JRAHS Trial HSC 2006 Extension 2

Question 1 (15 Marks) **Marks**

(a) Find $\int \frac{2x}{1+2x}\, dx$. **2**

(b) How many different ways are there of choosing a cricket 11 from a team of 15 players, if there can only be at most, one of the 2 Lee brothers and 2 of 3 Abey brothers? **3**

(c) Find $\lim_{x \to -5} \frac{\sqrt{20-x}-5}{5+x}$. **2**

(d) Find the sum of all the coefficients of powers of z in the expansion $(1 + z)^8$. **2**

(e) (i) Find the equation of the tangent to the hyperbola $3x^2 - y^2 = 12$ at the point $T(x_1, y_1)$. **3**

(ii) This tangent meets the line d ($x = 1$) in the point M. Prove that $FM \perp FT$, where F is the focus $(4, 0)$. **3**

Question 2 (15 Marks) **START A NEW PAGE**

(a) (i) Express $z = \sqrt{2} - i\sqrt{2}$ in modulus – argument form. **2**

(ii) Hence, write z^{22} in the form of $a + ib$, where a and b are real. **3**

(b) If a, b, c are real and unequal and that $a^2 + b^2 > 2ab$ deduce that

(i) $a^2 + b^2 + c^2 > ab + bc + ca$ **2**

(ii) If $a + b + c = 6$ show that $ab + bc + ca < 12$ **2**

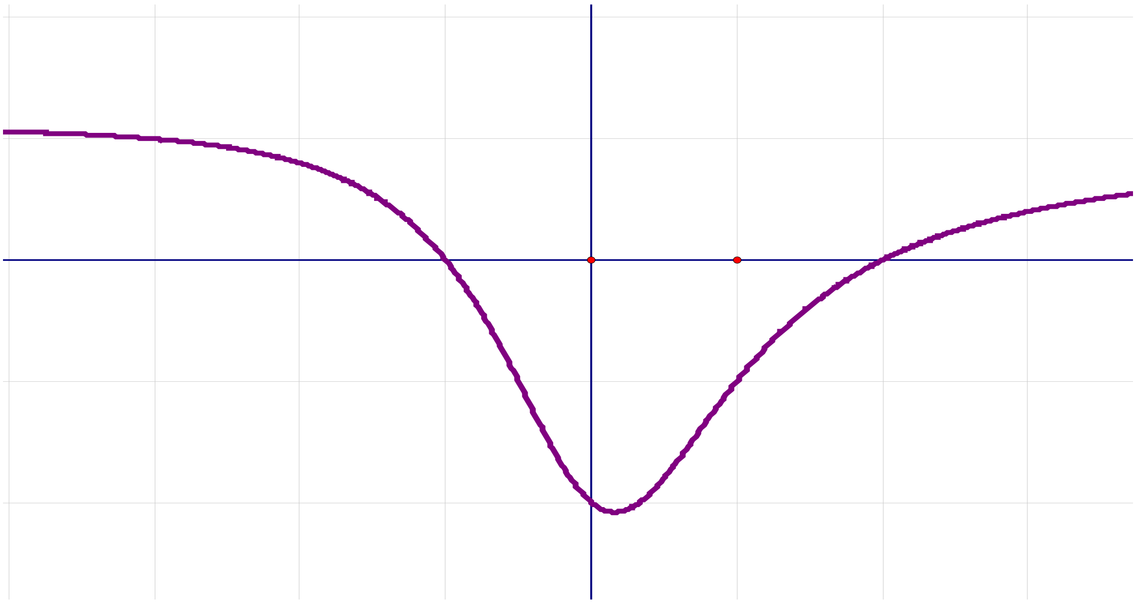
(c) Let $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x}\, dx$ for $n \geq 0$.

(i) Use integration by parts to show that $I_n = I_{n-1} - \frac{e^{-1}}{n!}$ **3**

(ii) Hence, evaluate I_4. **3**

Question 3 (15 Marks) **START A NEW PAGE** **Marks**

(a) The diagram below is a sketch of the function $y = f(x)$.



On separate diagrams sketch neatly:

(i) $y = |f(x)|$ **2**

(ii) $y = e^{f(x)}$ **2**

(iii) $y = \ln(f(x))$ **2**

(b) Reduce the polynomial $x^4 - 2x^2 - 15$ over the rational and the complex field. **2**

(c) If z_1, z_2 are 2 complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$. **4**
With the aid of a diagram, show that arg (z_1) and arg (z_2) differ by either $\frac{\pi}{2} \text{ or } \frac{3\pi}{2}$.

(d) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\, dx$. **3**

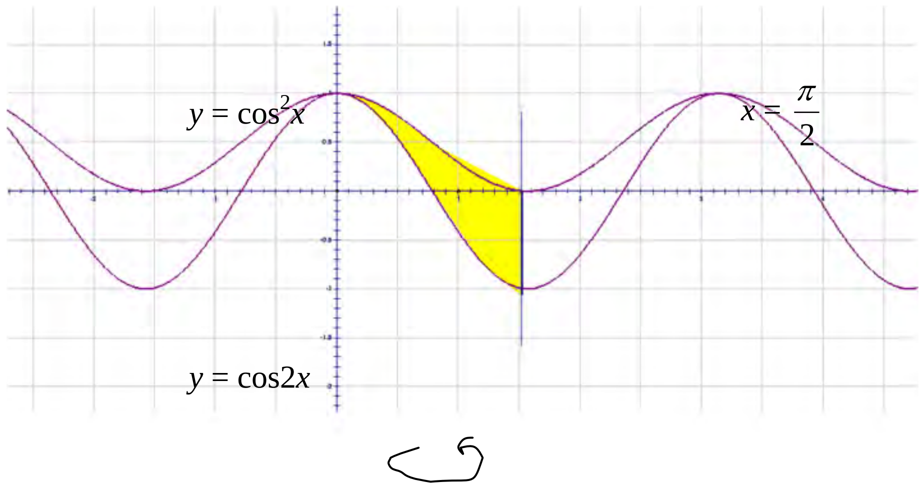
Question 4 (15 Marks) **START A NEW PAGE** **Marks**

(a) A string 50 *cm* of length can just sustain a weight of mass 20 *kg* without breaking. A mass of 4 *kg* is attached to one end of the string and revolves uniformly on a smooth horizontal table; the other end is being fixed to a point on the table. **3**

Find the greatest number of complete revolutions the mass can make in a minute without breaking the string. [Use acceleration due to gravity as 9.8 m/s^2]

(b) Evaluate $\int_a^{a^2} \frac{dx}{x \ln x}$, leaving your answer in simplified, exact form. **3**

(c)



The area between the graphs of $y = \cos^2 x$ and $y = \cos 2x$ for $0 \leq x \leq \frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is rotated about the y – axis. By considering cylindrical shells, find volume of revolution of the solid formed, in terms of π. **6**

(d) Use the properties of odd and even functions to evaluate $\int_{-4}^{4} \cos x\left(e^x - e^{-x}\right)dx$. **3**

Question 5 (15 Marks) **START A NEW PAGE**

(a) (i) Find A and B for which $\dfrac{x^4}{x^2+1} = A\left(x^2-1\right) + \dfrac{B}{x^2+1}$. **2**

(ii) Show that this result may be used in the integration of the function $x^3\tan^{-1}x$. **5**

Hence, evaluate the integral $\int_0^1 x^3 \tan^{-1} x \, dx$.

(b) A particle P is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of it's distance from the centre of the Earth.

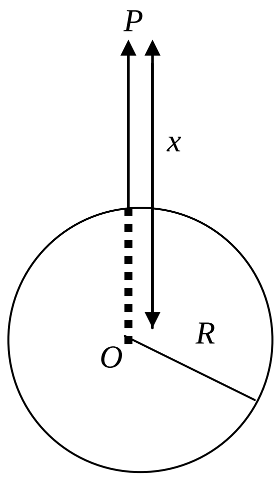


Diagram not to scale

(i) Prove that the speed v in any position x is given by $v^2 = u^2 - 2gR + \dfrac{2gR^2}{x}$, **4**

where R is the radius of the Earth and g is the acceleration due to gravity at the surface of the Earth.

(ii) If $u = \sqrt{2gR}$, find the time taken to reach a height $3R$ above the surface of the earth, in terms of g and R. **4**

Question 6 (15 Marks) **START A NEW PAGE** **Marks**

(a) A total of five players is selected at random from four sporting teams. Each of the teams consists of 10 players numbered from 1 to 10. What is the probability that the five selected players contain at least four players from the same team? Clearly explain your answer. **4**

(b) A particle P of mass m is attached by two equal, light inextensible strings of length l, to two points Q and S, distant h units apart in the same vertical line; S is directly below Q. P rotates in a horizontal circle with uniform angular velocity, ω.

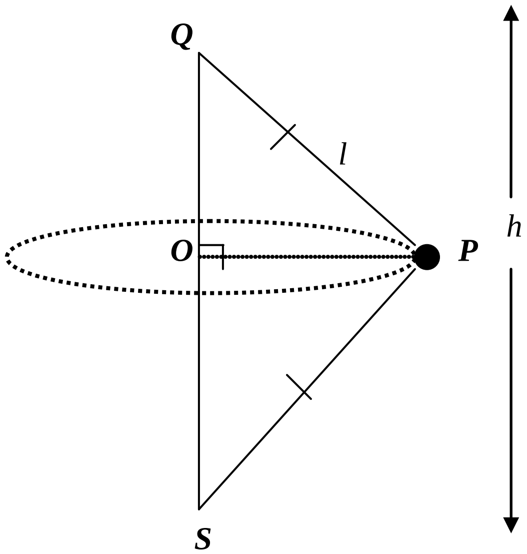


Diagram not to scale

(i) Prove that the tension in the string PQ is $ml\left(\frac{1}{2}\omega^2 + \frac{g}{h}\right)$ where g is the acceleration due to gravity. **5**

(ii) Find the tension in the string PS. **1**

(iii) Show that for both strings to remain stretched then $\omega > \sqrt{\frac{2g}{h}}$. **2**

(iv) If the tensions in the string are in the ration of 2:1, find the period of the motion of the particle. **3**

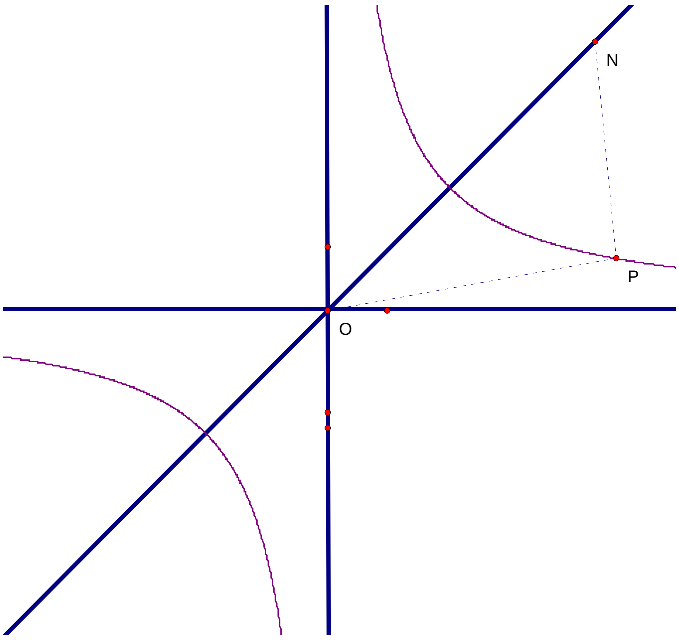
Question 7 (15 Marks) **START A NEW PAGE** **Marks**

(a) α, β, γ are non – zero roots of the equation $x^3 + px + q = 0$.
Find the equation whose roots are

(i) $\alpha^3, \beta^3, \gamma^3$ **3**

(ii) $\dfrac{\alpha}{\beta\gamma}, \dfrac{\beta}{\alpha\gamma}, \dfrac{\gamma}{\alpha\beta}$ **4**

(b) The diagram below shows the hyperbola $xy = c^2$. The point $P\left(ct, \dfrac{c}{t}\right)$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N.
O is the origin.



(i) Prove that the equation of the normal at P is **1**

$$y = t^2 x + \frac{c}{t} - c^3.$$

(ii) Find the coordinates of N. **1**

(iii) Show that ΔOPN is isosceles. **2**

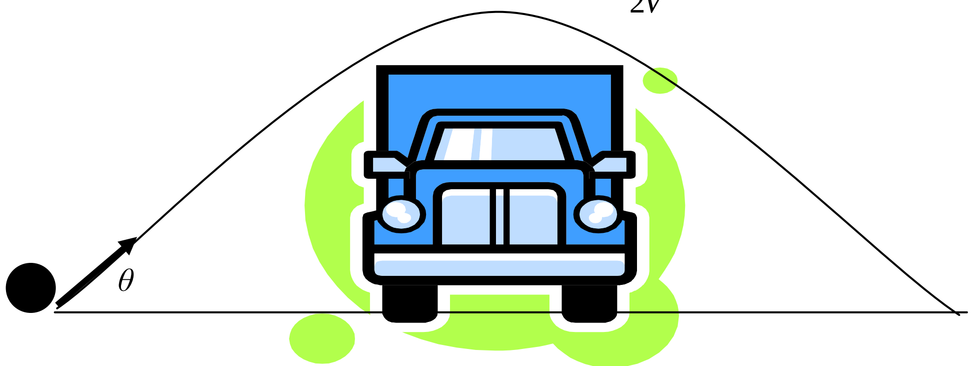
(c) (i) Draw a neat sketch of the graph of $y = \dfrac{4}{x} - x$. **2**

(ii) Draw a neat sketch of the curve $y = \sqrt{f(x)}$. **2**

Question 8 (15 Marks) **START A NEW PAGE** **Marks**

(a) Ben's Hot Wheel travels along an inclined ramp to jump over a truck. The Hot Wheel travels at an initial speed V m/s inclined at an angle θ to the horizontal at O, and acceleration due to gravity is g m/s^2.

The Wheel's trajectory is given by $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{2V^2}$.



(i) Write this equation in the general form of a parabola: $(x-h)^2 = 4a(y-k)$, where a, h, k are constants. **3**

(ii) Calculate the angle of projection, θ, if the range is three times the width of the truck and the top of the truck passes through the focus of equation in part (a). **3**

(b) Given p and q are positive real numbers,

(i) Prove that: $\dfrac{1}{2}(p+q) \geq \sqrt{pq}$. **1**

(ii) Hence, deduce that: $\sqrt{p} \leq \dfrac{1}{2}\left(\dfrac{p}{\sqrt{q}} + \sqrt{q}\right)$. **1**

Question 8 Part (c) continued on the next pag

(c) Given $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are positive real numbers, where $A_n = a_1 + a_2 + a_3 + \ldots + a_n$ and $B_n = b_1 + b_2 + b_3 + \ldots + b_n$, are such that $a_1, a_2, \ldots\ldots.., a_n > 0$, $b_1, b_2, \ldots\ldots.., b_n > 0$ and $A_r \le B_r$, for $r = 1, 2, 3, \ldots, n$.

(i) Prove, by mathematical induction for $n = 1, 2, 3, \ldots$, that: **3**

$$\frac{1}{\sqrt{b_n}} B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) B_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) B_{n-2} + \ldots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1$$
$$= \sqrt{b_1} + \sqrt{b_2} + \sqrt{b_3} + \ldots + \sqrt{b_n}.$$

(ii) Hence, given: $\dfrac{a_1}{\sqrt{b_1}} + \dfrac{a_2}{\sqrt{b_2}} + \dfrac{a_3}{\sqrt{b_3}} + \ldots + \dfrac{a_n}{\sqrt{b_n}} =$

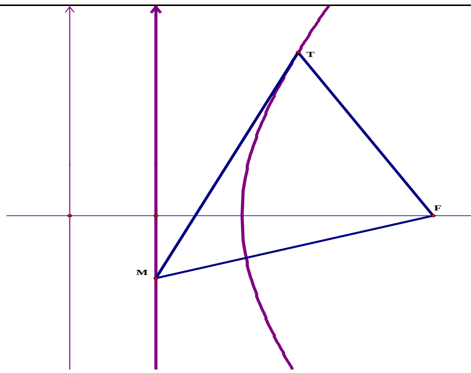
$$\frac{1}{\sqrt{b_n}} A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) A_{n-1} + \left(\frac{1}{\sqrt{b_{n-2}}} - \frac{1}{\sqrt{b_{n-1}}}\right) A_{n-2} + \ldots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) A_1,$$

show that: $\displaystyle\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} \le \sum_{r=1}^{n} \sqrt{b_r}$. **1**

(iii) Deduce that: $\displaystyle\sum_{r=1}^{n} \sqrt{a_r} \le \sum_{r=1}^{n} \sqrt{b_r}$. **3**

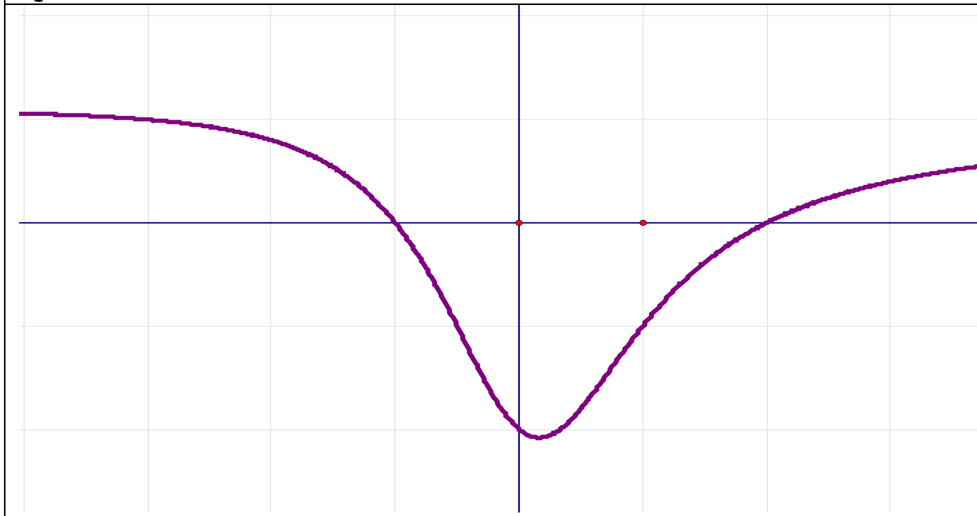
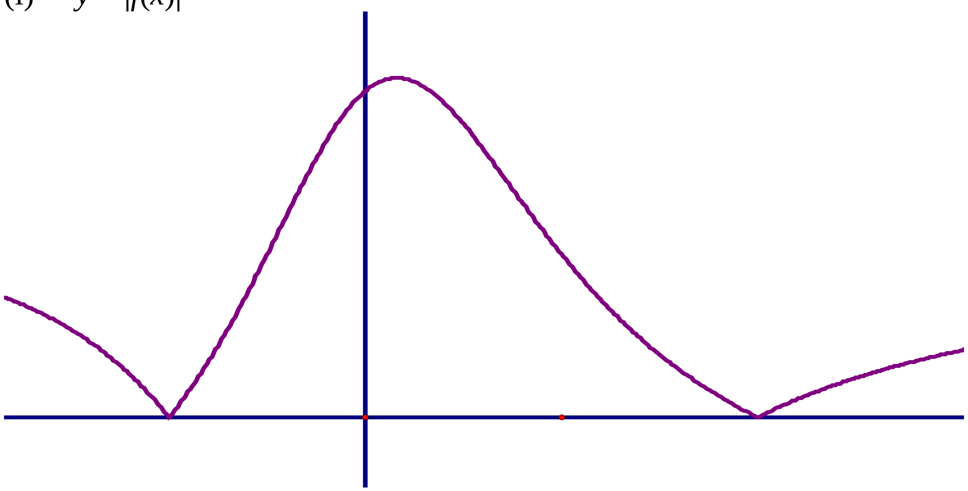
Year 12 Mathematics Extension 2 TRIALS 2006 – Suggested Solutions (LK)

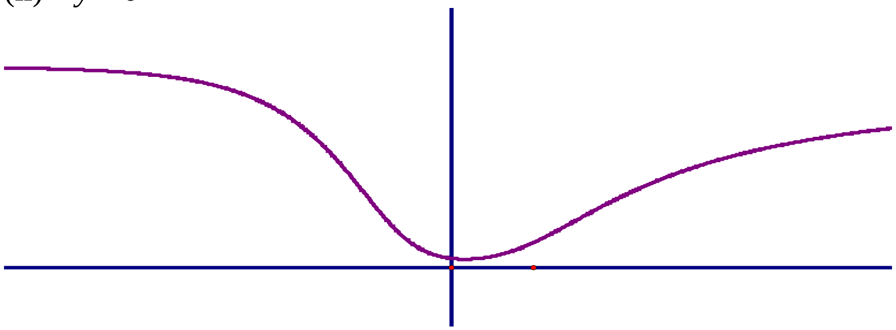
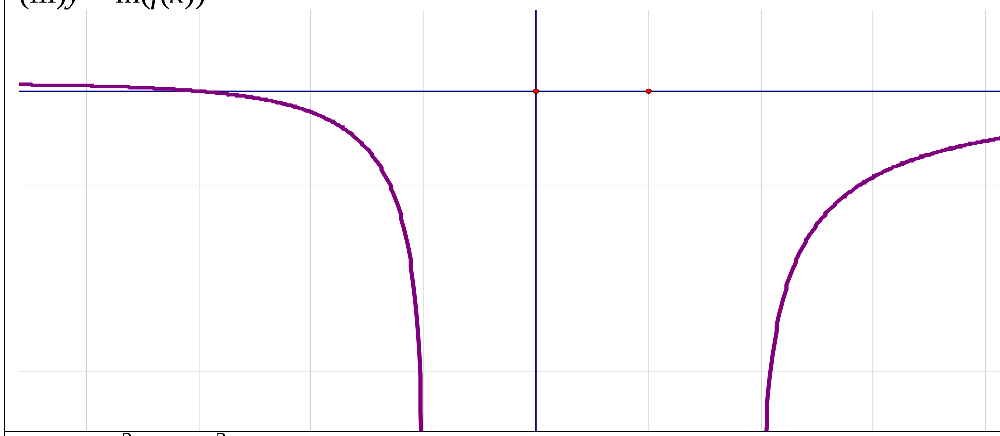
<table>
<tr><th>Solutions to Questions</th><th>Marking Scheme</th><th>Comments</th></tr>
<tr><td>Question 1</td><td></td><td></td></tr>
<tr><td>(a) $\int \frac{2x}{1+2x}\,dx = \int 1-\frac{1}{1+2x}dx$
$= x-\frac{1}{2}\ln\left|1+2x\right|+c$</td><td>1 for changing the integral
1 correct integration</td><td></td></tr>
<tr><td>(b) Let the Lee brothers be L_1 and L_2
Let the Abey brothers be $A_{1,}$ A_2 and $A_{3.}$
Then there are 10 others.
$\therefore$ # possibilities is 365 ways.
<table><tr><th>L</th><th>A</th><th>Others</th></tr><tr><td>1</td><td>0</td><td>10</td></tr><tr><td>1</td><td>1</td><td>9</td></tr><tr><td>1</td><td>2</td><td>8</td></tr><tr><td>0</td><td>1</td><td>10</td></tr><tr><td>0</td><td>2</td><td>9</td></tr></table>$\therefore$ 2 + 60 + 270 + 3 + 30 = 365 [using ${}^nC_r$]</td><td>1 for correct answer
2 for justification</td><td></td></tr>
<tr><td>(c) $\lim_{x\to -5}\frac{\sqrt{20-x}-5}{5+x}\times\frac{\sqrt{20-x}+5}{\sqrt{20-x}+5}$
$= \lim_{x\to -5}\frac{20-x-25}{(5+x)\left(\sqrt{20-x}+5\right)}$
$= \lim_{x\to -5}\frac{-1}{\sqrt{20-x}+5}$
$= -\frac{1}{10}$</td><td>1 for multiplying by conjugate
1 correct simplification & answer</td><td></td></tr>
</table>

Solutions to Questions	Marking Scheme	Comments
Question 1 continued		
(d) $(1+z)^8 = \binom{8}{0} + \binom{8}{1}z + \binom{8}{2}z^2 + \cdots + \binom{8}{7}z^7 + \binom{8}{8}z^8$ Let $z = 1$ then $2^8 = \binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \cdots + \binom{8}{8}$ $\therefore$ sum of coefficients = **2^8 or 256**.	1 correct expansion 1 correct substitution of $z = 1$ and the correct answer.	
 (e) (i) Since $3x^2 - y^2 = 12$ then we get $6x - 2y\dfrac{dy}{dx} = 0 \quad \therefore \dfrac{dy}{dx} = \dfrac{3x_1}{y_1}$ at T. $\therefore$ equation of tangent: $y - y_1 = \dfrac{3x_1}{y_1}(x - x_1)$ (ii) since this tangent meets the line $x = 1$ $\therefore y = \dfrac{3x_1 - 12}{y_1}$ & since $F(4, 0)$ $\therefore FM_{\text{grad}} = -\dfrac{(x_1 - 4)}{y_1}$ & $FT_{\text{grad}} = \dfrac{y_1}{x_1 - 4}$ $\therefore FM \times FT = -1 \qquad \therefore FM \perp FT$.	1 correct differential 1 gradient at T. 1 correct tangent equation 1 + 1 for correct gradients 1 correct justification of $FM \perp FT$.	

Solutions to Questions	Marking Scheme	Comments
Question 2		
(a)(i) $z = 2 \text{ cis } (-\frac{\pi}{4})$ (ii) $z^{22} = \left[2cis\left(-\frac{\pi}{4}\right)\right]^{22}$ by De Moivre's Theorem $= 2^{22}\left[\cos\left(-\frac{22\pi}{4}\right) + i\sin\left(-\frac{22\pi}{4}\right)\right]$ $= 2^{22}\left[\cos\left(\frac{11\pi}{2}\right) - i\sin\left(\frac{11\pi}{4}\right)\right]$ $= 2^{22}(0 - -i) = \mathbf{2^{22}\ i.}$	1 for modulus 1 for argument 1 correct use of De Moivre's theorem 1 correct simplification 1 correct answer	
(b)(i) Since $a^2 + b^2 > 2ab$ then similiarly $a^2 + c^2 > 2ac$ and $b^2 + c^2 > 2bc$. $\therefore\ 2(a^2 + b^2 + c^2) > 2(ab + bc + ac)$ $\therefore\ a^2 + b^2 + c^2 > ab + bc + ac$ as required (ii) Since $a + b + c = 6$ Then $(a + b + c)^2 = 36$ $\therefore\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 36$ But from part (i) $a^2 + b^2 + c^2 > ab + bc + ac$ $\therefore\ a^2 + b^2 + c^2 + 2ab + 2bc + 2ac > 3(ab + bc + ac)$ $\therefore\ 36 > 3(ab + bc + ac)$ $\therefore\ ab + bc + ac < 12$ as required.	1 correctly writing the other two inequalities. 1 correct inequality 1 correct expansion 1 correct substitution	

Solutions to Questions	Marking Scheme	Comments
Question 2 continued		
(c)(i) $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x}dx,\ n \ge 0$ By integration by parts		
$= \frac{1}{n!}\left[-x^n e^{-x}\right]_0^1 - \frac{1}{n!}\int_0^1 \left[-nx^{n-1}e^{-x}\right]dx$	1 + 1 marks	
$= \frac{1}{n!}\left[-e^{-1}\right] + \frac{n}{n!}\int_0^1 \left[x^{n-1}e^{-x}\right]dx$	1 mark	
$= -\frac{e^{-1}}{n!} + \frac{1}{(n-1)!}\int_0^1 \left[x^{n-1}e^{-x}\right]dx$		
$= I_{n-1} - \frac{e^{-1}}{n!}$ as required.		
(ii) $I_4 = I_3 - \frac{e^{-1}}{4!} \quad = I_2 - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= I_1 - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= I_0 - \frac{e^{-1}}{1!} - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$	1 up to this line	
$= \frac{1}{0!}\int_0^1 e^{-x}dx - \frac{e^{-1}}{1!} - \frac{e^{-1}}{2!} - \frac{e^{-1}}{3!} - \frac{e^{-1}}{4!}$		
$= -[e^{-x}]_0^1 - e^{-1}\left(\frac{41}{24}\right)$	1 for this line	
$= 1 - \frac{65}{24e}$	1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 3		
 (i) $y = \lvert f(x) \rvert$ 	1 shape 1 critical points	

Solutions to Questions	**Marking Scheme**	**Comments**
Question 3 continued		
(ii) $y = e^{f(x)}$  (iii) $y = \ln(f(x))$ 	1 shape 1 asymptote / turning point 1 shape 1 x intercept / asymptote	
(b) $(x^2 - 5)(x^2 + 3)$ over rational field $\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right)\left(x-3i\right)\left(x+\sqrt{3}i\right)$ over complex field	1 correct factorization 1 correct factorisation	

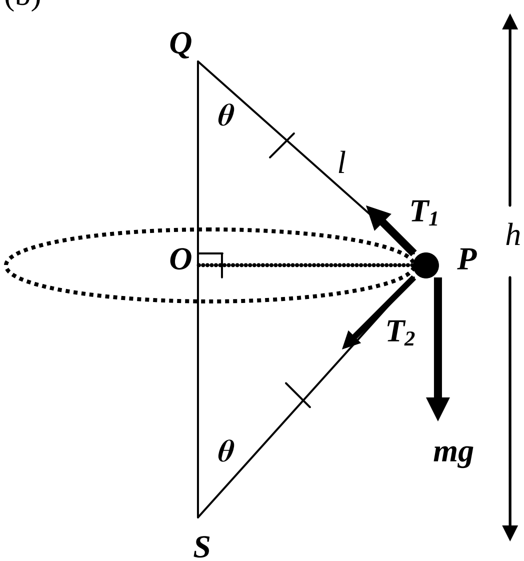
Solutions to Questions	Marking Scheme	Comments
Question 3 continued		
(c)		
(d) $\int_0^{\frac{\pi}{2}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$ $= \left[\ln\lvert\cos x + \sin x\rvert\right]_0^{\frac{\pi}{2}}$ $= 0$	1 correct conversion to sin & cos 1 correct integration + answer	

Solutions to Questions	Marking Scheme	Comments
Question 4		
(a) $T \le 20 \times 9.8$ But $T = m\omega^2 r$ $\therefore m\omega^2 r \le 20 \times 9.8$ $\therefore 4 \times \omega^2 \times ½ \le 20 \times 9.8$ $\therefore \omega^2 \le 98$ rads/ sec $\therefore \omega \le \sqrt{98} \times 60 \times \frac{1}{2\pi}$ revs / min $\therefore$ greatest no. of revolutions 94 revs/ min	1 $T \le 20 \times 9.8$ 1 1 converting to revs/ min & correct answer	
(b) $\int_a^{a^2} \frac{dx}{x\ln x} = \left[\ln(\ln x)\right]_a^{a^2}$ $= \ln(\ln a^2) - \ln(\ln a)$ $= \ln\left(\frac{\ln a^2}{\ln a}\right)$ $= \ln 2$	1 correct integration 1 correct use of log. Rule 1 correct answer	

(c) (i) $\partial V = 2\pi xy\partial x$ $\therefore$ Total volume = $\int_0^{\frac{\pi}{2}} 2\pi x\left(\cos^2 x - \cos 2x\right)dx$ $= \int_0^{\frac{\pi}{2}} 2\pi\left(x - x\cos^2 x\right)dx$ $= \left[\frac{1}{2}x^2 - \frac{1}{2}x\left(x + \frac{1}{2}\cos 2x\right)\right]_0^{\frac{\pi}{2}}$ $= \frac{\pi^2}{8} - \frac{1}{2}\left(\frac{\pi^2}{8} - \frac{1}{4} - \frac{1}{4}\right)$ $= \left(\frac{\pi^3}{8} + \frac{\pi}{2}\right)$ units^3	1 showing correct volume of slice 1 correct integral + limits 1 correct simplified integral 1 correct use of IBP 1 correct integration by parts 1 correct answer in terms of π.	
(d) $\int_{-4}^{4} \cos x\left(e^x - e^{-x}\right)dx$ Since $\cos x$ is an EVEN function & $e^x - e^{-x}$ is an ODD function, EVEN $\times$ ODD $\Rightarrow$ ODD $\therefore \int_{-4}^{4} \cos x\left(e^x - e^{-x}\right)dx = 0$	1 + 1 for correct justification 1 correct answer	**Note: students do not need to physically find the integral!**

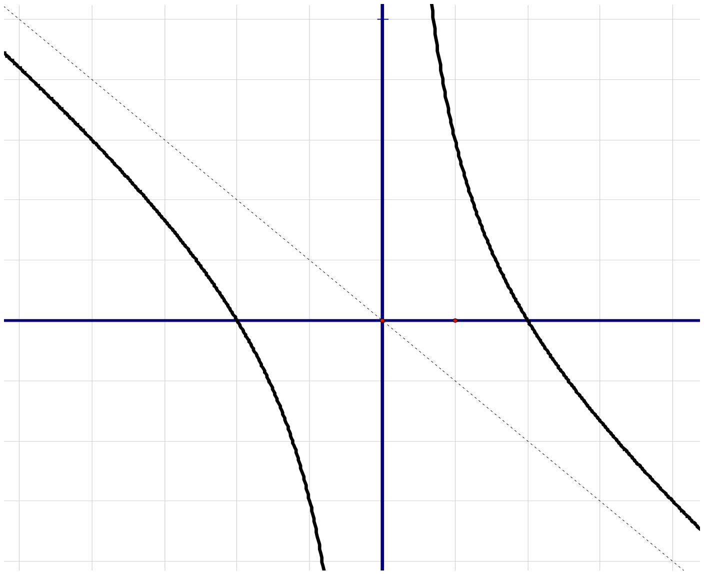
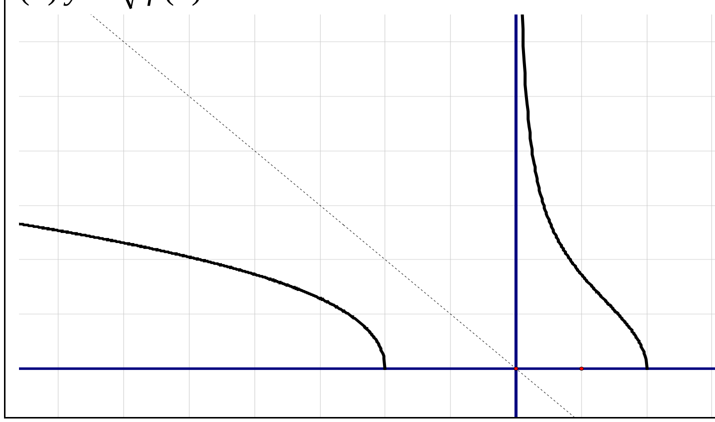
Solutions to Questions	**Marking Scheme**	**Comments**
Question 5		
(a) (i) $\dfrac{x^4}{x^2+1} = A(x^2-1) + \dfrac{B}{x^2+1}$ Let $x = 0 \Rightarrow B - A = 0$ Let $x = 1 \Rightarrow \frac{1}{2} = \frac{1}{2} \times B$ $\therefore B = 1$ and $A = 1$	1 substituting a reasonable x values 1 correct answers for A and B.	
(ii) $\displaystyle\int_0^1 x^3 \tan^{-1} x \, dx$ $= \left[\dfrac{x^4}{4}\tan^{-1} x\right]_0^1 - \dfrac{1}{4}\displaystyle\int_0^1 \dfrac{x^4}{1+x^2}dx$ $= \dfrac{\pi}{16} - \dfrac{1}{4}\left[\dfrac{x^3}{3} - x + \tan^{-1} x\right]_0^1$ $= \dfrac{\pi}{16} + \dfrac{1}{6} - \dfrac{\pi}{6}$ $= \dfrac{1}{6}$	1 + 1 for integration by parts 1 for integration 1 substitution 1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 5 continued		
(b)(i) particle moving upwards & acceleration is acting downwards $\therefore\ a = -\dfrac{k}{x^2}$		
When $x = R$, $a = g$ $\therefore\ k = gR^2$.	1 for finding k in terms of g and R.	
$\therefore\ \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{-gR^2}{x^2}$ since $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$	1 for finding a in terms of g and R	
$\therefore\ \dfrac{1}{2}v^2 = \dfrac{gR^2}{x} + c$	1 integration	
Now when $x = R$, $v = u$ initially $\therefore\ c = \dfrac{1}{2}u^2 - gR$	1 for c	
$\therefore\ v^2 = u^2 - 2gR + \dfrac{2gR^2}{x}$		
(ii) when $u = \sqrt{2gR}$ ➔ $v^2 = \dfrac{2gR^2}{x}$	1 v^2 equation	
$\therefore\ v = \dfrac{dx}{dt} = \sqrt{\dfrac{2gR^2}{x}}$, $v > 0$		
$\therefore\ \dfrac{dt}{dx} = \dfrac{\sqrt{x}}{\sqrt{2gR^2}}$	1 for $\dfrac{dt}{dx}$	
$\therefore\ t = \displaystyle\int_R^{4R} \dfrac{x^{\frac{1}{2}}}{\sqrt{2gR^2}}dx = \dfrac{1}{\sqrt{2gR^2}}\left[\dfrac{2}{3}x^{\frac{3}{2}}\right]_R^{4R}$	1 correct integration for t	
$\therefore\ t = \dfrac{14}{3}\sqrt{\dfrac{R}{2g}}$	1 correct answer	

Solutions to Questions	Marking Scheme	Comments
Question 6		
(a) **4 players** There are ${}^4C_1$ ways of choosing the team Then we need four players of the 10: ${}^{10}C_4$ ways The 5th player is chosen from the remaining 30 in ${}^{30}C_1$ way $\therefore$ total number of ways = ${}^4C_1 \times {}^{10}C_4 \times {}^{30}C_1$ = **25200** **5 players** ➔ ${}^4C_1 \times {}^{10}C_5 = 1008$ ways $\therefore$ p(at least 4 players from same team) $= \dfrac{25200+1008}{\binom{40}{5}} = \dfrac{1}{25}$	1 correct answer + 1 justification 1 correct answer 1 correct probability	
(b) 		

Solutions to Questions	Marking Scheme	Comments
Question 6 continued		
(b) (i) By trig. $R = l\sin\theta$ and $h = 2l\cos\theta$.	1 $\sin\theta$ and $\cos\theta$ relationships	
Resolving forces:		
Vertically: $0 = mg + T_2\cos\theta - T_1\cos\theta$.	1 vertical force	
$\therefore\ mg = (T_1 - T_2)\cos\theta$. ➔ ①		
Horizontally: $m\omega^2 r = T_1\sin\theta + T_2\sin\theta$	1 horizontal force	
$\therefore\ m(l\sin\theta)\ \omega^2 = (T_1 + T_2)\sin\theta$. ➔ ②		
From ① ➔ $T_1 - T_2 = \dfrac{mg}{\cos\theta} = \dfrac{2mgl}{h}$		
From ② ➔ $T_1 + T_2 = ml\omega^2$	1 rearrangements of equation	
① + ② ➔ $T_1 = ml(\tfrac{1}{2}\omega^2 + \dfrac{g}{h})$ → tension in PQ.	1 correct addition & simplification	
(ii) ② − ① ➔ $T_2 = ml\left(\dfrac{1}{2}\omega^2 - \dfrac{g}{h}\right)$	1 for answer	
(iii) String PS (hence PQ) will only remain stretched is $T_2 > 0\ \therefore \left(\dfrac{1}{2}\omega^2 - \dfrac{g}{h}\right) > 0$	1 knowing $T_2 > 0$	
$\therefore\ \omega > \sqrt{\dfrac{2g}{h}}$	1 showing answer is true	
(iv) If $T_1 : T_2 = 2 : 1$ then $\omega^2 = \dfrac{6g}{h}$	1 getting ω^2	
$\therefore\ \omega = \sqrt{\dfrac{6g}{h}}$ $\therefore$ period of motion = $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{\dfrac{h}{6g}}}$	1 for ω + 1 period	

Solutions to Questions	Marking Scheme	Comments
Question 7		
(a)(i) $\alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)^3 + 3\alpha\beta\gamma$ $\therefore (\alpha+\beta+\gamma)^3 + 3\alpha\beta\gamma$ $= (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma) + 3\alpha\beta\gamma$ $= 0 + 3(-q) = -3q$ (ii) Let $X = \dfrac{\alpha}{\beta\gamma} = \dfrac{\alpha^2}{\alpha\beta\gamma} = -\dfrac{\alpha^2}{q}$ $\therefore \alpha^2 = -qX$ But $\alpha^3 + p\alpha + q = 0$ as α is a root $\therefore -qX\alpha + p\alpha + q = 0$ $(qX - p)\alpha = q$ $(qX-p)^2a^2 = q^2$ $\therefore (qX-p)^2(-qX) = q^2$ $\therefore X(qX-p)^2 = -q$	(1) correct relationship (1) correct expansion (1) correct answer (1) a^2 equation (1) (1) correct solution	
(b) (i) $\therefore$ gradient of normal at P is t^2 $\therefore$ equation is given by $\left(y - \dfrac{c}{t}\right) = t^2(x - ct)$ Which leads to the required equation. (ii) $N\left(\dfrac{c-tc^3}{t(t^2-1)}, \dfrac{c-tc^3}{t(t^2-1)}\right)$ (iii) $\tan\ \angle NOP = \left\|\dfrac{1+\frac{1}{t^2}}{1-\frac{1}{t^2}}\right\| = \left\|\dfrac{1+t^2}{t^2-1}\right\| = \left\|\dfrac{1+t^2}{1-t^2}\right\|$ $\tan\ \angle ONP = \left\|\dfrac{1+t^2}{1-t^2}\right\| \quad \therefore\ \angle NOP = \angle ONP$	(1) correct gradient & use of point – gradient formula. (1) correct coordinates. (1) Finding angles + (1) conclusion $\therefore\ \Delta ONP$ is isosceles (equal base angles)	**Can also show *PN* = *OP***

Solutions to Questions	Marking Scheme	Comments
Question 7 continued		
(c) (i) $y = \dfrac{4}{x} - x$  (ii) $y = \sqrt{f(x)}$ 	(1) correct shape (1) correct roots & Asymptotes (1) + (1) for each arm	

Solutions to Questions	Marking Scheme	Comments
Question 8		
(a)(i) Since $y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$ (÷ by $gsec^2\theta / 2V^2$)		
$\therefore x^2 - \frac{2V^2\tan\theta}{g\sec^2\theta}x = \frac{-2V^2}{g\sec^2\theta}y$	(1)	
By completing the square:		
$\therefore x^2 - \frac{2V^2}{g}\sin\theta\cos\theta x + \left(\frac{V^2}{g}\sin\theta\cos\theta\right)^2$	(1) completing the square	
$= \frac{-2V^2\cos^2\theta}{g}y + \frac{V^4}{g^2}\sin^2\theta\cos^2\theta$		
$\therefore \left[x - \frac{V^2}{g}\sin\theta\cos\theta\right]^2 = -\frac{2V^2\cos^2 x}{g}\left[y - \frac{V^2}{2g}\sin^2\theta\right]$	(1) putting in the form $(x - h)^2 = 4a(y - k)$.	
Which is of the form $(x - h)^2 = 4a(y - k)$.		
(ii) The **focal length** is $\frac{1}{4} \times -\frac{2V^2\cos^2\theta}{g} = \frac{V^2\cos\theta}{2g}$	(1) focal length	
Horizontal range when $y = 0$		
$\therefore x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2} = 0$ ➔ $x\left(\tan\theta - \frac{gx\sec^2\theta}{2V^2}\right) = 0$		
$\therefore$ ignore $x = 0$, then $x = \frac{V^2\sin 2\theta}{g}$ $\therefore$ since $\frac{2V^2\cos\theta}{2g} = \frac{V^2\sin 2\theta}{g}$	(1) for RANGE	
we get $\cos\theta = \sin 2\theta = 2\sin\theta\cos\theta$; $\boldsymbol{\cos\theta \neq 0}$		
$\therefore \sin\theta = ½$ ➔ $\theta = \frac{\pi}{3}$	(1) answer	

Solutions to Questions	Marking Scheme	Comments
Question 8 continued		
(b)(i) $\frac{1}{2}(p+q) \geq \sqrt{pq}$. RTP ¼ $(p+q)^2 - pq \geq 0$ LHS = ¼ $(p^2 + 2pq + q^2 - 4pq)$ = ¼ $(p-q)^2 \geq 0$ (ii) $\frac{1}{2}(p+q) \geq \sqrt{pq}$. now ÷ by $\sqrt{q} > 0$ $\Rightarrow \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \frac{q}{\sqrt{q}}\right) \geq \sqrt{p}$ $\Rightarrow \frac{1}{2}\left(\frac{p}{\sqrt{q}} + \sqrt{q}\right) \geq \sqrt{p}$ as required	(1) mark (1) mark	
(c)(i) see attached		

James Ruse 2008 MX2 Trial Q8(c) Solution

Q8 (c)(i)

<u>For $n = 1$:</u> This case is trivial.

<u>For $n = 2$:</u>

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{b_2}}\,B_2 + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \\
&= \frac{1}{\sqrt{b_2}}(b_1 + b_2) + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) b_1 \\
&= \frac{b_1}{\sqrt{b_2}} + \sqrt{b_2} + \sqrt{b_1} - \frac{b_1}{\sqrt{b_2}} \\
&= \sqrt{b_2} \quad \text{as required. Therefore, true for } n = 2
\end{aligned}$$

Assume result holds up to some $n = k$ (strong induction), that is

$$\frac{1}{\sqrt{b_k}}\,B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 = \sum_{r=1}^{k} \sqrt{b_r}$$

<u>For $n = k + 1$</u>

$$\begin{aligned}
\text{LHS} &= \frac{1}{\sqrt{b_{k+1}}}(B_{k+1} + \left(\frac{1}{\sqrt{b_k}} - \frac{1}{\sqrt{b_{k+1}}}\right) B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \\
&= \frac{1}{\sqrt{b_{k+1}}}(B_{k+1} - B_k) + \left(\frac{1}{\sqrt{b_k}}\,B_k + \left(\frac{1}{\sqrt{b_{k-1}}} - \frac{1}{\sqrt{b_k}}\right) B_{k-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1\right) \\
&= \frac{1}{\sqrt{b_{k+1}}}(b_{k+1}) + \sum_{r=1}^{k} \sqrt{b_r} \\
&= \sqrt{b_{k+1}} + \sum_{r=1}^{k} \sqrt{b_r} \\
&= \sum_{r=1}^{k+1} \sqrt{b_r}
\end{aligned}$$

Therefore, since the initial case and two consecutive cases hold, by the principle of Mathematical Induction, the proposition is true $\quad \square$.

Q8 (c)(ii)

$$\begin{aligned}
\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}} &= \frac{1}{\sqrt{b_k}}\,A_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) A_{n-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) A_1 \\
&\le \frac{1}{\sqrt{b_k}}\,B_n + \left(\frac{1}{\sqrt{b_{n-1}}} - \frac{1}{\sqrt{b_n}}\right) B_{n-1} + \cdots + \left(\frac{1}{\sqrt{b_1}} - \frac{1}{\sqrt{b_2}}\right) B_1 \quad \text{since } A_i \le B_i \text{ for all } 0 < i \le n \\
&= \sum_{r=1}^{n} \sqrt{b_r}
\end{aligned}$$

Q8 (c)(iii)

Since all $a_r > 0$ and $b_r > 0$ for all $0 < r \leq n$, therefore from Q8(b)(ii)

$$\sqrt{a_r} < \frac{1}{2}\left(\frac{a_r}{\sqrt{b_r}} + \sqrt{b_r}\right), \quad \text{for all } r \text{ where } 0 < r \leq n$$

Summing over n terms gives:

$$\begin{aligned}
\sum_{r=1}^{n} \sqrt{a_r} &\leq \sum_{r=1}^{n} \frac{1}{2}\left(\frac{a_r}{\sqrt{b_r}} + \sqrt{b_r}\right) \\
&= \frac{1}{2}\left(\sum_{r=1}^{n} \frac{a_r}{\sqrt{b_r}}\right) + \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) \quad \text{(rearranging the sum)} \\
&\leq \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) + \frac{1}{2}\left(\sum_{r=1}^{n} \sqrt{b_r}\right) \quad \text{(using the result from Q8(c)(ii))} \\
&= \sum_{r=1}^{n} \sqrt{b_r} \quad \text{as required}
\end{aligned}$$