JAMES RUSE AGRICULTURAL HIGH SCHOOL

TRIAL HSC 4 UNIT 2000

QUESTION 1 .

(a) Integrate :

(i) $\int e^x \sin e^x \, dx$

(ii) $\int \frac{dx}{\sqrt{x^2 - 9}}$

(iii) $\int x \cos 2x \, dx$

(b) Graph $y^2 = x^2 (1 - x)$ and evaluate the enclosed area .

(c) Use De Moivre's theorem to show that :

$\cos 3\theta = 4 \cos^3\theta - 3 \cos\theta$ and $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$

QUESTION 2 : START A NEW PAGE

(a) A symmetrical pier of height 5 metres has an elliptical base with equation $\frac{x^2}{25} + \frac{y^2}{4} = 1$ and slopes to a parallel elliptical top with equation $\frac{x^2}{9} + y^2 = 1$.

If the cross sections of the area parallel to the base are also elliptical find the volume of the pier given that the area of an ellipse with semi -major axis a and semi-minor axis b is πab.

(b) Find the volume of rotation when the region bounded by the x and y axes , $x = 2$ and the curve $y = \frac{1}{x^2 - 4x + 13}$ is rotated about the y axis .

(c) A party of 10 people is divided at random into 5 groups of 2 people.

Find the probability of 2 particular people being in the same group.

QUESTION 3 : START A NEW PAGE

(a) (i) If $z = x + iy$ and $w = u + iv$ express u and v as real functions of x and y when $w = \dfrac{z}{1+z}$.

(ii) If $\mathrm{Re}(w) = 0$ describe the locus of z .

(b) (i) Find the square roots of $24 + 10i$

(ii) Solve $z^2 + (1 + 3i)z - 8 - i = 0$

(iii) Describe the locus $\left| z - 2 + i \right| = \left| z^2 + (1 + 3i)z - 8 - i \right|$

QUESTION 4 : START A NEW PAGE

(a) The equation of a conic is given by $\dfrac{x^2}{8} - \dfrac{y^2}{8} = 1$.

(i) Determine the magnitude of the eccentricity , the location of the focii, and the equations of the directrices and asymptotes .

(ii) The conic is rotated 45^0 to the new (X,Y) plane.

Derive the equation of the conic in the X - Y plane .

(b) Points P ($cp, \dfrac{c}{p}$) and Q ($cq, \dfrac{c}{q}$) lie on the rectangular hyperbola $xy = c^2$.

(i) Derive the equation of the tangent at the point P.

(ii) State the equation of the tangent at Q, hence show that the intersection point R of the tangents is $\left(\dfrac{2cpq}{p+q} , \dfrac{2c}{p+q} \right)$

(iii) If the intersection point R of the tangents lies on a directrix find the relation between p and q , stating any restrictions on p and q.

QUESTION 5 : START A NEW PAGE

(a) A circular bitumen road 6 metres wide is installed on a hill which slopes at 7^0 .

If the inner radius of the road is 40 metres then:

(i) show that the velocity of a motor bike when the motor bike is in the centre of the road and no lateral force on the tyres is $\sqrt{Rg \tan\theta}$ where R is the radius of the road, g is the acceleration due to gravity of 9.8 m/s^2 , and θ is the slope of the road, hence evaluate the velocity.(2 dec pl)

(ii) If the friction force on the tyres is 0.2 times the magnitude of the normal force find

the maximum speed (to 2 decimal places) of the motor bike at the outer radius.

(b) Prove by induction that $u_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $u_1 = 1$ and $u_2 = 1$

given the recurrence relation $u_{n+2} = u_n + u_{n+1}$

QUESTION 6 : START A NEW PAGE

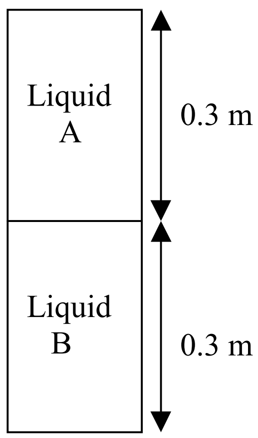
A container is filled with liquid A of height 0.3 m on top of liquid B of height 0.3 m.

A steel ball of mass 10 grams is released from rest at the top of liquid A .

It falls experiencing a resistive force in liquid A of $0.04v^2$ Newtons and a resistive force

of 0.05v Newtons in liquid B ,where v is the velocity (m/s) of the steel ball .

Assuming that no mixing of the liquids occurs, and the acceleration due to gravity is 10 m/s^2 then



(i) show that the velocity of the steel ball when it passes from liquid A to liquid B is 1.51 m/s.

(ii) show that the final velocity of the steel ball satisfies the equation: $v + 2\ln(2 - v) + 1.42 = 0$

(iii) show that the final velocity is approximately 1.80 m/s

(iv) find the total time to reach the bottom of liquid B.

QUESTION 7 : START A NEW PAGE

(a) A particle is projected with velocity V and angle of elevation θ from a point O on the top of a cliff of height h above sea level.

(i) Derive the equation of the trajectory and show that the range x of the particle before landing in the sea is given by the solution of the equation :

$$h + x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2} = 0$$

(ii) Implicitly differentiate the equation to find $\frac{dx}{d\theta}$ and show that the greatest horizontal distance D the particle can travel before landing in the sea is :

$$D = \frac{V}{g}\sqrt{V^2 + 2gh}$$

(DO NOT TEST TO CONFIRM MAXIMUM)

(b) If $\int \sec x \, dx = \ln(\sec x + \tan x)$ find $\int \frac{dx}{(4x^3 - 3x)\sqrt{1 - x^2}}$

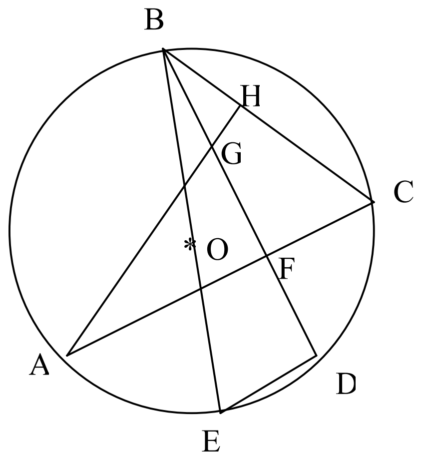
QUESTION 8 : START A NEW PAGE

(a) (i) Show $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{2}\sin x} = \frac{2\pi}{3\sqrt{3}}$

(ii) Show $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^{a} [\,f(x) + f(2a - x)\,]\,dx$

hence evaluate $\displaystyle\int_0^{\pi} \frac{x\,dx}{1 + \frac{1}{2}\sin x}$

(b)



A , B, C, D, and E are points on a circle centre O with diameter BE and AC || DE .

AH $\perp$ BC , and BD intersect AH and AC at G and F respectively .

(i) Prove $\angle BFC = 90^0$

(ii) Prove CFGH is a cyclic quadrilateral .

(iii) Prove AB . BG = BE . BH

END OF EXAM