JAMES RUSE A H S

GESTA NON VERBA

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2004

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

All questions may be attempted

All questions are of equal value

Department of Education approved calculators are permitted

In every question, show all necessary working

Marks may not be awarded for careless or badly arranged work

No grid paper is to be used unless provided with the examination paper

QUESTION 1 **Marks**

(a) The complex number z is given by $z = -1 + i\sqrt{3}$. 2

(i) Show that $z^2 = 2\bar{z}$.

(ii) Evaluate $|z|$ and Arg z. 2

(iii) Show that z is a root of the equation $z^3 - 8 = 0$. 2

(b) (i) Find $\int x\sec^2(x^2)dx$. 3

(ii) Find $\int \frac{x^4}{x^2+1}dx$. 3

(iii) Evaluate $\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x}dx$. 3

QUESTION 2 (Start a new page)

(a) (i) Sketch the graph of $y = \frac{x+3}{x+4}$ clearly showing all points of intersection with the x-axis and y-axis, and the equations of all the asymptotes. 2

(ii) On separate axes, sketch the graphs of:

(α) $y = \left(\frac{x+3}{x+4}\right)^2$ 2

(β) $y^2 = \left(\frac{x+3}{x+4}\right)$ 2

Part (b) on the next page.

(b) A railway track has been constructed around a circular curve of radius 500 metres. The distance across the track between the rails is 1.5 metres and the outer rail is 0.1 metres above the inner rail. A train of mass m travels on the track at a speed of $v = v_0$ metres/second and no lateral forces.

(i) Draw a diagram showing all the forces on the train. **1**

(ii) Show that $v_o^2 = 500g \tan \theta$, where θ is the angle the track makes with the horizontal. **2**

The train now travels on the track at a speed of v metres/second, where $v > v_o$.

(iii) Draw a diagram showing all the forces on the train. **1**

(iv) Show that the lateral force, F, exerted by the rail on the wheel is given by $F = \frac{mv^2}{500}\cos\theta - mg\sin\theta$. **3**

(v) Deduce that F is one fifth of the weight of the train when $v = 2v_o$. **2**

QUESTION 3 (Start a new page)

(a) (i) If α is a double root of a polynomial $P(x)$, show that α is a zero of $P'(x)$. **2**

(ii) Find integers m and n such that $(x+1)^2$ is a factor of x^5+2x^2+mx+n. **4**

(b) Sketch the region on an Argand diagram whose points z satisfy both inequalities $|z-\bar{z}| \le 4$ and $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{3}$. **3**

(c) The equation of motion of a particle moving x metres along a straight line after t seconds is given by $x = 2v - \tan^{-1} v$. Initially its velocity is 1 metre/second. Find the exact time when its velocity is 7 metres/second. **6**

QUESTION 4 (Start a new page)

(a) Beach volleyball is played with two teams where each team has two players.

(i) In how many ways can four players be grouped in pairs to play a game of beach volleyball. **2**

(ii) The eight members of a beach volleyball club meet to play two games at the same time on two separate courts. In how many different ways can the club members be selected to play these two games. **3**

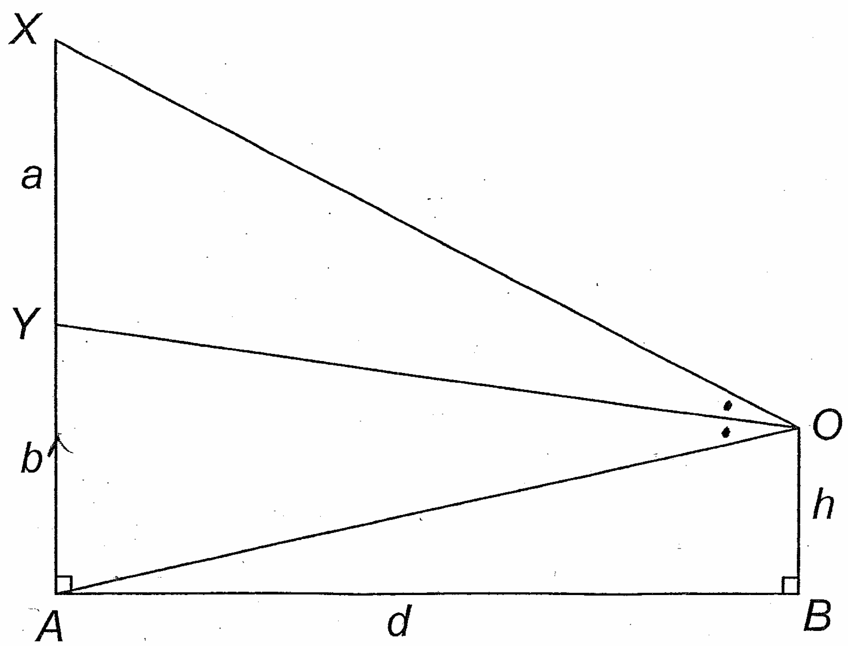
(b) (i) Use the substitution $x = t - y$, where t is a constant, to show that $\int_0^t f(x)dx = \int_0^t f(t-x)dx$. **3**

(ii) Hence, or otherwise, evaluate $\int_0^1 x(1-x)^{2004}\,dx$. **2**

(c) In the diagram below, AX and OB are perpendicular to AB and OY bisects $\angle XOA$. If $XY = a$, $YA = b$, $AB = d$ and $OB = h$, show that

(i) $\dfrac{OX}{OA} = \dfrac{a}{b}$. **2**

(ii) $(a-b)d^2 = (a+b)b^2 - 2b^2h - (a-b)h^2$. **3**



QUESTION 5 (Start a new page)

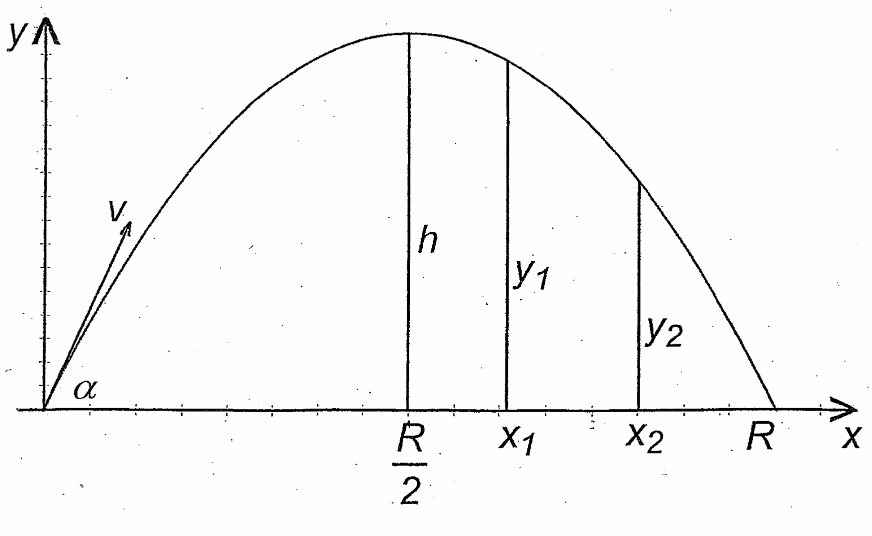
(a) A tank contains *100* Litres of brine (salt water) whose concentration is *3* grams/Litre. Three Litres of brine whose concentration is *2* grams/Litre flow into the tank each minute, and at the same time *3* Litres of mixture flows out each minute. If Q is the quantity of salt in the mixture after a time t minutes,

(i) show that the rate of increase of the quantity of salt, $\frac{dQ}{dt}$, for $t > 0$ is given by $\frac{dQ}{dt} = \left(6 - \frac{3Q}{100}\right)$ grams/minute. **3**

(ii) Show that the quantity of salt in the tank is always between *200* grams and *300* grams. **4**

(b) A cricketer is capable of catching a ball with equal ease at any height from level ground between y_1 and y_2 where $y_1 > y_2$ as shown in the diagram below. For a hit which gives a ball a range R and greatest height h, show that he should estimate his position on the field in the plane within an interval of length $\frac{R}{2}\left[\left(\sqrt{1-\frac{y_2}{h}}\right)-\left(\sqrt{1-\frac{y_1}{h}}\right)\right]$. **8**

(You may assume the equation $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha$)



QUESTION 6 (Start a new page)

(a) Find $\int \frac{5}{16+9\cos^2 x} dx$ **5**

(b) The complex number z is a function of the real number t, given that $z = \frac{t-i}{t+i}$ for $0 \le t \le 1$. Evaluate $|z|$ and hence describe the locus of z in an Argand diagram. **5**

(c) Find the equation of the tangent to the curve $xy\,(x+y) + 16 = 0$ at point on the curve where the gradient of the tangent is -1. **5**

QUESTION 7 (Start a new page)

(a) (i) Show that $\tan\left(A + \frac{\pi}{2}\right) = -\cot A$. **2**

(ii) Use mathematical induction to prove that $\tan\left[(2n+1)\frac{\pi}{4}\right] = (-1)^n$ for n a positive integer. **4**

(b) A polynomial $P(x)$ is divided by $x^2 - a^2$ where $a \neq 0$, and the remainder is $px + q$.

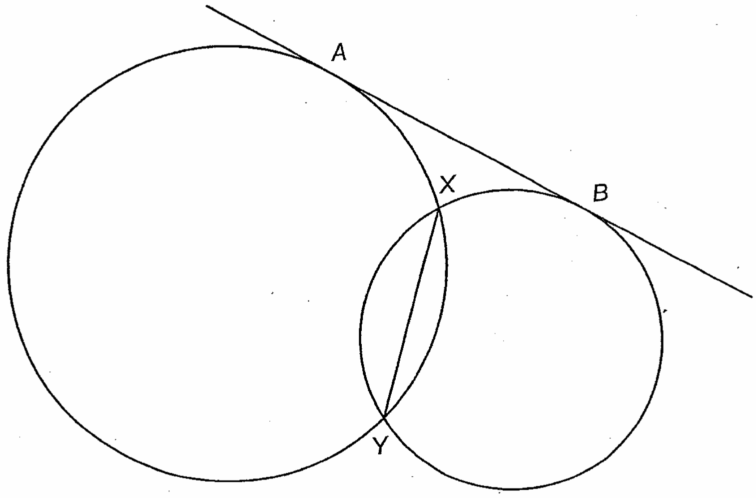
(i) Show that $p = \frac{1}{2a}[P(a) - P(-a)]$ and $q = \frac{1}{2}[P(a) + P(-a)]$. **2**

(ii) Find the remainder when $P(x) = x^n - a^n$, for n a positive integer, is divided by $x^2 - a^2$. **3**

(c) A particle moving with a speed of v metres/second experiences air resistance of kv^2 per unit mass, where k is a constant. Falling from rest in a vertical line through a distance d, prove that it will acquire a speed of $v = V\sqrt{1-e^{-2kd}}$ metres/second, where $V = \sqrt{\frac{g}{k}}$ and g the constant acceleration due to gravity. **4**

QUESTION 8 (Start a new page)

(a) In the diagram below, AB is a common tangent and XY is a common chord. Extend BX to meet AY at Q and extend AX to meet BY at P.



(i) Copy the diagram onto your answer sheet showing all the information given. **1**

(ii) Prove that $PXQY$ is a cyclic quadrilateral. **3**

(iii) Prove that AB is parallel to PQ. **2**

(iv) Prove that XY bisects PQ. **3**

(b) (i) If k is an integer where $k \geq 3$ and $(k-1)(k+1) < k^2$, show that $\dfrac{1}{(k-1)k(k+1)} > \dfrac{1}{k^3}$. **1**

(ii) Given that $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \ldots\ldots + \dfrac{1}{n^3} = \sum_{3}^{n} \dfrac{1}{k^3}$, use partial fractions in part (i) or otherwise to prove that $S_n < \dfrac{1}{12}$. **5**

END of PAPER

SOLUTIONS TO TRIAL H.S.C. EXTENSION II 2004.

QUESTION 1

(a)(i) $z^2 = (-1+\sqrt{3}\,i)^2$
$= 1 - 2\sqrt{3}\,i + 3i^2$
$= -2 - 2\sqrt{3}\,i$
$= 2(-1-\sqrt{3}\,i)$
$\therefore z^2 = 2\bar{z}$

(ii) $|z| = \sqrt{(-1)^2+(\sqrt{3})^2}$
$\therefore |z| = 2$
$\arg z = \tan^{-1}(-\sqrt{3})$
$\therefore \arg z = \dfrac{2\pi}{3}$

(iii) $z^3 = z \cdot z^2$
$= z \cdot 2\bar{z}$ from (i)
$= 2|z|^2$
$\therefore z^3 = 8$ since $|z| = 2$
$\therefore z^3 - 8 = 0$

(b)(i) $\int x \sec^2(x^2)\,dx$
$= \frac{1}{2}\int 2x \sec^2(x^2)\,dx$
$= \frac{1}{2}\int \sec^2(x^2)\,d(x^2)$
$= \frac{1}{2}\int d\tan(x^2)$
$= \frac{1}{2}\tan(x^2) + C$

(ii) $\int \dfrac{x^4}{x^2+1}\,dx$
$= \int\left(\dfrac{x^4-1+1}{x^2+1}\right)dx$
$= \int \dfrac{(x^2-1)(x^2+1)+1}{(x^2+1)}\,dx$
$= \int\left[(x^2-1) + \dfrac{1}{x^2+1}\right]dx$
$= \dfrac{x^3}{3} - x + \tan^{-1}x + C$

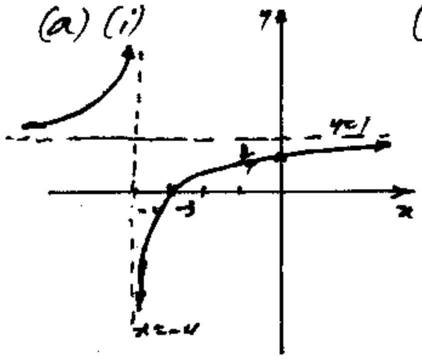
(iii) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x}\,dx$
$= \displaystyle\int_0^{\frac{\pi}{3}} \left(\frac{1-\cos^2 x}{\cos^2 x}\right)\sin x\,dx$
$= \displaystyle\int_0^{\frac{\pi}{3}} \left(\frac{1-\cos^2 x}{\cos^2 x}\right)(-d\cos x)$

Let $U = \cos x$
when $x = 0$ $U = 1$
$x = \frac{\pi}{3}$ $U = \frac{1}{2}$

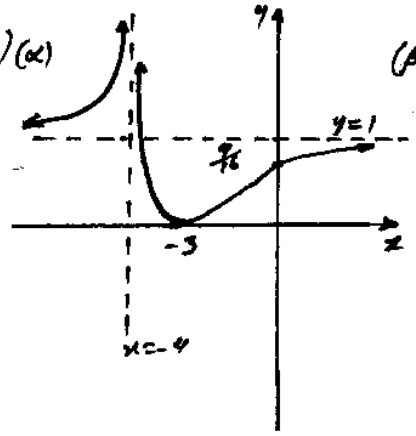
$= \displaystyle\int_1^{\frac{1}{2}} \left(\frac{U^2-1}{U^2}\right)dU$
$= \displaystyle\int_1^{\frac{1}{2}} \left(1 - \frac{1}{U^2}\right)dU$
$= \left[U + \dfrac{1}{U}\right]_1^{\frac{1}{2}}$
$= 2\frac{1}{2} - 2$
$= \dfrac{1}{2}$

QUESTION 2

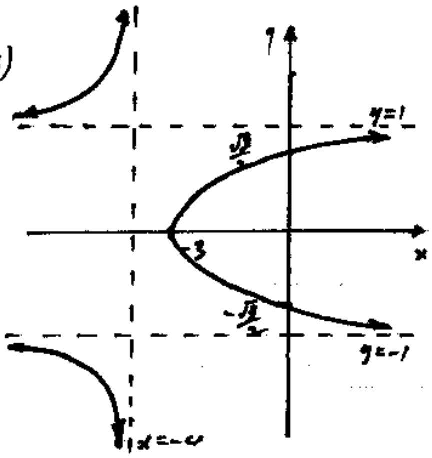
(a) (i)



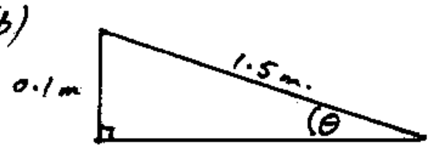
(ii) (α)



(β)



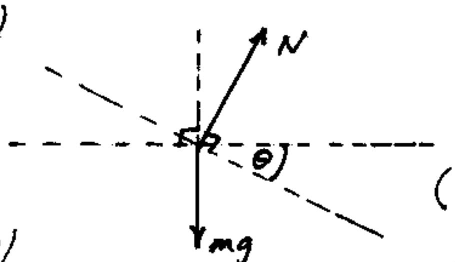
(b)



Motion is in a horizontal circle radius r and constant speed v m/s

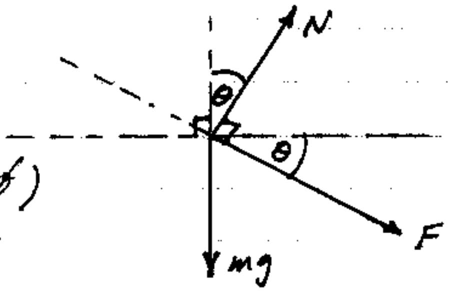
$r = 500$ m.

(i)



(Let weight of train be mg)

(iii)



(ii) (No lateral force on wheels)

$N\cos\theta = mg$ (since vertical component zero)

$N\sin\theta = \frac{mv_0^2}{r}$ (since horizontal component $\frac{mv^2}{r}$)

$\therefore \tan\theta = \frac{mv_0^2}{r} \div mg$

$\therefore \tan\theta = \frac{v_0^2}{gr}$

$\therefore v_0^2 = gr\tan\theta$

$\therefore v_0^2 = 500g\tan\theta$

(iv)

$N\cos\theta - F\sin\theta = mg \quad —(1)$

$N\sin\theta + F\cos\theta = \frac{mv^2}{r} \quad —(2)$

(1) × $\sin\theta$:

$N\sin\theta\cos\theta - F\sin^2\theta = mg\sin\theta \quad —(3)$

(2) × $\cos\theta$:

$N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta \quad —(4)$

(4) − (3):

$F(\cos^2\theta + \sin^2\theta) = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

$\therefore F = \frac{mv^2}{500}\cos\theta - mg\sin\theta$

(v) Now $F = \frac{m\cos\theta}{500} \times 4 \times 500g\tan\theta - mg\sin\theta$ since $v^2 = 4v_0^2$

$\therefore F = 3mg\sin\theta$

$= 3mg \times \frac{1}{15}$

$\therefore F = \frac{1}{5}mg$

QUESTION 3

(a) (i) Let $P(x) = (x-\alpha)^2 Q(x)$

$$P'(x) = (x-\alpha)^2 Q'(x) + Q(x) \times 2(x-\alpha)$$
$$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$$
$$= 0 \text{ when } x = \alpha \quad \therefore \alpha \text{ is a zero of } P'(x).$$

(ii) Let $P(x) = x^5 + 2x^2 + mx + n$

$$P'(x) = 5x^4 + 4x + m$$

Since $(x+1)^2$ is a factor of $P(x)$

$\therefore P(-1) = (-1)^5 + 2(-1)^2 + m(-1) + n = 0$ and $P'(-1) = 5(-1)^4 + 4(-1) + m = 0$

$\therefore -1 + 2 - m + n = 0$ and $5 - 4 + m = 0$

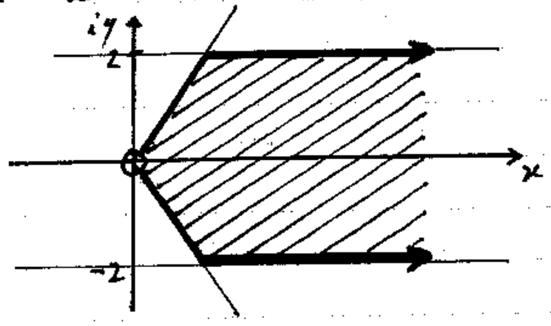
$\therefore m - n = 1 \quad \therefore n = -2 \quad \therefore m = -1$

(b) Let $z = x + iy$

$$|z - \bar{z}| = |(x+iy) - (x-iy)|$$
$$= |2iy|$$
$$= 2|y|$$

$\therefore z$ satisfies $|z - \bar{z}| \le 4$

iff $|y| \le 2$. also $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{3}$



(c) $x = 2v - \tan^{-1} v$

$\therefore \dfrac{dx}{dv} = 2 - \dfrac{1}{1+v^2}$

$\therefore \dfrac{dx}{dv} = \dfrac{1+2v^2}{1+v^2}$

$\therefore \dfrac{dv}{dx} = \dfrac{1+v^2}{1+2v^2}$

$\therefore \dfrac{dv}{dx} \cdot \dfrac{dx}{dt} = \dfrac{(1+v^2)v}{1+2v^2}$

$\therefore \dfrac{dv}{dt} = \dfrac{(1+v^2)v}{1+2v^2}$

$\therefore dt = \dfrac{1+2v^2}{(1+v^2)v}$

$\therefore t = \displaystyle\int \frac{1+2v^2}{(1+v^2)v}\, dv$

$= \displaystyle\int \left(\frac{1}{v} + \frac{v}{1+v^2}\right) dv$

$\therefore t = \ln v + \frac{1}{2}\ln(1+v^2) + C$

when $t = 0$, $v = 1 \quad \therefore C = -\frac{1}{2}\ln 2$

$\therefore t = \ln v + \frac{1}{2}\ln(1+v^2) - \frac{1}{2}\ln 2$

$= \ln\left[v\sqrt{\dfrac{1+v^2}{2}}\right]$

when $v = 7$

$t = \ln\left[7\sqrt{\dfrac{1+7^2}{2}}\right]$

$\therefore t = \ln 35$

QUESTION 4

(a) (i) Number of ways of choosing 2 from 4 is $^4C_2 = 6$.
If players are A, B, C and D then $(A,B) \rightarrow (C,D)$ is the same as $(C,D) \rightarrow (A,B)$

$\therefore \frac{1}{2} \times {}^4C_2 = \frac{6}{2}$
$= 3$

(ii) There are 4 groups of 2 to be selected, $\therefore$ number of combinations $= \dfrac{^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!}$
$= 105$

Let $(A,B), (C,D), (E,F), (G,H)$ be one set of four combinations. Now (A,B) can play any 3 of the others, leaving the other two pairs to play each other.
$\therefore 105 \times 3 = 315$ different selections.

(b) (i) Since $x = t - y$

$\therefore \frac{dx}{dy} = -1$

$\therefore \int_0^t f(x)\,dx = \int_t^0 f(t-y)\,\frac{dx}{dy}\cdot dy$

since $x = t - y$

$\therefore \int_0^t f(x)\,dx = -\int_0^t f(t-y)(-1)\,dy$

$= \int_0^t f(t-y)\,dy$

$\equiv \int_0^t f(t-x)\,dx$

$\therefore \int_0^t f(x)\,dx = \int_0^t f(t-x)\,dx$

(ii) $\int_0^1 x(1-x)^{2004}\,dx$

$= \int_0^1 (1-x)\,x^{2004}\,dx$ from (i)

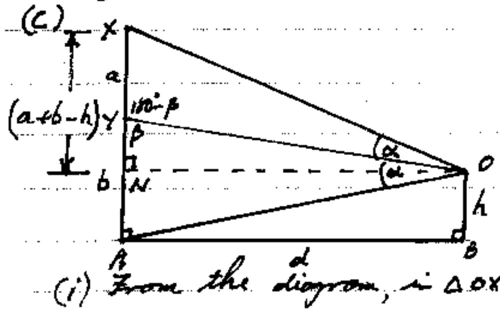
$= \int_0^1 (x^{2004} - x^{2005})\,dx$

$= \left[\dfrac{x^{2005}}{2005} - \dfrac{x^{2006}}{2006}\right]_0^1$

$= \dfrac{1}{2005} - \dfrac{1}{2006}$

$= \dfrac{1}{4{,}022{,}030}$

(c)



(i) From the diagram, in ΔOXY

$\dfrac{\sin\alpha}{a} = \dfrac{\sin(180°-\beta)}{OX} = \dfrac{\sin\beta}{OX}$

$\therefore \dfrac{OX\sin\alpha}{a} = \sin\beta$ — ①

In ΔOAY

$\dfrac{\sin\alpha}{b} = \dfrac{\sin\beta}{OA}$ $\therefore \dfrac{OA\sin\alpha}{b} = \sin\beta$ — ②

From (1) and (2)

$\dfrac{OX\sin\alpha}{a} = \dfrac{OA\sin\alpha}{b}$

$\therefore \dfrac{OX}{OA} = \dfrac{a}{b}$

(ii) Construct $ON \perp AX$ $\therefore XN = (a+b-h)$

Now $OA^2 = h^2 + d^2$ (by Pythagoras')

and $OX^2 = (a+b-h)^2 + d^2$

$\therefore \dfrac{(OX)^2}{(OA)^2} = \dfrac{(a+b-h)^2 + d^2}{h^2+d^2} = \dfrac{a^2}{b^2}$ from (i)

$\therefore \dfrac{(a+b)^2 - 2h(a+b) + h^2 + d^2}{h^2+d^2} = \dfrac{a^2}{b^2}$

$\therefore b^2(a+b)^2 - 2hb^2(a+b) + h^2b^2 + b^2d^2 = a^2h^2 + a^2d^2$

$\therefore b^2(a+b)^2 - 2hb^2(a+b) + b^2h^2 - a^2h^2 = a^2d^2 - b^2d^2$

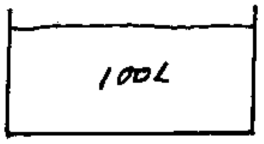
$\therefore b^2(a+b)^2 - 2hb^2(a+b) + h^2(b^2-a^2) = d^2(a^2-b^2)$

Divide both sides by $(a+b)$

$\therefore b^2(a+b) - 2hb^2 - h^2(a-b) = (a-b)d^2$

QUESTION 5

(a)(i)



Now 3L of brine concentration 2 gms/L ie; 6 gms. of salt / minute. Also each litre contains $\frac{Q}{100}$ gms. of salt. Since 3 litres of mixture flows out each minute, ∴ the rate of outflow of salt is $\frac{Q}{100}$ gms/L × 3 L/minute ∴ $\frac{3Q}{100}$ gms/minute.

Since $\frac{dQ}{dt}$ = rate of inflow − rate of outflow

∴ $\frac{dQ}{dt} = \left(6 - \frac{3Q}{100}\right)$ gms/minute.

(ii) Now $\frac{dQ}{dt} = -\frac{3}{100}(Q-200)$

$\therefore \int \frac{dQ}{Q-200} = \int (-.03)\,dt$

$\therefore \ln(Q-200) = -.03t + C$

when $t=0$, $Q = 300$ (100L with conc. of 3 gms/L)

$\therefore C = \ln 100$

$\therefore \ln\left(\frac{Q-200}{100}\right) = -.03t$

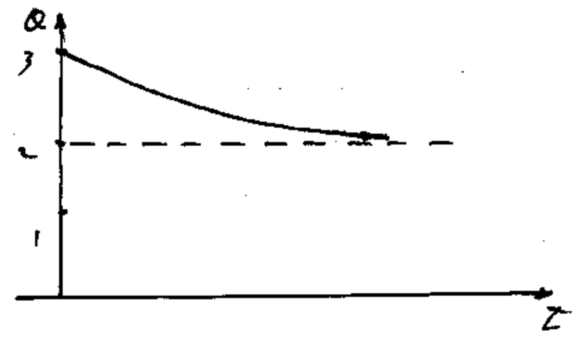
$\therefore \frac{Q-200}{100} = e^{-.03t}$

$\therefore Q = 200 + 100e^{-.03t}$

when $t=0$ $Q = 300$

$t \to \infty$ $Q \to 200$

hence quantity of salt always between 200 gms. and 300 gms.



OR

$\frac{dQ}{dt} = -.03(Q-200)$

$\therefore Q = 200 + Ae^{-.03t}$

[Solution to $\frac{dQ}{dt} = K(Q-Q_0)$ is $Q = Q_0 + Ae^{Kt}$]

when $t=0$, $Q = 300$(....)

$\therefore 300 = 200 + A$

$\therefore A = 100$

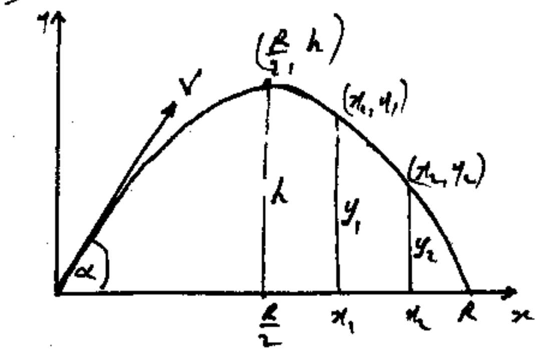
$\therefore Q = 200 + 100e^{-.03t}$

As $t \to 0$ $Q \to 300$

As $t \to \infty$ $Q \to 200$

∴ Q is always between 200 gms and 300 gms.

5(b)



Let $x = Vt\cos\alpha$ — (i)

$y = Vt\sin\alpha - \frac{gt^2}{2}$ — (ii)

Solving (i) and (ii)

$$y = x\tan\alpha - \frac{g x^2}{2V^2}\sec^2\alpha$$

Let $b = \tan\alpha$ and $c = \frac{g\sec^2\alpha}{2V}$

$\therefore y = bx - cx^2$ — (1)

Substituting $(\frac{R}{2}, h)$, (x_1, y_1), (x_2, y_2) and $(R, 0)$ into (1):

$h = \frac{bR}{2} - \frac{cR^2}{4}$ —— (2)

$y_1 = x_1 b - cx_1^2$ —— (3)

$y_2 = x_2 b - cx_2^2$ —— (4)

$0 = Rb - R^2 c$ —— (5)

For $R > 0$, $b = Rc$ from (5)

Subst. into (2)

$h = \frac{R}{2} \times Rc - \frac{R^2}{4} \times c$

$\therefore h = \frac{R^2 c}{4}$

Let the required distance be $x_2 - x_1$.

From (3): $x_1^2 c - x_1 b + y_1 = 0$ From (4): $x_2^2 c - x_2 b + y_2 = 0$

$\therefore x_1 = \frac{b \pm \sqrt{b^2 - 4cy_1}}{2c}$ $\therefore x_2 = \frac{b \pm \sqrt{b^2 - 4cy_2}}{2c}$

Now $x_2 - x_1 = \left(\frac{b \pm \sqrt{b^2 - 4cy_2}}{2c}\right) - \left(\frac{b \pm \sqrt{b^2 - 4cy_1}}{2c}\right)$

(Note that only positive root required, since $x_1 > \frac{R}{2}$ and $x_2 > \frac{R}{2}$)

$\therefore x_2 - x_1 = \left(\frac{b + \sqrt{b^2 - 4cy_2}}{2c}\right) - \left(\frac{b + \sqrt{b^2 - 4cy_1}}{2c}\right)$

$= \sqrt{\frac{b^2 - 4cy_2}{4c^2}} - \sqrt{\frac{b^2 - 4cy_1}{4c^2}}$

$= \sqrt{\frac{b^2}{4c^2} - \frac{y_2}{c}} - \sqrt{\frac{b^2}{4c^2} - \frac{y_1}{c}}$

$= \sqrt{\frac{R^2}{4} - \frac{y_2 \cdot R^2}{4h}} - \sqrt{\frac{R^2}{4} - \frac{y_1 \cdot R^2}{4h}}$ (since $b = Rc$ and $h = \frac{R^2 c}{4}$)

$\therefore x_2 - x_1 = \frac{R}{2}\left[\sqrt{1 - \frac{y_2}{h}} - \sqrt{1 - \frac{y_1}{h}}\right]$

QUESTION 6

(a)
$$\int \frac{5}{16+9\cos^2 x}\,dx$$
$$= \int \frac{5}{16(\sin^2 x+\cos^2 x)+9\cos^2 x}\,dx$$
$$= \int \frac{5}{16\sin^2 x+25\cos^2 x}\,dx$$
$$= \int \frac{5/\cos^2 x}{(16\sin^2 x+25\cos^2 x)/\cos^2 x}\,dx$$
$$= \int \frac{5\sec^2 x}{16\tan^2 x+25}\,dx$$
$$= \int \frac{5\,d\tan x}{16\tan^2 x+25}$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{4}{5}\tan x\right)+c$$

Note: Let $v=\tan x$

$$\int \frac{5\,dv}{25+16v^2} = \frac{5}{16}\int \frac{dv}{\left(\frac{5}{4}\right)^2+v^2}$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{4v}{5}\right)+c$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{4\tan x}{5}\right)+c$$

(b)
$$|z| = \left|\frac{t-i}{t+i}\right|$$
$$= \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}}$$
$\therefore |z| = 1$

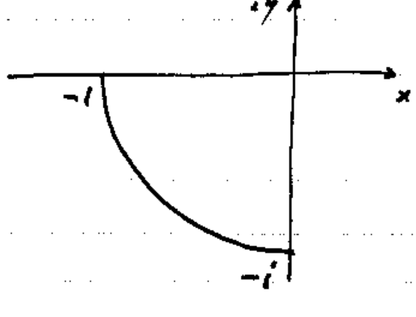
$\therefore$ z lies on the unit circle centre O.

Now $z = \frac{t-i}{t+i}\times\frac{t-i}{t-i} = \frac{t^2-1-2ti}{t^2+1}$

Let $z = X+iY$ where $X=\frac{t^2-1}{t^2+1}$ and $Y=\frac{-2t}{t^2+1}$

$\therefore$ X and Y are both non-positive for $0\le t\le 1$. Hence the locus is in the 3rd quadrant.

Also $\tan\arg z = \frac{Y}{X} = \frac{2t}{1-t^2}$. Hence for $0\le t\le 1$ $\tan\arg z$ varies from 0 to ∞, $\therefore$ $\arg z$ varies from π to $\frac{3\pi}{2}$: $\therefore$ the required locus is part of the unit circle in the 3rd quadrant where, for $0\le t\le 1$, z travels from -1 to $-i$.



QUESTION 7

(a) (i) $LHS = \tan\left(A + \frac{\pi}{2}\right)$ $\quad R.H.S = -\cot A$

$= \dfrac{\sin\left(A + \frac{\pi}{2}\right)}{\cos\left(A + \frac{\pi}{2}\right)}$

$= \dfrac{\sin A \cos\frac{\pi}{2} + \cos A \sin\frac{\pi}{2}}{\cos A \cos\frac{\pi}{2} - \sin A \sin\frac{\pi}{2}}$

$= -\cot A = RHS$.

(ii) Test for $n = 1$

$LHS = \tan\frac{3\pi}{4}$

$= -1 = RHS$.

Assume true for $n = K$

$\tan\left[(2K+1)\frac{\pi}{4}\right] = (-1)^K$

Prove true for $n = K+1$

$LHS = \tan\left\{\left[2(K+1)+1\right]\frac{\pi}{4}\right\}$

$= \tan\left[(2K+3)\frac{\pi}{4}\right]$

$= \tan\left[(2K+1)\frac{\pi}{4} + \frac{\pi}{2}\right]$

$= -\cot\left[(2K+1)\frac{\pi}{4}\right]$ from (a) where $A = (2K+1)\frac{\pi}{4}$

$= -\dfrac{1}{\tan\left[(2K+1)\frac{\pi}{4}\right]}$

$= -\dfrac{1}{(-1)^K} = (-1)\cdot\dfrac{1}{(-1)^{-K}(-1)^{2K}}$

$= (-1)^{K+1} = RHS$

Since proven true for $n=1$ $\therefore$ true for $n=2$. Since true for $n=K$, proven true for $n=K+1$ $\therefore$ true for all $n = 1, 2, 3 \cdots$ by M.I.

(b)(i) $P(x) = (x^2 - a^2)\,Q(x) + px + q$

$= (x-a)(x+a)\,Q(x) + px + q$

$\therefore P(a) = pa + q \quad — (1)$

and $P(-a) = -pa + q \quad — (2)$

$(1)-(2)$: $\quad P(a) - P(-a) = 2pa$

$\therefore p = \frac{1}{2a}\left[P(a) - P(-a)\right]$

$(1)+(2)$: $\quad 2q = P(a) + P(-a)$

$\therefore q = \frac{1}{2}\left[P(a) + P(-a)\right]$

(ii) when $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for n EVEN

$\therefore P(a) = a^n - a^n = 0$, $P(-a) = (-a)^n - a^n = 0$

$\therefore$ remainder is zero, since

$px + q = \frac{1}{2a}[0-0]x + \frac{1}{2}[0+0] = 0$

If n is ODD, then

$P(a) = (+a)^n - a^n = 0$ and

$P(-a) = (-a)^n - a^n = -a^n - a^n = -2a^n$

and remainder is

$px + q = \frac{1}{2a}\left[0 - -2a^n\right]x + \frac{1}{2}\left[0 - 2a^n\right]$

$\therefore px + q = a^{n-1}x - a^n$

6 (c)

$$xy(x+y)+16=0$$

$$\therefore x^2y+xy^2+16=0$$

Differentiating w.r.t. x:

$$\therefore x^2\frac{dy}{dx}+2xy+2xy\frac{dy}{dx}+y^2=0$$

$$\therefore (x^2+2xy)\frac{dy}{dx}=-(2xy+y^2)$$

When $\frac{dy}{dx}=-1$ $\quad (x^2+2xy)(-1)=-2xy-y^2$

$$\therefore -x^2-\cancel{2xy}=-\cancel{2xy}-y^2$$

$$\therefore x^2=y^2$$

$$\therefore x=\pm y$$

If $x=-y$

$\therefore xy(x+y)+16\neq 0$

$\therefore x\neq -y$

If $x=y$

$\therefore x^2(2x)+16=0$

$\therefore x^3=-8$

$\therefore x=-2$

at $x=-2$ $\quad (-2y)(-2+y)+16=0$

$$\therefore y^2-2y-8=0$$

$$\therefore (y-4)(y+2)=0$$

$$\therefore y=4 \text{ or } -2$$

Since $x=y$ $\therefore y=-2$

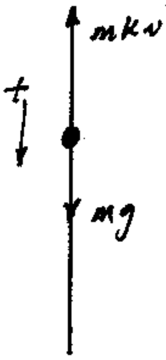
$\therefore$ required point is $(-2,-2)$

Equation of tangent at $(-2,-2)$ is

$$y+2=-1(x+2)$$

$$\therefore x+y+4=0$$

7 (c)



From Newton's 2nd Law:

$$F = m\ddot{x} = mg - mkv^2$$

$$\therefore \ddot{x} = g - kv^2 \quad \text{for unit mass}$$

$$\therefore v\frac{dv}{dx} = g - kv^2$$

$$\therefore \frac{v\,dv}{g - kv^2} = dx$$

$$\therefore \int \frac{v\,dv}{g - kv^2} = \int dx$$

$$\therefore -\frac{1}{2k}\int \frac{-2kv\,dv}{g - kv^2} = \int dx$$

$$\therefore -\frac{1}{2k}\ln(g - kv^2) = x + c$$

When $x = 0$, $v = 0$ $\therefore$ $c = -\frac{1}{2k}\ln g$

$$\therefore -\frac{1}{2k}\ln(g - kv^2) = x - \frac{1}{2k}\ln g$$

$$\therefore -\frac{1}{2k}\ln\left(\frac{g - kv^2}{g}\right) = x$$

$$\therefore \ln\left(\frac{g - kv^2}{g}\right) = -2kx$$

$$\therefore \frac{g - kv^2}{g} = e^{-2kx}$$

$$\therefore g - kv^2 = g e^{-2kx}$$

$$\therefore v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$$

When $x = d$ and $v > 0$

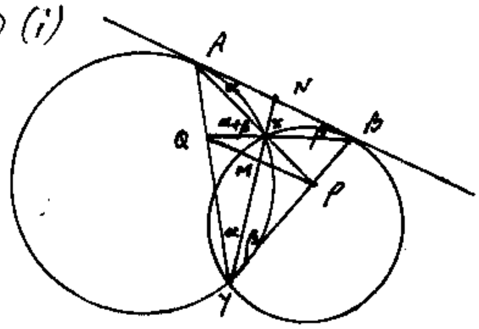
$$v = \sqrt{\frac{g}{k}}\sqrt{1 - e^{-2kd}}$$

Since $V = \sqrt{\frac{g}{k}}$

$$\therefore v = V\sqrt{1 - e^{-2kd}}$$

QUESTION 8

(a) (i)



(ii) $B\hat{A}X = A\hat{Y}X$ ($B\hat{A}X$ formed by chord AX and tangent equal to $A\hat{Y}X$ in alternate segment)

Similarly $A\hat{B}X = B\hat{Y}X$

$\therefore Q\hat{Y}P = \alpha + \beta$

In $\triangle ABX$, $A\hat{X}Q = \alpha+\beta$ (Exterior $A\hat{X}Q$ equal to interior opposite $A\hat{B}X$ and $B\hat{A}X$)

$\therefore PXQY$ is a cyclic quad. (Exterior $A\hat{X}Q$ equal to interior remote $P\hat{Y}Q$)

(iii) $B\hat{A}X = X\hat{Y}Q$ from (ii)

$X\hat{Y}Q = A\hat{P}Q$ (angles at circumference in same segment)

$\therefore B\hat{A}X = A\hat{P}Q$

$\therefore AB \parallel PQ$ (alternate angles equal)

(iv) Let YX intersect PQ at M.

Extend YX to meet AB at N.

Now $AN^2 = YN \cdot NX = BN^2$ (Square of tangent equal to product of intercepts of intersecting secant)

$\therefore AN = BN$ ie; N bisects AB

Since $\triangle ABY \,|||\, \triangle QPY$, $\therefore$ M bisects PQ.

(b)(i) Since $(K-1)(K+1) < K^2 \quad K \geq 3$

$\therefore (K-1)K(K+1) < K^3$

$\therefore \dfrac{1}{(K-1)K(K+1)} > \dfrac{1}{K^3}$

(ii) Let $\dfrac{1}{(K-1)K(K+1)} = \dfrac{A}{(K-1)K} + \dfrac{B}{K(K+1)} = \dfrac{\left(\frac{1}{2}\right)}{(K-1)K} - \dfrac{\left(\frac{1}{2}\right)}{K(K+1)}$ —— (1)

Now $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \dots + \dfrac{1}{n^3}$

$\therefore S_n < \dfrac{1}{2\cdot3\cdot4} + \dfrac{1}{3\cdot4\cdot5} + \dfrac{1}{4\cdot5\cdot6} + \dots + \dfrac{1}{(n-1)n(n+1)} = \displaystyle\sum_{K=3}^{n} \dfrac{1}{(K-1)K(K+1)}$

ie; $S_n < \dfrac{1}{2} \displaystyle\sum_{K=3}^{n} \left[\dfrac{1}{(K-1)K} - \dfrac{1}{K(K+1)}\right]$ from (1)

$\therefore 2S_n < \left[\left(\dfrac{1}{2\cdot3} - \dfrac{1}{3\cdot4}\right) + \left(\dfrac{1}{3\cdot4} - \dfrac{1}{4\cdot5}\right) + \dots + \left(\dfrac{1}{(n-1)n} - \dfrac{1}{n(n+1)}\right)\right]$

Now $\left(\dfrac{1}{2\cdot3} - \dfrac{1}{3\cdot4}\right) + \left(\dfrac{1}{3\cdot4} - \dfrac{1}{4\cdot5}\right) + \dots + \left(\dfrac{1}{(n-1)n} - \dfrac{1}{n(n+1)}\right) = \dfrac{1}{2\cdot3} - \dfrac{1}{n(n+1)}$

$\therefore 2S_n < \dfrac{1}{6} - \dfrac{1}{n(n+1)}$

$\therefore 2S_n < \dfrac{1}{6}$ for $n \geq 3$

$\therefore S_n < \dfrac{1}{12}$