TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2003

MATHEMATICS

EXTENSION II

Time Allowed – 3 Hours
(Plus 5 minutes reading time)

All questions may be attempted

All questions are of equal value

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

Standard integral tables are provided for your convenience.
Approved silent calculators may be used.

The answers to all questions are to be returned in separate bundles clearly labelled Question 1, Question 2 , etc. Each bundle must show your candidate number.

Question 1:

(a) The complex numbers $z_1 = \dfrac{a}{1+i}$ and $z_2 = \dfrac{b}{1+2i}$ where a and b are real, satisfy the condition $z_1 + z_2 = 1$. Find the value of a and b. **3**

(b) The complex number z has modulus r and argument θ where $0 \le \theta \le \pi$. Find in terms of r and θ the modulus and arguments of

(i) z^2 **1**

(ii) $\dfrac{1}{z}$ **1**

(iii) iz **1**

(c) (i) Sketch (without using calculus) the curve $y = \dfrac{x^2+2x-3}{x-2}$ clearly showing its intercepts with the coordinate axes and the position of all its asymptotes. **5**

(ii) Find the area bounded by the curve $y = \dfrac{x^2+2x-3}{x-2}$ and the x-axis. **4**

Question 2: (START A NEW PAGE)

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \cos\theta \sin^3\theta \; d\theta$. **2**

(ii) $\displaystyle\int_0^3 \frac{\sqrt{x}}{1+x}\, dx$. $\left(\text{Let } u^2 = x\right)$. **3**

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos\theta}\, d\theta$ **4**

(b) Given that $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x \; dx$.

(i) Prove that $I_n + n(n-1)I_{n-2} = n\left(\dfrac{\pi}{2}\right)^{n-1}$. **4**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^4 \sin x \; dx$. **2**

Question 3: (START A NEW PAGE)

(a) Sketch the ellipse $9x^2+25y^2=225$ clearly showing: **4**

(i) the coordinates of the intercepts with the x and y-axes,

(ii) the coordinates of the foci,

(iii) the equation of the directrices.

(b) Prove that the curves $x^2-y^2=c^2$ and $xy=c^2$ meet at right angles. **4**

(c) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the ellipse meets the x-axis at the points A and A'.

(i) Prove that the tangent at P has the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

(ii) The tangent at P meets the tangents from A and A' at points Q and Q' respectively. Find the coordinates of Q and Q'. **2**

(iii) Prove that the product $AQ\times A'Q'$ is independent of the position of P. **2**

Question 4: (START A NEW PAGE)

(a) Prove that $\frac{d}{dx}\left[\sqrt{bx-x^2}+\frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right)\right]=-\sqrt{\frac{x}{b-x}}$ for $x\geq 0$. **3**

(b) A particle of mass m is attracted towards the origin by a force of magnitude $\frac{\mu m}{x^2}$ for $x\neq 0$, where the distance from the origin is x and μ is a positive constant.

(i) If the particle starts from rest at a distance b to the right of the origin, show that its velocity v is given by $v^2=2\mu\left(\frac{b-x}{bx}\right)$. **3**

(ii) Find the time required for the particle to reach a point halfway towards the origin. **4**

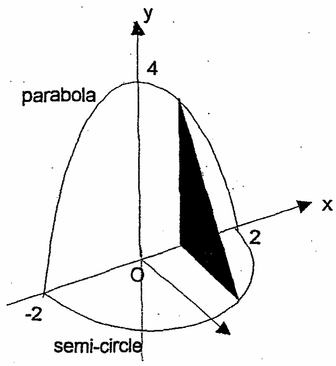
(c) Using the Principle of mathematical induction, prove that $(x+1)^n-nx-1$ is divisible by x^2 for all integer $n\geq 2$. **5**

Question 5: (START A NEW PAGE)

(a) (i) Using the substitution $x = 2\sin\theta$, prove that $\int_0^2 \left(4-x^2\right)^{\frac{3}{2}} dx = 16\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$ 2

(ii) A solid (see diagram) sits on a semi-circular base of radius 2 units. Vertical cross-sections perpendicular to the diameter of the semi-circle are right-angled triangles with their heights being bounded by the parabola $y = 4 - x^2$. By slicing the solid perpendicular to the x-axis, show that the volume $(V \text{ unit}^3)$ of the solid formed is given by 2

$$V = \int_0^2 \left(4-x^2\right)^{\frac{3}{2}} dx$$



(iii) Find the volume of the solid. 5

(b) A tourist is walking along a straight road. At one point he observes a vertical tower standing on a large flat plain. The tower is on a bearing 053° with an angle of elevation of 21°. After walking 230 metres, the tower is on a bearing 342° with an angle of elevation of 26°.

(i) Draw a neat diagram showing the above information. 1

(ii) Find the height of the tower correct to the nearest metre. 5

Question 6: (START A NEW PAGE)

(a) The tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ meets the x and y axes at F and G respectively and the normal at T meets the line $y = x$ at H.

(i) Show that the tangent at T is $x + t^2 y = 2ct$. 3

(ii) Show that the normal at T is $t^3 x - ty = c\left(t^4 - 1\right)$. 2

(iii) Prove that $FH \perp HG$. 6

(b) The area bounded by the curve $y = \frac{\ln x}{\sqrt{x}}$ and the x-axis for $1 \le x \le e$ is rotated through one revolution about the y-axis. Using the method of cylindrical shells, find the volume of the solid formed. 4

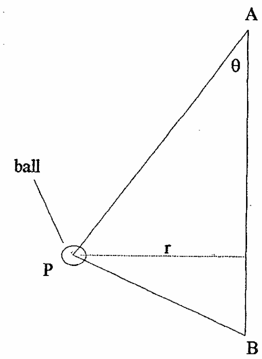
Question 7: **(START A NEW PAGE)**

(a) In a state swimming championships, 12 swimmers (including the Jones twins) are chosen to represent their club and are divided into three teams of four swimmers to form 3 relay teams. Find the number of ways this can be done:

(i) with no restrictions. **2**

(ii) if the Jones twins (Angela and Bethany) are not to be in the same relay team. **3**

(b) The ends of a light string are fixed at 2 points A and B with B directly below A, as shown in the diagram. The string passes through a small ball of mass m which is then fastened to the string at point P. The angle PAB is θ and the distance from P to AB is r.



Suppose now that the ball revolves in a horizontal circle about the vertical through AB with constant angular velocity ω and while this happens both sections (AP and BP) of the string are taut and the angle APB is a right angle.

(i) Draw a diagram showing the forces acting on the ball. **2**

(ii) Show that the tensions T_1 and T_2 in the sections of the string AP and BP respectively are $T_1 = m(r\omega^2 \sin\theta + g\cos\theta)$ and $T_2 = m(r\omega^2 \cos\theta - g\sin\theta)$. **4**

(iii) Given that AB = 100cm and AP = 80cm, show that $\omega^2 > \dfrac{25g}{16}$. **2**

(iv) Suppose that the ball is free to slide on the string. Show that the condition for the ball to remain at point P on the string is $\omega^2 = \dfrac{175g}{12}$. **2**

Question 8: **(START A NEW PAGE)**

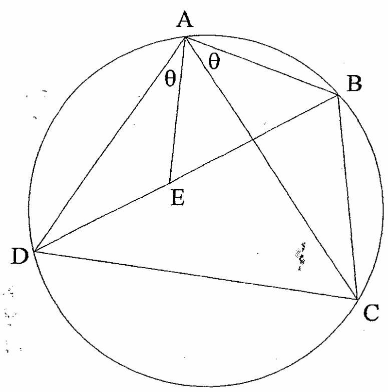
(a) (i) If $t = \tan x$ prove that $\tan 4x = \dfrac{4t(1-t^2)}{t^4 - 6t^2 + 1}$. **2**

(ii) If $\tan x \tan 4x = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$. **1**

(iii) Prove that $x = 18°$ and $x = 54°$ satisfy the equation $\tan x \tan 4x = 1$. **2**

(iv) Deduce that $\tan 54° = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$. **3**

(b) ABCD is a cyclic quadrilateral and E is on BD such that $\angle DAE = \angle BAC$.



(i) Copy the diagram onto your answer sheet and prove that ΔABE and ΔADC are similar. **2**

(ii) Prove that $AB \times CD = AC \times BE$. **1**

(iii) Hence by proving that another pair of triangles are similar, deduce that $AB \times CD + AD \times BC = AC \times BD$. **4**

THE END

QUESTION 1

(a)(i) $\dfrac{a}{1+i} + \dfrac{b}{1+2i} = 1$

$a(1+2i) + b(1+i) = (1+i)(1+2i)$

$(a+b) + i(2a+b) = -1 + 3i$

$a + b = -1$ —— ①

$2a + b = 3$ —— ②

② − ① $a = 4$

from ① $4 + b = -1$

$b = -5$

$\therefore a = 4,\ b = -5$

(b) (i) $z^2 = r^2 \operatorname{cis} 2\theta$

$|z^2| = r^2$, $\arg(z^2) = 2\theta$ or $|z^2| = |z|^2 = r^2$, $\arg(z^2) = 2\arg z = 2\theta$

(ii) $\frac{1}{z} = z^{-1} = r^{-1} \operatorname{cis}(-\theta)$ or $\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{r}$ $\arg\left(\frac{1}{z}\right) = \arg(1) - \arg z = -\theta$.

$\left|\frac{1}{z}\right| = \frac{1}{r}$ $\arg\left(\frac{1}{z}\right) = -\theta$

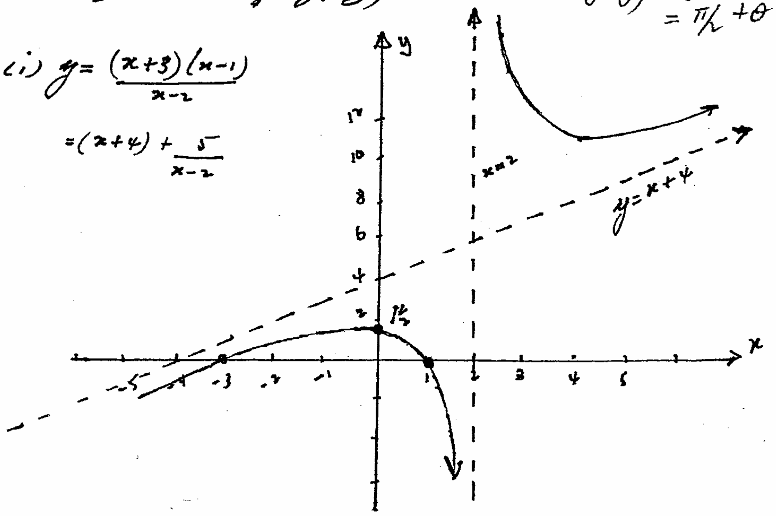
(iii) $iz = \operatorname{cis}\frac{\pi}{2} \cdot r\operatorname{cis}\theta$ or $|iz| = |i||z| = r$

$= r\operatorname{cis}\left(\theta + \frac{\pi}{2}\right)$

$|iz| = r$, $\arg(iz) = \theta + \frac{\pi}{2}$ $\arg(iz) = \arg(i) + \arg z = \frac{\pi}{2} + \theta$

(c) (i) $y = \dfrac{(x+3)(x-1)}{x-2}$

$= (x+4) + \dfrac{5}{x-2}$



Q1(c)(ii) $A = \int_{-3}^{1} x+4 + \frac{5}{x-2} \, dx$

$= \left[\frac{1}{2}x^2 + 4x + 5\ln(2-x) \right]_{-3}^{1}$

$= \left(\frac{1}{2} + 4 + 5\ln 1 \right) - \left(\frac{9}{2} - 12 + 5\ln 5 \right)$

$= 12 - 5\ln 5$

Area $= 12 - 5\ln 5 \; u^2$

QUESTION 2

(a) (i) $\left[\frac{1}{4} \sin^4\theta \right]_0^{\pi/6} = \frac{1}{4}\left(\sin^4 \frac{\pi}{6} - \sin^4 0 \right)$

$= \frac{1}{4}\left(\frac{1}{2}^4 - 0 \right)$

$= \frac{1}{64}$

(ii) $u^2 = x$ $\qquad x=0,\ u=0$

$2u\,du = dx$ $\qquad x=3,\ u=\sqrt{3}$

$\int_0^3 \frac{\sqrt{x}}{1+x}\,dx = \int_0^{\sqrt{3}} \frac{u}{1+u^2} \cdot 2u\,du$

$= 2\int_0^{\sqrt{3}} \frac{u^2}{1+u^2}\,du$

$= 2\int_0^{\sqrt{3}} 1 - \frac{1}{1+u^2}\,du$

$= 2\left[u - \tan^{-1}u \right]_0^{\sqrt{3}}$

$= 2\left\{ (\sqrt{3} - \tan^{-1}\sqrt{3}) - (0 - \tan^{-1}0) \right\}$

$= 2(\sqrt{3} - \pi/6)$

(iii) let $t = \tan\theta/2$, $\cos\theta = \frac{1-t^2}{1+t^2}$ $\qquad d\theta = \frac{2dt}{1+t^2}$

$\theta = 0 \quad t = 0, \quad \theta = \pi/2, \quad t = 1$

$\int_0^1 \frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2} = 2\int_0^1 \frac{dt}{5(1+t^2)+3(1-t^2)}$

Q2(a)(iii)

$$= 2\int_0^1 \frac{dt}{8+2t^2}$$

$$= \int_0^1 \frac{dt}{4+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{u}{2}\right]_0^1$$

$$= \frac{1}{2}\tan^{-1}\frac{1}{2}.$$

(b)(i) $$I_n = \int_0^{\pi/2} x^n \cdot \frac{d}{dx}(-\cos x)\,dx$$

$$= \left[-x^n\cos x\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x \cdot n x^{n-1}\,dx$$

$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$

$$= n\int_0^{\pi/2} x^{n-1} \cdot \frac{d}{dx}(\sin x)\,dx$$

$$= n\left[x^{n-1}\sin x\right]_0^{\pi/2} - n\cdot\int_0^{\pi/2}\sin x\cdot(n-1)x^{n-2}\,dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

$$I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

(ii) $$\int_0^{\pi/2} x^4\sin x\,dx = I_4$$

$$I_4 = -4(3)I_2 + 4\left(\frac{\pi}{2}\right)^3$$

$$I_2 = -2(1)I_0 + 2\left(\frac{\pi}{2}\right)^1$$

$$I_0 = \int_0^{\pi/2}\sin x\,dx$$

$$= 1$$

$$\therefore I_2 = -2(1) + \pi$$

Q2(b)(ii)

$$I_4 = -12(-2+\pi) + \frac{\pi^3}{2}$$
$$= 24 - 12\pi + \frac{\pi^3}{2}$$

QUESTION 3

(a)(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ $\quad a=5 \quad b=3$

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

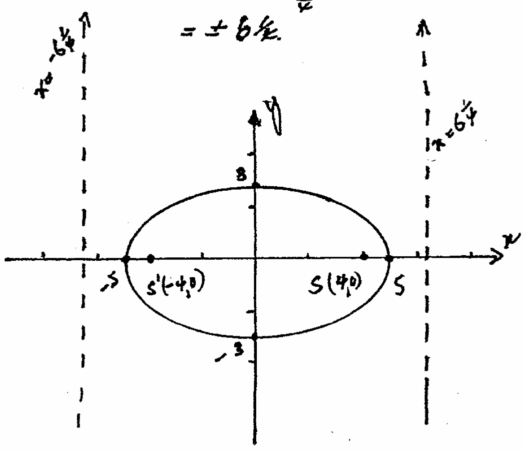
$e = 4/5$.

x-int $(\pm 5, 0)$

y-int $(0, \pm 3)$

(ii) foci $(\pm ae, 0) = (\pm 5 \cdot \frac{4}{5}, 0)$
$= (\pm 4, 0)$

(iii) directrices $x = \pm a/e$
$= \pm 5 \cdot \frac{5}{4}$
$= \pm 6\frac{1}{4}$



(b) $x^2 - y^2 = c^2$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y} = m_1$

$xy = c^2$

$y + x\frac{dy}{dx} = 0$

Q3(b) $\frac{dy}{dx} = -y/x = m_2$

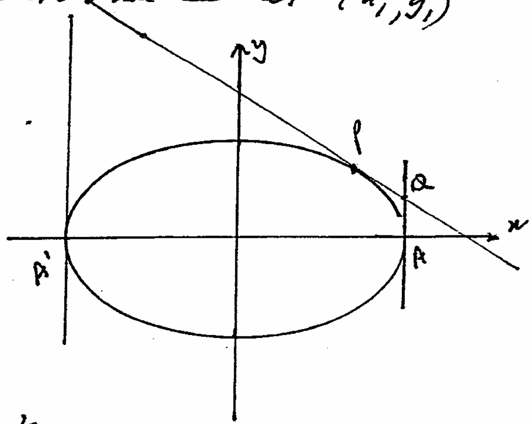
at (x_1, y_1) $m_1 = \frac{x_1}{y_1}$ & $m_2 = \frac{-y_1}{x_1}$

$m_1 m_2 = \frac{x_1}{y_1} \cdot \frac{-y_1}{x_1}$

$= -1$

$\therefore$ curves are $\perp$ at (x_1, y_1)

(c)



(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{a^2} \cdot \frac{b^2}{2y}$

$= \frac{-b^2}{a^2} \frac{x}{y}$

at P $\frac{dy}{dx} = \frac{-b^2}{a^2} \cdot \frac{a\cos\theta}{b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

tangent $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$a\sin\theta\, y - ab\sin^2\theta = -b\cos\theta\, x + ab\cos^2\theta$

$b\cos\theta\, x + a\sin\theta\, y = ab(\sin^2\theta + \cos^2\theta)$

$\therefore b\cos\theta\, x + a\sin\theta\, y = ab$

Q3(c)(ii) $\quad b\cos\theta\, x + a\sin\theta\, y = ab$

at A, $x = a$ $\therefore ab\cos\theta + a\sin\theta\, y = ab$

$$y = \frac{ab(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{b(1-\cos\theta)}{\sin\theta}$$

Q is $\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$

at A', $x = -a$ $\therefore -ab\cos\theta + a\sin\theta\, y = ab$

$$y = \frac{ab(1+\cos\theta)}{a\sin\theta}$$

$\therefore Q'$ is $\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$

(iii) $AQ.A'Q' = \frac{b(1-\cos\theta)}{\sin\theta} \cdot \frac{b(1+\cos\theta)}{\sin\theta}$

$$= \frac{b^2(1-\cos^2\theta)}{\sin^2\theta}$$

$$= \frac{b^2(\sin^2\theta)}{\sin^2\theta}$$

$$= b^2.$$

$\therefore AQ.A'Q'$ is independent of θ.

QUESTION 4.

(a) $\frac{d}{dx}\left[(6x-x^2)^{\frac{1}{2}} + \frac{6}{2}\cos^{-1}\left(\frac{2x-6}{6}\right)\right]$

$$= \frac{1}{2}(6x-x^2)^{-\frac{1}{2}} \cdot (6-2x) + \frac{6}{2} \cdot \frac{-2/6}{\sqrt{1-\left(\frac{2x-6}{6}\right)^2}}$$

$$= \frac{6-2x}{2\sqrt{6x-x^2}} - \frac{1}{\sqrt{\frac{6^2-(2x-6)^2}{6^2}}}$$

$$= \frac{6-2x}{2\sqrt{6x-x^2}} - \frac{6}{\sqrt{6^2-4x^2+4 6x-6^2}}$$

Q4 (a)

$$= \frac{6-2x}{2\sqrt{6x-x^2}} - \frac{6}{\sqrt{4\cdot 6x-4x^2}}$$

$$= \frac{6-2x-6}{2\sqrt{6x-x^2}}$$

$$= \frac{-x}{\sqrt{6x-x^2}}$$

$$= -\sqrt{\frac{x^2}{6x-x^2}} \qquad \text{for } x \geq 0$$

$$= -\sqrt{\frac{x}{6-x}}$$

(b)(i) $m\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\mu m x^{-2}$ $\qquad (\ddot{x} < 0)$

$$\frac{1}{2}v^2 = \mu x^{-1} + c$$

$t=0, x=6, v=0$

$$0 = \frac{\mu}{6} + c$$

$$c = -\frac{\mu}{6}$$

$$\frac{1}{2}v^2 = \frac{\mu}{x} - \frac{\mu}{6}$$

$$v^2 = 2\mu\left(\frac{6-x}{6x}\right)$$

(ii) since $v \neq 0$ for $x > 0$ then particle does not stop.
$\therefore$ motion is in same dirn. as initial motion
& since $\ddot{x} < 0$ when $x = 6$ then direction of motion
is towards origin for $x > 0$

$$\therefore v = -\sqrt{\frac{2\mu(6-x)}{6x}}$$

$$\frac{dx}{dt} = -\sqrt{\frac{2\mu}{6}} \cdot \sqrt{\frac{6-x}{x}}$$

Q4(b)(ii) $\frac{dt}{dx} = -\sqrt{\frac{b}{2\mu}} \cdot \sqrt{\frac{x}{b-x}}$

$$t = \int_b^{b/2} -\sqrt{\frac{b}{2\mu}} \cdot \sqrt{\frac{x}{b-x}}\, dx$$

$$= \sqrt{\frac{b}{2\mu}} \left[\sqrt{bx-x^2} + \frac{b}{2}\cos^{-1}\left(\frac{2x-b}{b}\right) \right]_b^{b/2}$$

$$= \sqrt{\frac{b}{2\mu}} \left\{ \left(\sqrt{\frac{b^2}{2} - \frac{b^2}{4}} + \frac{b}{2}\cos^{-1}(0) \right) - \left(\sqrt{b^2-b^2} + \frac{b}{2}\cos^{-1}(1) \right) \right\}$$

$$= \sqrt{\frac{b}{2\mu}} \left\{ \sqrt{\frac{b^2}{4}} + \frac{b}{2}\cdot\frac{\pi}{2} - 0 \right\}$$

$$= \sqrt{\frac{b}{2\mu}} \left(\frac{b}{2} + \frac{b\pi}{4} \right)$$

$$\text{time} = \frac{b(2+\pi)}{4} \cdot \sqrt{\frac{b}{2\mu}}$$

(c) when $n=2$, $(x+1)^2 - 2x - 1 = x^2 + 2x + 1 - 2x - 1$
$= x^2$

$\therefore$ true for $n=2$

Assume true for $n \leq k$ (k an integer)

ie $(x+1)^k - kx - 1 = x^2 P(x)$

To prove true for $n=k+1$

ie $(x+1)^{k+1} - (k+1)x - 1 = x^2 Q(x)$

$LHS = (x+1)^{k+1} - (k+1)x - 1$

$= (x+1).(x+1)^k - (k+1)x - 1$

$= (x+1)\left[x^2 P(x) + kx + 1\right] - (k+1)x - 1$

$= x^3 P(x) + kx^2 + x + x^2 P(x) + kx + 1 - kx - x - 1$

$= x^3 P(x) + x^2 k + x^2 P(x)$

$= x^2 (x P(x) + P(x) + k)$

$= x^2 Q(x)$

$\therefore$ if true for $n=k$ then true for $n=k+1$ & since true for $n=2$ then true for all integer $n \geq 2$.

QUESTION 5

(a) (i) $x = 2\sin\theta$ $\quad$ $x=0,\ \theta=0$

$dx = 2\cos\theta\, d\theta$ $\quad$ $x=2,\ \cos\theta = 1$

$\theta = \pi/2$

$$\int_0^2 (4-x^2)^{3/2}\,dx = \int_0^{\pi/2} (4-4\sin^2\theta)^{3/2}\,.\,2\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 8(1-\sin^2\theta)^{3/2}\,.\,2\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/2} \cos^3\theta\,.\cos\theta\,d\theta$$

$$= 16\int_0^{\pi/2} \cos^4\theta\,d\theta$$

(ii) $AB = y = 4-x^2$

$BC^2 = 2^2 - x^2$

$BC = \sqrt{4-x^2}$ $\quad (BC>0)$

Area $\Delta ABC = \frac{1}{2}(4-x^2)\sqrt{4-x^2}$

$= \frac{1}{2}(4-x^2)^{3/2}$

ΔV = area cross-section $\times$ thickness

$\doteqdot \frac{1}{2}(4-x^2)^{3/2}\cdot\Delta x$

$$V = \lim_{\Delta x\to 0}\sum_{-2}^{2} \tfrac{1}{2}(4-x^2)^{3/2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

$$V = \int_0^2 (4-x^2)^{3/2}\,dx \quad \text{since function is even.}$$

(iii) $V = \int_0^2 (4-x^2)^{3/2}\,dx$

$= 16\int_0^{\pi/2} \cos^4\theta\,d\theta$ $\qquad$ $\cos 2\alpha = 2\cos^2\alpha - 1$

$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$= 16\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$

$= 4\int_0^{\pi/2} 1 + 2\cos 2\theta + \cos^2 2\theta\,d\theta$

Q5(a)(iii) $V = 4\int_0^{\pi/2} 1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \, d\theta$ ⑩

$= 2\int_0^{\pi/2} 3 + 4\cos 2\theta + \cos 4\theta \, d\theta$

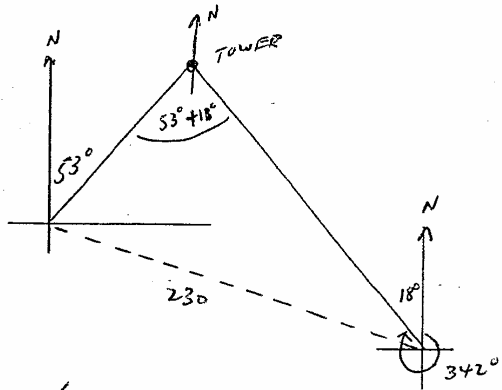
$= 2\left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$

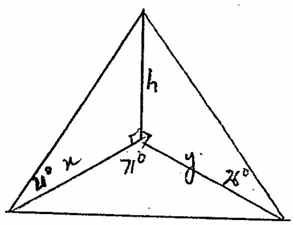
$= 2\left\{\frac{3\pi}{2} + 2\sin\pi + \frac{1}{4}\sin 2\pi - 0\right\}$

$= 3\pi$

Volume $= 3\pi \ u^3$

(b)





$\frac{h}{x} = \tan 21°$ $\qquad \frac{h}{y} = \tan 26°$

$x = \frac{h}{\tan 21°}$ $\qquad y = h\cot 26°$

$= h\cot 21°$

$230^2 = (h\cot 21°)^2 + (h\cot 26°)^2 - 2(h\cot 21°)(h\cot 26°)\cos 71°$

$h = \frac{230}{\sqrt{\cot^2 21° + \cot^2 26° - 2\cot 21°\cot 26°\cos 71°}}$

$\doteqdot 83.91$ m

Height $= 84$ m (to nearest m)

QUESTION 6

(a) (i) $y = c^2/x$

$y' = -c^2/x^2$

at C, $y' = \frac{-c^2}{c^2t^2}$

$= \frac{-1}{t^2}$

Tangent: $y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

(ii) Normal : $tx^2 - y = k$

at C : $t^2(ct) - \frac{c}{t} = k$

$k = \frac{c(t^4-1)}{t}$

Normal : $t^2x - y = \frac{c(t^4-1)}{t}$

$t^3x - y = c(t^4-1)$

(iii) at F, $y=0$ $\therefore x = 2ct$ $F(2ct, 0)$

at G, $x=0$ $t^2y = 2ct$

$y = \frac{2c}{t}$ $(t \neq 0$ $G(0, \frac{2c}{t})$

for H $y = x$ —— ①

$t^3x - ty = c(t^4-1)$ —— ②

sub ① into ②

$t^3x - tx = c(t^4-1)$

$x = \frac{c(t^2-1)(t^2+1)}{t(t^2-1)} = \frac{c(t^2+1)}{t}$ $(t \neq 0)$

$\therefore H$ is $\left[\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right]$

$$m(FH) = \frac{\frac{c(t^2+1)}{t} - 0}{\frac{c(t^2+1)}{t} - 2ct}$$

$$= \frac{c(t^2+1)}{c(t^2+1) - 2ct^2}$$

$$= \frac{c(t^2+1)}{c(1-t^2)}$$

$$= \frac{1+t^2}{1-t^2}$$

$$m(GH) = \frac{\frac{c(t^2+1)}{t} - \frac{2c}{t}}{\frac{c(t^2+1)}{t}}$$

$$= \frac{c(t^2+1) - 2c}{c(t^2+1)}$$

$$= \frac{t^2-1}{t^2+1}$$

$$m(FH) \cdot m(GH) = \frac{1+t^2}{1-t^2} \cdot \frac{t^2-1}{t^2+1}$$

$$= \frac{t^2-1}{1-t^2}$$

$$= \frac{t^2-1}{-(t^2-1)}$$

$$= -1$$

$\therefore FH \perp GH$ (prod. slopes $= -1$)

Q6 (b) $V = 2\pi \int_1^e x \, \frac{\ln x}{\sqrt{x}} \, dx$

$= 2\pi \int_1^e \sqrt{x} \ln x \, dx$

$= 2\pi \int_1^e \ln x \cdot \frac{d}{dx}\left(\frac{2}{3} x^{3/2}\right) dx$

$= 2\pi \left\{ \left[\frac{2}{3} x^{3/2} \ln x \right]_1^e - \int_1^e \frac{2}{3} x^{3/2} \cdot \frac{1}{x} \, dx \right\}$

$= 2\pi \left\{ \left(\frac{2}{3} e^{3/2} \ln e - 0 \right) - \frac{2}{3} \int_1^e x^{1/2} \, dx \right\}$

$= 2\pi \left\{ \frac{2}{3} e^{3/2} - \frac{2}{3} \left[\frac{2}{3} \cdot x^{3/2} \right]_1^e \right\}$

$= 2\pi \left\{ \frac{2}{3} e^{3/2} - \frac{4}{9} \left(e^{3/2} - 1 \right) \right\}$

Vol. $= \frac{4\pi}{9} \left(e^{3/2} + 2 \right) \, u^3$

QUESTION 7

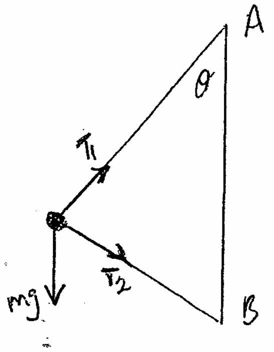
(a)(i) N^o ways $= \dfrac{{}^{12}C_4 \cdot {}^{8}C_4 \cdot {}^{4}C_4}{3!}$

$= 5775$

(ii) N^o ways $= {}^{10}C_3 \cdot {}^{7}C_3 \cdot {}^{4}C_4$

$= 4200$

(b) (i)

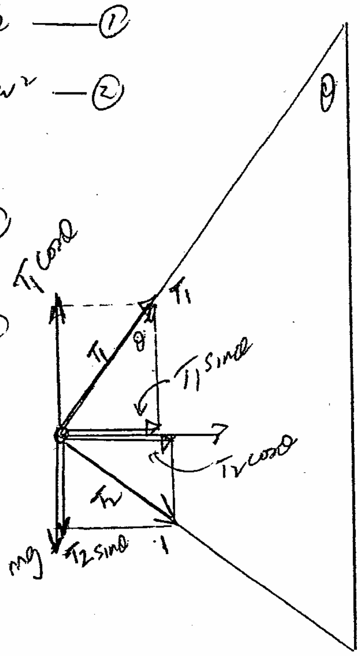


(14)

Q7(b) (ii)

Vertically: $T_1\cos\theta = mg + T_2\sin\theta$ —— ①

Horizontally: $T_1\sin\theta + T_2\cos\theta = mr\omega^2$ —— ②



① × cos θ

$T_1\cos^2\theta - T_2\sin\theta\cos\theta = mg\cos\theta$ —③

② × sin θ

$T_1\sin^2\theta + T_2\sin\theta\cos\theta = mr\omega^2\sin\theta$ —④

③ + ④

$T_1(\sin^2\theta+\cos^2\theta) = mg\cos\theta + mr\omega^2\sin\theta$

$T_1 = mg\cos\theta + mr\omega^2\sin\theta$

from ① $T_2\sin\theta = T_1\cos\theta - mg$

$= mg\cos^2\theta + mr\omega^2\cos\theta\sin\theta - mg$

$= mg(1-\sin^2\theta) + mr\omega^2\cos\theta\sin\theta - mg$

$= mg - mg\sin^2\theta + mr\omega^2\cos\theta\sin\theta - mg$

$= mr\omega^2\cos\theta\sin\theta - mg\sin^2\theta$

$T_2 = mr\omega^2\cos\theta - mg\sin\theta$

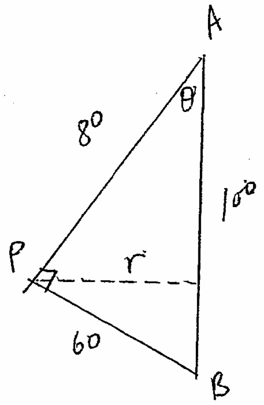
(iii) $\sin\theta = 0.6$

$\cos\theta = 0.8$ $\quad AP = 0.8$

$r = 0.8\sin\theta$

$= 0.8 \times 0.6$

$= 0.48$



if $T_2 > 0$

$m(r\omega^2\cos\theta - g\sin\theta) > 0$

$r\omega^2\cos\theta > g\sin\theta$

$\omega^2 > \dfrac{g\sin\theta}{r\cos\theta}$

Q7(b)(iii) (cont)

$$\omega^2 > \frac{g \times 0.6}{0.48 \times 0.8}$$

$$\omega^2 > \frac{25g}{16}$$

(iv) If free to move $T_1 = T_2$

$$m(r\omega^2 \sin\theta + g\cos\theta) = m(r\omega^2\cos\theta - g\sin\theta)$$

$$r\omega^2\sin\theta + g\cos\theta = r\omega^2\cos\theta - g\sin\theta$$

$$g(\cos\theta + \sin\theta) = r\omega^2(\cos\theta - \sin\theta)$$

$$\omega^2 = \frac{g(\cos\theta+\sin\theta)}{r(\cos\theta - \sin\theta)}$$

$$= \frac{g(0.8+0.6)}{0.48(0.8-0.6)}$$

$$= \frac{175g}{12}$$

QUESTION 8

(a)(i) $\tan 4x = \dfrac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2} \quad \text{where } t = \tan x$$

$$= \frac{\frac{4t}{1-t^2}}{\frac{(1-t^2)^2-(2t)^2}{(1-t^2)^2}}$$

Q8(a)(i) $\tan 4x = \dfrac{4t}{1-t^2} \times \dfrac{(1-t^2)^2}{1-2t^2+t^2-4t^2}$ (18)

$$= \frac{4t(1-t^2)}{1-6t^2+t^4}$$

(ii) $\tan x \tan 4x = 1$

$$\frac{t(4t)(1-t^2)}{t^4-6t^2+1} = 1$$

$$4t^2(1-t^2) = t^4-6t^2+1$$

$$4t^2-4t^4 = t^4-6t^2+1$$

$$5t^4-10t^2+1 = 0$$

(iii) $x=18^\circ$, $\tan x \tan 4x = \tan 18^\circ \tan 72^\circ$

$$= \tan 18^\circ \cot 18^\circ$$

$$= 1$$

$x=54^\circ$, $\tan x \tan 4x = \tan 54^\circ \tan 216^\circ$

$$= \tan 54^\circ \tan 36^\circ$$

$$= \tan 54^\circ \cot 54^\circ$$

$$= 1.$$

(iv) $5t^4-10t^2+1=0$

$$t^2 = \frac{10 \pm \sqrt{100-20}}{10}$$

$$= \frac{5 \pm 2\sqrt{5}}{5}$$

Now the roots of the above equation are $\pm \tan 18^\circ$ and $\pm \tan 54^\circ$ and since $\tan 18^\circ < \tan 54^\circ$ then $\tan 54^\circ = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$

QUESTION 8 (b)

(i) In ΔABE & ΔACD

$A\hat{B}E = A\hat{C}D$ (angles at circumference on same arc. AD)

$E\hat{A}B = D\hat{A}C$ ($E\hat{A}C + \theta$)

$\therefore \Delta ABE \parallel\!\mid \Delta ACD$ (equiangular)

(ii) $\dfrac{AB}{AC} = \dfrac{AE}{AD} = \dfrac{BE}{DC}$ (ratio of corresponding sides in similar triangles)

$AB \cdot CD = AC \cdot BE$

(iii) In ΔABC & ΔAED

$B\hat{A}C = D\hat{A}E$ (both θ)

$B\hat{C}A = E\hat{D}A$ (angles at circumference on same arc AB)

$\therefore \Delta ABC \parallel\!\mid \Delta ADE$ (equiangular)

$\dfrac{BC}{DE} = \dfrac{AC}{AD}$ $\left(= \dfrac{AB}{AE}\right)$ (ratio of corresponding sides in similar triangles)

$BC \cdot AD = DE \cdot AC$.

$BC \cdot AD + AB \cdot CD = DE \cdot AC + BE \cdot AC$ (from (ii)+(iii))

$= AC(DE + BE)$

$AB \cdot CD + BC \cdot AD = AC \cdot BD$.