 3y – 12 = 0 d = $\left\lvert \frac{0+0-12}{\sqrt{4^2+(-3)^2}} \right\rvert = \left\lvert \frac{12}{5} \right\rvert = \frac{12}{5}$	**A**
7	Graph A	**A**
8	$F = mr\omega^2$ $240 = 5 \times 3 \times \omega^2$ $240 = 15\omega^2$ $16 = \omega^2$ $\omega = 4$	**B**
9	$P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$ Let $x = 1 + i$ $x^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$ $x^3 = 2i\,(1+i) = 2i + 2i^2 = 2i - 2$ Remainder = $P(1+i) = 2i - 2 + 2i - (1+i) + 1$ $= 4i - 1 - 1 - i$ $= 3i - 2$	**B**
10	$\frac{x+1}{x-3} \le \frac{x+3}{x-2}$ $x \ne 3\ or\ 2$ Then $(x+1)(x-2) = (x+3)(x-3)$ $x^2 - 2x + x - 2 = x^2 - 9$ $x^2 - x - 2 = x^2 - 9$ $-x = -7$ $x = 7$ By inspection, $2 < x < 3 \cap x \ge 7$	**D**

1.	A ●	B ○	C ○	D ○
2.	A ○	B ○	C ●	D ○
3.	A ○	B ○	C ○	D ●
4.	A ○	B ○	C ●	D ○
5.	A ●	B ○	C ○	D ○
6.	A ○	B ○	C ●	D ○
7.	A ●	B ○	C ○	D ○
8.	A ○	B ●	C ○	D ○
9.	A ○	B ●	C ○	D ○
10.	A ○	B ○	C ○	D ●

Question 11		2015	
	Solution	Marks	Allocation of marks
(a)	$A = 3 + 3\sqrt{3}i$ and $B = -5 - 12i$. (i) $\bar{B} = \overline{-5 - 12i}$ $= -5 + 12i$	1	Answer
	(ii) $\frac{A}{B} = \frac{3+3\sqrt{3}i}{-5-12i}$ $\frac{A}{B} = \frac{3+3\sqrt{3}i}{-5-12i} \times \frac{-5+12i}{-5+12i}$ $= \frac{-15+36i-15\sqrt{3}i-36\sqrt{3}}{25-144i^2}$ $= \frac{(-15-36\sqrt{3})+(36-15\sqrt{3})i}{169}$	2	1 – correct product 1 – correct answer
	(iii) $\sqrt{B} = \sqrt{-5 - 12i}$ $Let\ (x+iy)^2 = -5-12i$ $\therefore\ x^2 + 2ixy - y^2 = -5 - 12i$ $\therefore\ x^2 - y^2 = -5$ ----------------(1) $and\ \ 2xy = -12$ $(x^2+y^2)^2 = (x^2-y^2)^2 + 4ixy$ $= (-5)^2 + (-12)^2$ $= 169$ $\therefore\ x^2 + y^2 = 13$ ------------------- (2) (1) + (2) $2x^2 = 8 \rightarrow x^2 = 4 \rightarrow x = \pm 2$ (2) − (1) $2y^2 = 18 \rightarrow y^2 = 9 \rightarrow y = \pm 3$ Since $2xy = -12$ $\sqrt{B} = \sqrt{-5-12i} = \pm(2-3i)$	2	1 – working 1 – Answer
	(iv) Modulus $(r) = \sqrt{(3)^2 + (3\sqrt{3})^2} = \sqrt{36} = 6$ Argument: $tan\,\theta = \frac{3\sqrt{3}}{3} = \sqrt{3}\ ,\ \ \theta = \frac{\pi}{3}$	2	1 - modulus 1 - argument
	(v) $A^4 = \left(6\ cis\ \frac{\pi}{3}\right)^4 = 1296\ cis\ \frac{4\pi}{3} = 1296\ cis\ \frac{-2\pi}{3}$	1	Correct answer

Question 11		**2015**	
	Solution	Marks	Allocation of marks
(b)	(i) $2x^3 - 3x^2 + 4x - 5 = 0$ Let $X = \frac{1}{x}$, $\therefore x = \frac{1}{X}$ Therefore equation is $2\left(\frac{1}{X}\right)^3 - 3\left(\frac{1}{X}\right)^2 + 4\left(\frac{1}{X}\right) - 5 = 0$ i.e. $\frac{2}{X^3} - \frac{3}{X^2} + \frac{4}{X} - 5 = 0$ Multiply by X^3 $2 - 3X + 4X^2 - 5X^3 = 0$ ie $5x^3 - 4x^2 + 3x - 2 = 0$	**2**	1 – correct substitution 1 – correct equation
	(ii)) $2x^3 - 3x^2 + 4x - 5 = 0$ Let $X = 2x$ $\therefore x = \frac{X}{2}$ *Therefore equation is* $2\left(\frac{X}{2}\right)^3 - 3\left(\frac{X}{2}\right)^2 + 4\left(\frac{X}{2}\right) - 5 = 0$ $2\left(\frac{X^3}{8}\right) - 3\left(\frac{X^2}{4}\right) + \frac{4X}{2} - 5 = 0$ $\frac{X^3}{4} - \frac{3X^2}{4} + 2X - 5 = 0$ $X^3 - 3X^2 + 8X - 20 = 0$ i.e. $x^3 - 3x^2 + 8x - 20 = 0$	**2**	1 – correct substitution 1 – correct equation

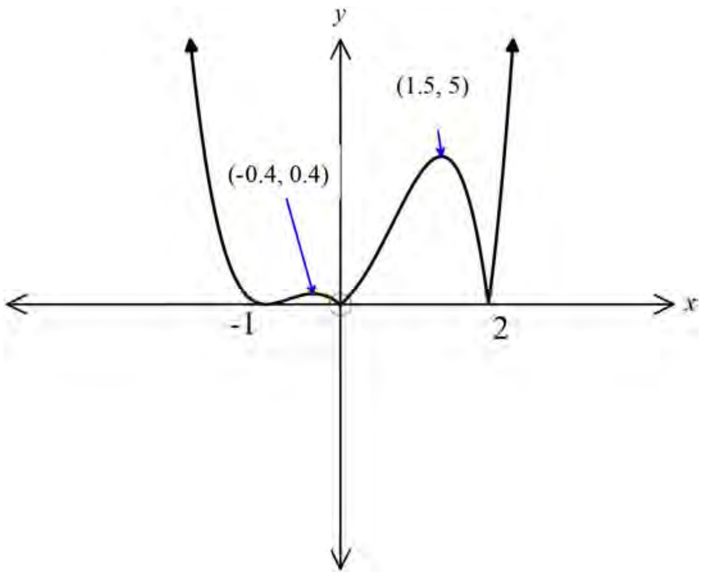
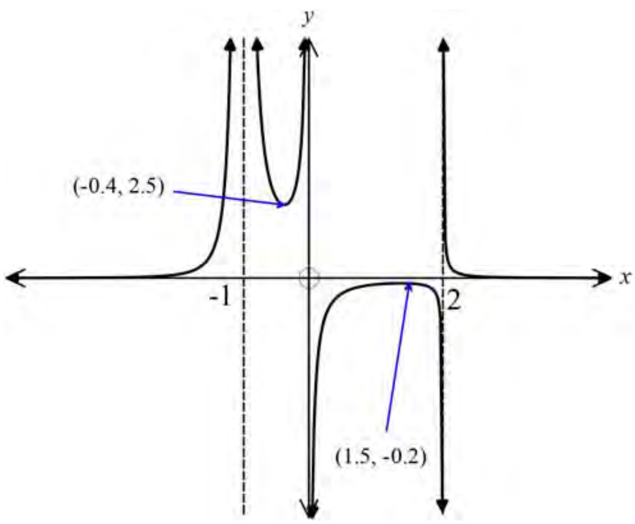
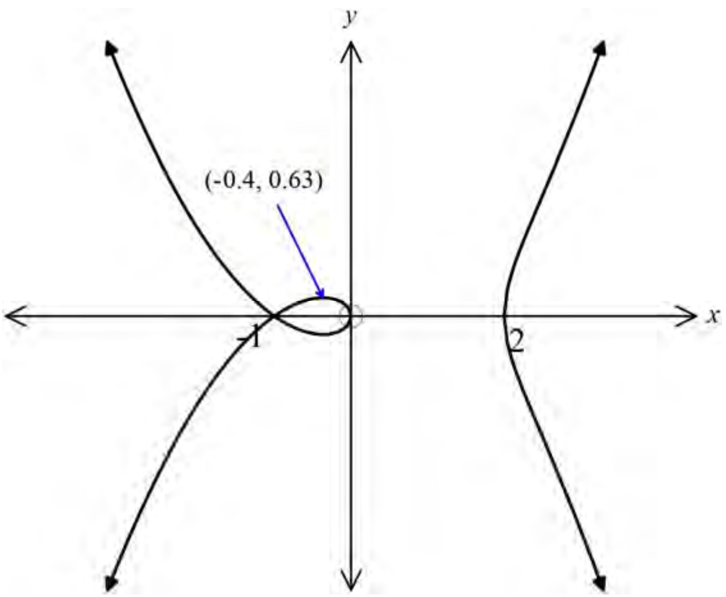
Question 11		2015	
	Solution	Marks	Allocation of marks
(c)	$\int \frac{dx}{\sqrt{9 + 16x - 4x^2}}$ $9 + 16x - 4x^2 = 9 - 4(x^2 - 4x)$ $= 9 - 4(x^2 - 4x + 4) + 16$ $= 25 - 4(x - 2)^2$ $\int \frac{dx}{\sqrt{9 + 16x - 4x^2}} = \int \frac{dx}{\sqrt{25 - 4(x-2)^2}}$ $= \frac{1}{5} \int \frac{dx}{\sqrt{1 - \frac{4}{25}(x - 2)^2}}$ $u = \frac{2(x-2)}{5} + c$ $du = \frac{2}{5} dx$ $dx = \frac{5}{2} du$ $= \frac{1}{5} \int \frac{\frac{5}{2} du}{\sqrt{1 - u^2}}$ $= \frac{1}{2} \int \frac{du}{\sqrt{1 - u^2}}$ $= \frac{1}{2} sin^{-1} u$ $= \frac{1}{2} sin^{-1} \left(\frac{2(x-2)}{5}\right)$	**3**	1 – correct manipulation 1 – correct substitution 1 – correct answer

Question 12		2015	
	Solution	Marks	Allocation of marks

Question 12		**2015**	
	Solution	Marks	Allocation of marks
(a)	$\int_0^{\frac{\sqrt{\pi}}{2}} 3x \ sin(x^2)\ dx$ Substitute $u = x^2$ $\frac{du}{dx} = 2x$ $du = 2x\ dx$ $\frac{3}{2} du = 3x\ dx$ $x = 0 \Rightarrow u = 0^2 = 0$ $x = \frac{\sqrt{\pi}}{2} \Rightarrow u = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$ $\int_0^{\frac{\sqrt{\pi}}{2}} 3x \ sin(x^2)\ dx = \int_0^{\frac{\pi}{4}} \frac{3}{2}\ sin\ u\ du$ $= \frac{3}{2}\int_0^{\frac{\pi}{4}} sin\ u\ du$ $= \frac{3}{2}\left[-cos\ u\right]_0^{\frac{\pi}{4}}$ $= -\frac{3}{2}\left(cos\left(\frac{\pi}{4}\right) - cos(0)\right)$ $= -\frac{3}{2}\left(\frac{1}{\sqrt{2}} - 1\right)$ $= -\frac{3}{2}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$ $= \frac{3\sqrt{2}-3}{2\sqrt{2}}$ $= \frac{6-3\sqrt{2}}{4}$ $\int_0^{\frac{\sqrt{\pi}}{2}} 3x\sin\left(x^2\right)dx$ $=-\frac{3}{2}\left[\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ $=-\frac{3}{2}\left[\cos\frac{\pi}{4}-\cos 0\right]$ $=-\frac{3}{2}\left(\frac{1}{\sqrt{2}-1}\right)$	**3**	USING A SUBSTITUTION 1 – changing limits and variable 1 – integral including correct limits 1 – substitution and simplification to get answer WITHOUT A SUBSTITUTION 1 Correct integration 1 correct working Correct answer

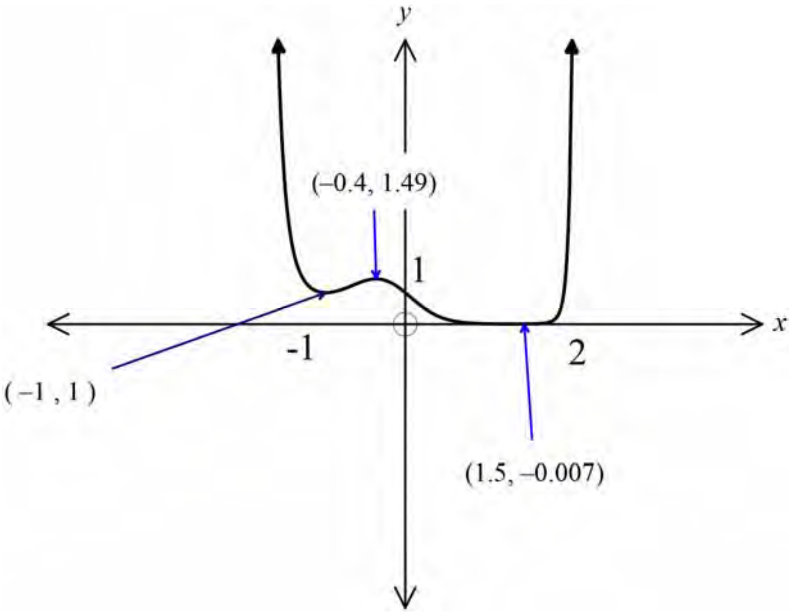
Question 12		2015	
	Solution	Marks	Allocation of marks
(b)	(i) $\frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$ $\therefore\ 4x^2 - 3x - 4 = A(x-1)(x+2) + B\,x\,(x+2) + C\,x\,(x-1)$ When $x = 0, \quad -4 = -2A \qquad \therefore A = 2$ $x = -2, \quad 18 = 6C \qquad \therefore C = 3$ $x = 1, \quad -3 = 3B \qquad \therefore B = -1$ $\therefore\ \frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}$ (ii) $\int \frac{4x^2-3x-4}{x^3+x^2-2x} = \int \left(\frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\right) dx$ $= 2ln\,x - ln(x-1) + 3\,ln(x+2) + c$	**2**	1 - Working 1 – correct values 1 – correct integral 1 – correct answer
(c)	$x^4 - 7x^3 + 17x^2 - x - 26 = 0$ $(3-2i)$ is a factor $\therefore\ (3+2i)$ is also a factor since coefficients are real $\therefore\ x^2 - 6x + 13$ is a factor. By division, $x^4 - 7x^3 + 17x^2 - x - 26 = (x^2 - 6x + 13)(x^2 - x - 2)$ $= (x^2 - 6x + 13)(x-2)(x+1)$ Therefore solution to $x^4 - 7x^3 + 17x^2 - x - 26 = 0$ is: $x = 3 \pm 2i, -1\ and\ 2$ OR USE SUMS AND PRODUCTS OF ROOTS $\alpha = 3-2i,\ \beta = 3+2i,\ \gamma = ?,\ \delta = ?$ $\sum \alpha = 6 + \gamma + \delta \rightarrow \gamma + \delta = 1$ $\prod \alpha = 13\gamma\delta = -26 \rightarrow \gamma\delta = -2$ $\delta = -\frac{2}{\gamma}$ so $\gamma - \frac{2}{\gamma} = 1$ $\gamma^2 - \gamma - 2 = 0$ $\gamma = 2, -1$ $\therefore$ roots are $3-2i, 3+2i, 2, -1$	**3**	1 – using conjugate theorem Method 1 1 – division 1 – answer Method 2 1 correct use of sums and products 1 answer

Question 12		2015	
	Solution	Marks	Allocation of marks
(d)	(i) $xy = c^2$ $P\left(ct, \frac{c}{t}\right)$ By implicit differentiation $y + x\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ At $P\left(ct, \frac{c}{t}\right)$ $\frac{dy}{dx} = -\frac{c}{t} \div ct$ $= -\frac{1}{t^2}$ $y - y_1 = m(x - x_1)$ $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ $t^2y - ct = -x + ct$ $x + t^2y - 2ct = 0$	**2**	1 – gradient of tangent 1 – equation of tangent
	(ii) When $y = 0$, $x + 0 - 2ct = 0$ $x = 2ct$ $\therefore A(2ct, 0)$ When $x = 0$, $0 + t^2y - 2ct = 0$ $y = \frac{2ct}{t^2} = \frac{2c}{t}$ $\therefore B\left(0, \frac{2c}{t}\right)$	**2**	One mark for each coordinate
	(iii) Now $OA = 2ct$ $OB = \frac{2c}{t}$ Area Triangle OAB $= \frac{1}{2}(2ct)\left(\frac{2c}{t}\right)$ $= 2c^2$ which is a constant as c is a constant.	**1**	Correct area

Question 13		**2014**	
	Solution	Marks	Allocation of marks
(a)	(i) 	**1**	1 – correct graph all coords shown
	(ii) 	**2**	1 – vertical asymptotes 1 – correct graph all coords shown
	(iii) 	**2**	1 –correct shape one side of axes 1 - correct graph both sides

Question 13		2014	
	Solution	Marks	Allocation of marks

(iv) — Marks: **2**



Allocation of marks:

1 – correct behaviour $x \to \infty$

1 - correct graph all coords shown

(b) — Marks: **3**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2b^2x}{2a^2y}$$

At $P(x_1, \ y_1)$ $\frac{dy}{dx} = \frac{-b^2x_1}{a^2y_1}$

Normal $m = \frac{a^2y_1}{b^2x_1}$

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$$

$$b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2y_1x_1$$

$$a^2y_1x - b^2x_1y = a^2y_1x_1 - b^2x_1y_1$$

$(\div \ x_1y_1)$

$$\therefore \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

Allocation of marks:

1 – gradient of normal

1 use of equation

1 – completion of proof

(c) (i) — Marks: **1**

$$m\ddot{x} = -mg - mkv^2$$

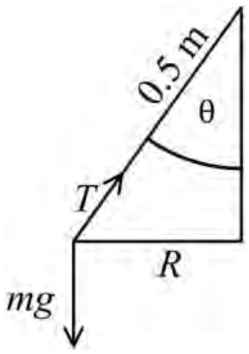
$$\ddot{x} = -g - kv^2$$

$$\ddot{x} = -(g + kv^2)$$

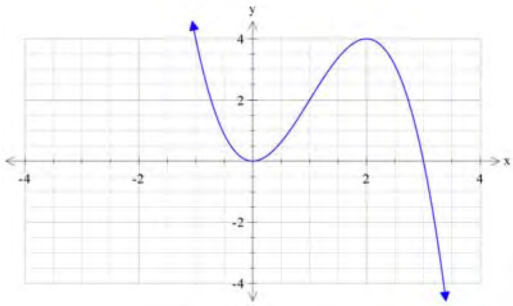
Allocation of marks: Correct answer

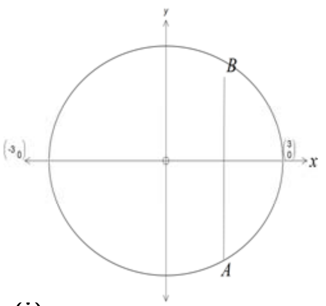
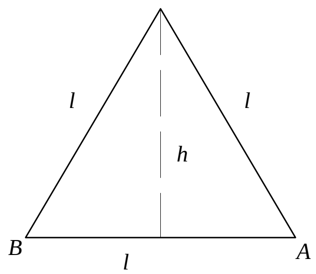
Question 13		2014	
	Solution	Marks	Allocation of marks
	(ii) $v\frac{dv}{dx} = -(g+kv^2)$ $\frac{dv}{dx} = \frac{-(g+kv^2)}{v}$ $$\frac{dx}{dv} = -\frac{v}{(g+kv^2)}$$ $$x = \int -\frac{v}{(g+kv^2)}\,dv$$ $$x = -\frac{1}{2k}\int \frac{2kv}{(g+kv^2)}\,dv$$ $$x = -\frac{1}{2k}\ln(g+kv^2)$$ $at\ x = 0, v = V$ $$0 = -\frac{1}{2k}\ln(g+kV^2) + C$$ $$\therefore C = \frac{1}{2k}\ln(g+kV^2)$$ $$x = -\frac{1}{2k}\ln(g+kv^2) + \frac{1}{2k}\ln(g+kV^2)$$ $$x = \frac{1}{2k}\ln\left\{\frac{(g+kV^2)}{(g+kv^2)}\right\}$$ Maximum height is obtained when $v = 0$. $$H = \frac{1}{2k}\ln\left\{\frac{(g+kV^2)}{g}\right\}$$ (iii) $$\frac{dv}{dt} = -(g+kv^2)$$ $$\frac{dt}{dv} = \frac{-1}{(g+kv^2)}$$ $$t = \int \frac{-1}{(g+kv^2)}\,dv$$ $$= -\frac{1}{k}\int \frac{1}{\frac{g}{k}+v^2}\,dv$$ $$t = -\frac{1}{k} \times \frac{1}{\sqrt{\frac{g}{k}}}\,tan^{-1}\left(\frac{v}{\sqrt{\frac{g}{k}}}\right) + C_1$$ A $$t = -\frac{1}{k} \times \frac{\sqrt{k}}{\sqrt{g}}\,tan^{-1}\left(\frac{\sqrt{k}v}{\sqrt{g}}\right) + C_1$$ $$A = -\frac{1}{k} \times \frac{\sqrt{k}}{\sqrt{g}}\,tan^{-1}\left(\frac{\sqrt{k}v}{g}\right) + C_1$$	**4**	1 – Evaluating integral *1 expression for H* *1 evaluating integral*

Question 13		2014	
	Solution	Marks	Allocation of marks
	$At\ t = 0, v = V$ $0 = -\frac{1}{\sqrt{kg}}\ tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)V + C_1$ $C_1 = \frac{1}{\sqrt{kg}}\ tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)V$ $t = -\frac{1}{\sqrt{kg}}\ tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)v + \frac{1}{\sqrt{kg}}\ tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)\ V$ Maximum height reached when $v = 0$, i.e. $T = \frac{1}{\sqrt{kg}}\ tan^{-1}\left(\frac{\sqrt{k}}{\sqrt{g}}\right)V$		1 expression for T

Question 14		**2014**	
	Solution	Marks	Allocation of marks
(a)	$\dfrac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$ $LHS = \dfrac{\cos A - \cos(A+2B)}{2\sin B}$ $= \dfrac{\cos A - (\cos A\cos 2B - \sin A\sin 2B)}{2\sin B}$ $= \dfrac{\cos A - \cos A\,(1-2\sin^2 B) + 2\sin A\sin B\cos B}{2\sin B}$ $= \dfrac{\cos A - \cos A + 2\sin^2 B\cos A + 2\sin A\sin B\cos B}{2\sin B}$ $= \dfrac{2\sin^2 B\cos A + 2\sin A\sin B\cos B}{2\sin B}$ $= \dfrac{2\sin B(\sin B\cos A + \sin A\cos B)}{2\sin B}$ $= \sin B\cos A + \sin A\cos B$ $= \sin A\cos B + \sin B\cos A$ $= \sin(A+B)$ $= RHS$ $\therefore \dfrac{\cos A - \cos(A+2B)}{2\sin B} = \sin(A+B)$	**3**	1 -Using cosine double angle 1 – working 1 – completion of proof
(b)	(i)  $\sin\alpha = \dfrac{R}{0.5}$ $R = 0.5\sin\alpha$	**1**	diagram
	(ii) $T\sin\theta = 5 \times (2\pi)^2 \times 0\cdot 5\sin\theta$ $T = 5 \times 4\pi^2 \times 0\cdot 5$ = 98.696 = 99N (nearest newton) OR $10\pi^2$N	**2**	1 – substitution 1 - answer
	(iv) $T\cos\theta = mg$ = (5) (10) = 50 $\cos\theta = \dfrac{50}{T}$ $= \dfrac{50}{98.696}$ $\theta = 59.562$ = 60° (nearest degree)	**1**	Correct working to answer

Question 14		2014	
	Solution	Marks	Allocation of marks
(c)	Step 1 Prove true for $n=1$ and $n=2$ $u_1=1<\left(\frac{7}{4}\right)^1$ $\therefore$ true for $n=1$ $u_2=1<\left(\frac{7}{4}\right)^2$ $\therefore$ true for $n=2$ Step 2 Let $n=k$ and $n=k-1$ be values for which the statement is true *ie* $u_k<\left(\frac{7}{4}\right)^k$ and $u_{k-1}<\left(\frac{7}{4}\right)^{k-1}$ Step 3 Prove true for $n=k+1$ *ie* $u_{k+1}<\left(\frac{7}{4}\right)^{k+1}$ $u_{k+1}=u_k+u_{k-1}$ $\therefore u_{k+1}<\left(\frac{7}{4}\right)^k+\left(\frac{7}{4}\right)^{k-1}$ from Step 2 $u_{k+1}<\left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}+1\right)$ $u_{k+1}<\left(\frac{7}{4}\right)^{k-1}\left(\frac{11}{4}\right)$ *now* $1<\frac{11}{4}<\left(\frac{7}{4}\right)^2$ so the value of the RHS has increased $\therefore u_{k+1}<\left(\frac{7}{4}\right)^{k-1}\left(\frac{49}{16}\right)$ $u_{k+1}<\left(\frac{7}{4}\right)^{k-1}\left(\frac{7}{4}\right)^2$ $u_{k+1}<\left(\frac{7}{4}\right)^{k+1}$ as required $\therefore$ true by mathematical induction	**4**	1 – *n = k, n=k-1 or similar* having tested for $n=1$ and $n=2$ 1 *n=k+1 or similar (following on logically from Step 2)* 1 – using assumption in Step 2 1 – proving true for *n = k + 1* including justifying truth of inequality

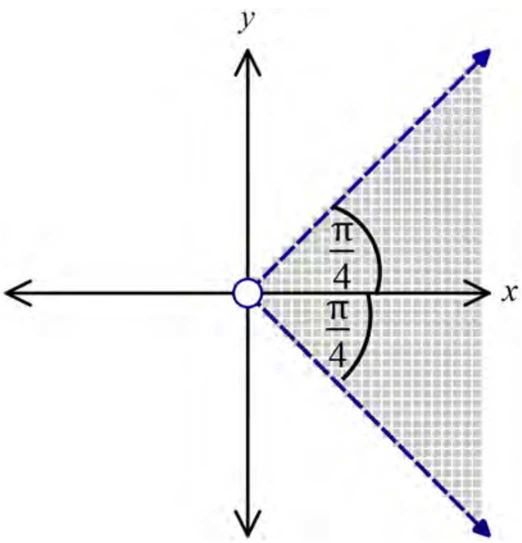
Question 14		**2014**	
	Solution	Marks	Allocation of marks
(d)	 $\partial V = 2\pi xy\partial x$ $= 2\pi x\left(3x^2 - x^3\right)\partial x$ $V = \lim_{\partial x \to \infty} \sum_{0}^{3} 2\pi x\left(3x^2 - x^3\right)\partial x$ Volume = $\int_a^b 2\pi xy\, dx$ $= \int_0^3 2\pi x(3x^2 - x^3)\, dx$ $= 2\pi \int_0^3 (3x^3 - x^4)\, dx$ $= 2\pi \left[\frac{3x^4}{4} - \frac{x^5}{5}\right]_0^3$ $= \frac{243\pi}{10}$ *cubic units*	**4**	2 – establishing integral 1 – integrating 1 - answer

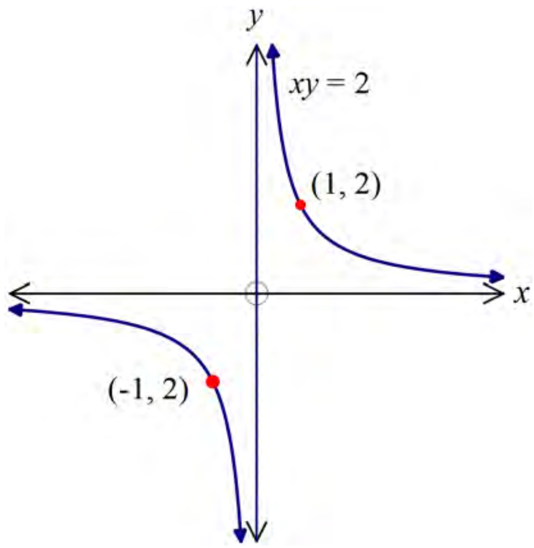
Question 15		2014	
	Solution	Marks	Allocation of marks
(a)	$\int x^n e^{-x^2} dx$ (i) Let $u = x^{n-1}$ $\quad v' = xe^{-x^2}$ $u' = (n-1)x^{n-2}$ $\quad v = -\frac{1}{2} e^{-x^2}$ $\int x^n e^{-x^2} dx = uv - \int vu'$ $= -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$	2	2 – integration by parts to derive reduction formula
	(ii) $\int_0^1 x^5 e^{-x^2} dx = \left[-\frac{1}{2} x^4 e^{-x^2}\right]_0^1 + \frac{4}{2} \int_0^1 x^3 e^{-x^2} dx$ $= \frac{-1}{2e} + 2 \int_0^1 x^3 e^{-x^2} dx$ $= \frac{-1}{2e} + 2 \left\{ \left[-\frac{1}{2} x^2 e^{-x^2}\right]_0^1 + 1 \int_0^1 x e^{-x^2} dx \right\}$ $= \frac{-1}{2e} - 2\left(\frac{1}{2e}\right) + 2\left[-\frac{1}{2} x^0 e^{-x^2}\right]_0^1 + \frac{1-1}{2} \int x^{1-2} e^{-x^2} dx$ $= \frac{-1}{2e} - \frac{1}{e} + \left[-e^{-x^2}\right]_0^1$ $= \frac{-1}{2e} - \frac{1}{e} - \frac{1}{e} + 1$ $= \frac{-1}{2e} - \frac{2}{2e} - \frac{2}{2e} + 1$ $= 1 - \frac{5}{2e}$	2	1 – first use of reduction formula 1 – simplifying to an answer
(b)	  (i) $x^2 + y^2 = 9$ $\quad \sin 60 = \frac{h}{l}$ $y = \sqrt{9 - x^2}$ $\quad h = l \sin 60$ $\therefore l = 2\sqrt{9 - x^2}$ $\quad h = \frac{\sqrt{3}}{2} l$	2	1 – expression for h

OR $A = = \frac{1}{2} l^2 \sin 60°$

$= \frac{\sqrt{3}}{4} l^2$

Question 15		**2014**	
	Solution	Marks	Allocation of marks
	$\therefore h = \sqrt{3}\,\sqrt{9 - x^2}$ $A(x) = \frac{1}{2}bh$ $= \frac{1}{2}\left(2\sqrt{9-x^2}\right)\left(\sqrt{3}\,\sqrt{9-x^2}\right)$ $= \sqrt{3}\left(9 - x^2\right)$ (ii) $V = \int_{-3}^{3} \sqrt{3}\,(9 - x^2)\,dx$ $= \sqrt{3}\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$ $= \sqrt{3}[(27-9) - (-27+9)]$ $= \sqrt{3}[18+18]$ $= 36\sqrt{3}$ $OR \quad V = 2\int_0^3 \sqrt{3}\,(9 - x^2$ $= 2\sqrt{3}\left[9x - \frac{x^3}{3}\right]_0^3$ $= 2\sqrt{3}\,[(27-9)-0]$ $= 2\sqrt{3} \times 18$ $= 36\sqrt{3}$	**2**	1 – expression for Area 1 - integral 1 - Answer

(c)	Let $f(x) = x^4 - 6x^3 + 9x^2 + 4x - 12$ $f'(x) = 4x^3 - 18x^2 + 18x + 4$ Double root when $f'(x) = f(x) = 0$ Test $x = \pm 1$ and $x = \pm 2$ (factors of 4) When $x = 2$, $f'(2) = 4(2^3) - 18(2^2) + 18(2) + 4$ $= 32 - 72 + 36 + 4 = 72 - 72 = 0$ $f(2) = (2^4) - 6(2^3) + 9(2^2) + 4(2) - 12$ $= 16 - 48 + 36 + 8 - 12 = 60 - 60 = 0$ $\therefore f'(2) = f(2) = 0$ $\therefore (x-2)$ is a repeated factor. $\therefore \alpha = 2$ is a double root.	**3**	1 – using double root theorem and finding the derivative 1 – testing for roots of $f'(x)$ 1 – testing in $f(x)$ and stating the value of α
(d)	(i) $\arg z = \theta$ where $\tan\theta = \frac{y}{x}$ If $\lvert\arg(z)\rvert < \frac{\pi}{4}$ then $-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$ 	**2**	1 – Graph 1 – showing main features

	(ii) $z = x + iy$ $z^2 = (x+iy)^2 = x^2 + 2xyi - y^2$ $= x^2 - y^2 + 2xyi$ $Im(z^2) = 2xy$ Graph required is $Im(z^2) = 4$ $2xy = 4$ ie $xy = 2$ or $y = \frac{2}{x}$  	**2**	1 – determining equation 1 – Graph

<table>
<tr><td colspan="2">Question 16</td><td colspan="2">2014</td></tr>
<tr><td></td><td>Solution</td><td>Marks</td><td>Allocation of marks</td></tr>
<tr><td>(a)</td><td>(i) $x = a\sec\theta$ $y = b\tan\theta$
$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\frac{dy}{d\theta} = b\sec^2\theta$
$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$
$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$
$= \frac{b\sec\theta}{a\tan\theta}$
$y - y_1 = m(x - x_1)$
$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$
$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$
$-ay\tan\theta + bx\sec\theta = ab(\sec^2\theta - \tan^2\theta)$
Since $\sec^2\theta - \tan^2\theta = 1$
$bx\sec\theta - ay\tan\theta = ab$</td><td>2</td><td>1 – deriving gradient of tangent

1 – using equation to complete proof</td></tr>
<tr><td></td><td>(ii) from (i) $m(tangent) = \frac{b\sec\theta}{a\tan\theta}$
$\therefore m\ (normal) = -\frac{a\tan\theta}{b\sec\theta}$
$y - y_1 = m(x - x_1)$
$y - b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a\sec\theta)$
$by\sec\theta - b^2\tan\theta\sec\theta = -ax\tan\theta + a^2\tan\theta\sec\theta$
By dividing by $\tan\theta\sec\theta$
$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$</td><td>2</td><td>1 – deriving gradient of normal

1 – equation of normal</td></tr>
<tr><td></td><td>(iii) Tangent:
$bx\sec\theta - ay\tan\theta = ab$
When $x = 0$ $y = \frac{-b}{\tan\theta}$ $\therefore A\left(0, \frac{-b}{\tan\theta}\right)$
Normal:
$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$
When $x = 0$ $y = \frac{(a^2 + b^2)\tan\theta}{b}$ $\therefore B\left(0, \frac{(a^2 + b^2)\tan\theta}{b}\right)$</td><td>2</td><td>1 for A

1 for B</td></tr>
</table>

Question 16		**2014**	
	Solution	Marks	Allocation of marks
	(v) Focus of hyperbola $= S\ (ae, 0)$ If AB is diameter of circle then angle ASB must be right angled. $m(AS) = \frac{0 - \frac{-b}{\tan\theta}}{ae - 0}$ $= \frac{b}{\tan\theta} \div ae$ $= \frac{b}{ae\tan\theta}$ $m(BS) = \frac{0 - \frac{(a^2+b^2)\tan\theta}{b}}{ae - 0}$ $= -\frac{(a^2+b^2)\tan\theta}{b} \div ae = -\frac{(a^2+b^2)\tan\theta}{abe}$ $m(AS) \times m(BS) = \frac{b}{ae\tan\theta} \times -\frac{(a^2+b^2)\tan\theta}{abe}$ $= \frac{-(a^2+b^2)}{a^2e^2}$ Now $e^2 - 1 = \frac{b^2}{a^2}$ $e^2 = \frac{b^2}{a^2} + 1$ $= \frac{b^2+a^2}{a^2}$ $\therefore m(AS) \times m(BS) = \frac{-(a^2+b^2)}{a^2} \div \frac{b^2+a^2}{a^2}$ $= \frac{-(a^2+b^2)}{a^2} \times \frac{a^2}{b^2+a^2}$ $= -1$ Therefore AB is diameter of circle passing through *S*, the foci of the hyperbola.	**3**	1 – gradients 1 - working 1 1 showing perpendicular

Question 16		2014	
	Solution	Marks	Allocation of marks
(b)	The letter ‘S’ occurs twice in CHRISTMAS **Case 1**: No S: Consider the letters CHRITMA Number of selections = ${}^7C_5$ and the possible arrangements of this selection is 5! Number with no S is ${}^7C_5 \times 5! = 2520$ **Case 2:** One ‘S’so 4 from the remaining 7letters = ${}^7C_4$ and the arrangements of this selection is 5!. Number with 1 S is ${}^7C_4 \times 5! = 4200$ Case 3: Two ‘S’ so 3 from the remaining 7 letters = ${}^7C_3$ and the Arrangements of this selection is $\frac{5!}{2!}$. Number with 2 ‘S’’s is ${}^7C_3 \times \frac{5!}{2!} = 2100$ Total number of distinct arrangements = 2520 + 4200 + 2100 = 8820	**2**	1 working 1 correct answer
(c)	$\angle PAD = \angle AQC$ (alternate segment theorem) $\angle AQC = \angle CPD$ (alternate angles AQ parallel to PB) ~~(i)~~ $\therefore \angle CPD = \angle PAD$ $\angle CDP = \angle PDA$ (common angle) $\therefore$ triangle CDP is similar to triangle PDA (equiangular)	**2**	1 – correct use of a circle geometry theorem 1 – correct proof
	~~(ii)~~ triangle CDP is similar to triangle PDA $\therefore \frac{CD}{PD} = \frac{PD}{AD}$ (ratio of corresponding sides in similar triangles) $\therefore PD^2 = AD \times CD$ $DB^2 = DC \times DA$ (product of intercepts on secant equals square of i.e $DB^2 = PD^2$ $PD = DB$ $\therefore AD$ bisects PB	**2**	1 – correctly establishes result 1-correctly establishes result