James Ruse Agricultural High School

4 unit mathematics

Trial HSC Examination 1989

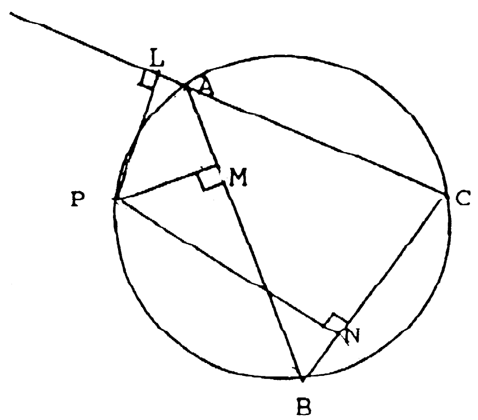
1. (a) Find the exact value of: **(i)** $\int_1^4 (\frac{\sqrt{x}-1}{\sqrt{x}})^2\ dx$ **(ii)** $\int_1^5 \frac{dx}{x^2-4x+3}$ **(iii)** $\int_0^2 \frac{dx}{(4+x^2)^2}$
(b) (i) Given that $T_n = \int_0^1 \frac{t^n\ dt}{t^2+1}$, show that $T_n = \frac{1}{n-1} - T_{n-2}$.
(ii) Hence evaluate $\int_0^1 \frac{t^5\ dt}{t^2+1}$

2. (a) There are 12 red, 12 green, 12 yellow and 12 blue cards in a pack. 5 cards are chosen at random. Write down expressions for the probability that:
(i) all 5 cards are red,
(ii) all 5 cards are the same colour,
(iii) at least one card of each colour is chosen.
(b) (i) Sketch $y = xe^{-x}$ showing the coordinates of all stationary points and points of inflexion.
(ii) Shade in the region R bounded by the x-axis, $y = xe^{-x}$ and $x = 2$.
(iii) This region R is rotated about the x-axis. Find the volume of the resultant solid of revolution.

3. (a) Given that $z_1 = 2 - i$ and $z_2 = 3 + 6i$, express in the form $A + iB$, where A and B are real: **(i)** $(z_1)^2$ **(ii)** $|z_1/\overline{z}_2|$
(b) Given that $w = 30 + 24\sqrt{6}i$, find the square roots of w in the form $A + iB$.
(i) Sketch in the Argand diagram the points corresponding to $|z - 2| = 2$.
(ii) On the same Argand diagram, sketch the curve given by $|z + 4| - |z - 4| = 2$
(iii) Shade in the region satisfying both $|z - 2| \leq 2$ and $|z + 4| - |z - 4| \geq 2$.

4. (a) Solve $\cos 3\theta + \sin 4\theta = 0$ for $0 \leq e \leq 2\pi$.
(b) By using two methods to evaluate $\int_0^1 (2 + x)^n\ dx$ prove that $\sum_{r=0}^{n} \binom{n}{r} \frac{2^{n-r}}{(r+1)} = \frac{3^{n+1}-2^{n+1}}{(n+1)}$.

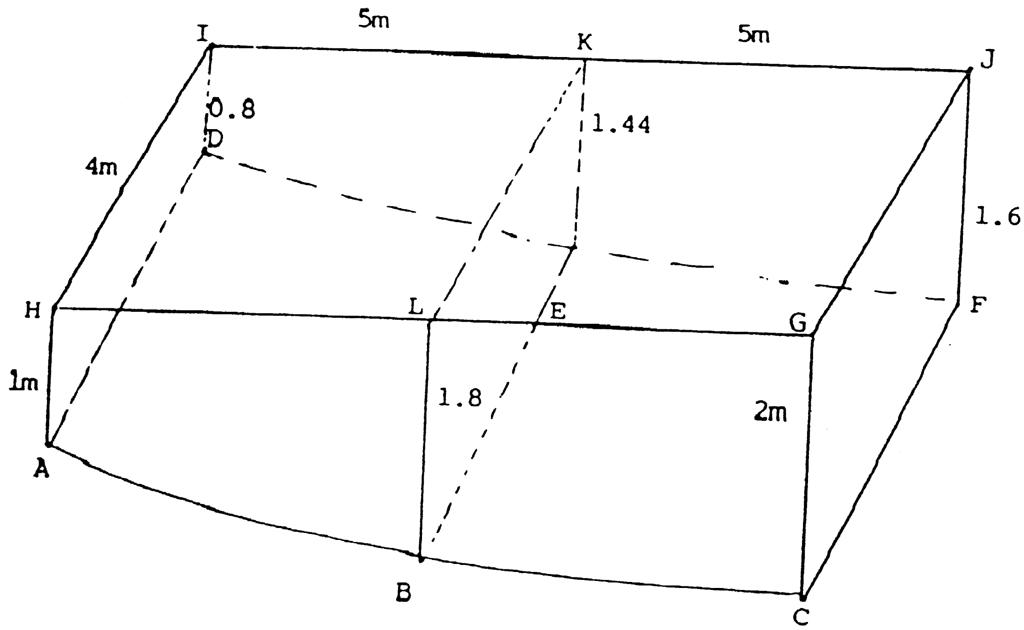
(c)



ABC is a triangle. P is a point on the minor arc $\overset{\frown}{AB}$. PL, PM and PN are perpendicular to AC, AB, BC or these sides produced.
Prove that **(i)** $P\hat{M}L = P\hat{B}C$ **(ii)** L, M, N are collinear.

5. An ellipse has equation $9x^2 + 25y^2 = 225$.
(i) Draw a neat sketch of this ellipse showing the position of its foci and directrices.
(ii) If S, S' are the foci and $P(x_1, y_1)$ is a point on the ellipse, prove that $SP+S'P = 10$. The line $y = mx + c$ cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P and Q. It cuts the asymptotes of the hyperbola at R and S (where R is the point of intersection with the asymptote closer to P). Prove that $PR = QS$.

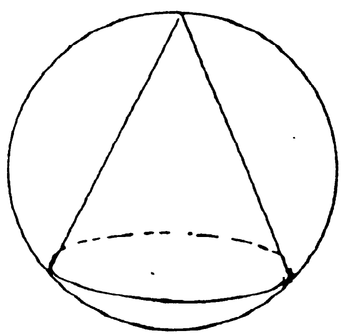
6. **(a)** The surface of a backyard swimming pool is 10 metres long and 4 metres wide.



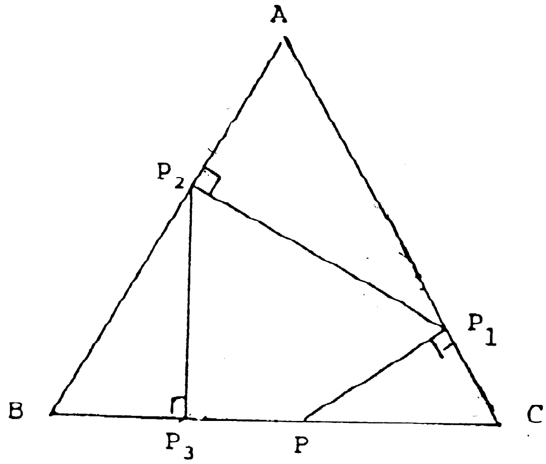
Cross-sections parallel to the end face $ADIH$ are all trapezia with one edge being 0.8 times the edge corresponding to it on the opposite face. The four side faces are all vertical. The depth at A is 1 metre. At B, which is halfway along the deep edge of the pool the depth is 1.8 metres and at C the depth is 2 metres.
(i) Taking the positive x-axis and positive y-axis as edges GH and GC respectively, prove that the equation of the parabolic edge ABC is $y = 2 + 0.02x - 0.012x^2$
(ii) By summing volumes of slices parallel to the face $ADIH$ find the capacity of the pool in litres if it is to be filled to within 10 centimetres of its top.
(b) A car is travelling at 80 kilometres per hour around a banked circular road with a radius of 400 metres. Find the angle at which the road must be banked if there is no sideways frictional force acting on the tyres.

7. (a) A ball is thrown which just clears two walls, each of height 4 metres. The walls are parallel to one another, are vertical and are 10 metres and 30 metres from the thrower. Prove that the angle of projection is $\tan^{-1}\frac{8}{15}$.
(b) The angles of elevation of the top of a hill from two points A and B are 36° and 26° respectively. The hill bears 046°T from A while B is 1000 metres from A and bears 110°T from A. Calculate the height of the hill to the nearest metre.

8. A cone is inscribed in a sphere of radius 10 centimetres as shown in the diagram below.



Find the radius of that cone which can be thus inscribed and which has the greatest possible volume.



ABC is an equilateral triangle with a side of length a centimetres. P is any point on the side BC. A perpendicular from P to AC meets that side at P_1. The distance CP_1 is x_1 centimetres. A perpendicular from P_1 to AB meets it at P_2.
(i) Given that AP_2 is x_2 centimetres, prove that $x_2 = \frac{1}{2}(a - x_1)$
(ii) A perpendicular from P_2 is now drawn to BC meeting it at P_3 which is x_3 units from B. A further perpendicular from P_3 meets AC at P_4 and this process is then continued on. Prove that if P_{n+1} and P_n are x_{n+1} and x_n centimetres from vertices of the triangle that $x_{n+1} = \frac{1}{2}(a - x_n)$
(iii) Hence deduce that as n increases indefinitely the shape $P_{n-1}P_nP_{n+1}$ tends to an equilateral triangle.
(iv) Find the length of the sides of this triangle.