TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2009

MATHEMATICS EXTENSION 2

Time Allowed – 3 Hours
(Plus 5 minutes Reading Time)

- *All* questions may be attempted
- *All* questions are of equal value
- Department of Education approved calculators are permitted
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged work
- No grid paper is to be used unless provided with the examination paper

The answers to all questions are to be returned in separate *stapled* bundles clearly labeled Question 1, Question 2, etc. Each question must show your Candidate Number.

Question 1 (15 Marks) Marks

(a) Find the modulus and principle argument of z if $z = 1 + i\sqrt{3}$. Hence find the smallest positive integer n such that z^n is a real number. 3

(b) (i) Find both complex roots of $\omega^2 = 3 + 4i$. 3

(ii) Hence find the complex roots of the equation $z^2 + (4 - 2i)z - 8i = 0$. 3

(c) Find all the complex numbers that satisfy $|z|^2 - iz = 36 + 4i$. 4

(d) Find $\int \frac{1}{x^2 + 2x + 5}\,dx$. 2

Question 2 (15 Marks) START A NEW PAGE Marks

(a) Hot liquid metal is poured into a cooling vat. The initial temperature, T_1, of the metal is $900°C$ and the initial temperature, T_2, of the vat is $36°C$. The temperatures of the metal and vat satisfy Newton's law of cooling according to the formulae

$\frac{dT_1}{dt} = -k(T_1 - T_2)$ and $\frac{dT_2}{dt} = 0.8k(T_1 - T_2)$ where $k > 0$.

(i) Show that $\frac{4}{5}\frac{dT_1}{dt} + \frac{dT_2}{dt} = 0$ and hence prove that $\frac{4}{5}T_1 + T_2 = 756$. 3

(ii) Find a formula for $\frac{dT_1}{dt}$ in terms of T_1 and hence show that $T_1 = 420 + Ae^{-\frac{9k}{5}t}$, where A and k are constants, satisfies this differential equation. 3

(iii) Four hours after the metal is poured into the vat, the metal has cooled to $500°C$. Find the temperature of the metal after a further 2 hours. Give your answer correct to the nearest degree. 3

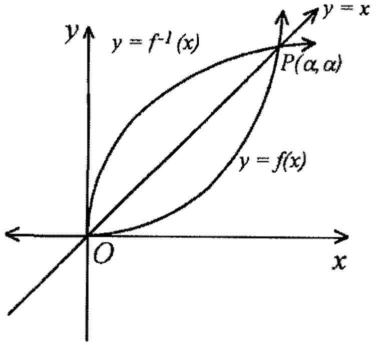
(b) (i) The area bounded by the graph of $y = x\sqrt{4 - x^2}$ and the x-axis is rotated one revolution about the y-axis to form a solid. Show, with the aid of a sketch of the curve $y = x\sqrt{4 - x^2}$, and using the method of Volume by Cylindrical Shells, that the volume, $V\ u^3$, of the solid is given by: 2

$$V = 4\pi\int_0^2 x^2\sqrt{4 - x^2}\,dx.$$

(ii) Using the substitution $x = 2\sin\theta$, find the volume of the solid that is formed. 4

Question 3 (15 Marks) START A NEW PAGE — Marks

(a) The diagram shows the graph of an increasing function $y = f(x)$ and its inverse $y = f^{-1}(x)$ for $x \ge 0$, and the line $y = x$. The graphs intersect at $(0,0)$ and $P(\alpha, \alpha)$. — 4



Use the substitution $u = f^{-1}(x)$ to show that $\int_0^{\alpha} f^{-1}(x)\,dx = \int_0^{\alpha} u f'(u)\,du$ and hence show that the area, A square units, bounded by $y = f(x)$ and $y = f^{-1}(x)$ for $x \ge 0$ is given by

$$A = \int_0^{\alpha} \{x f'(x) - f(x)\}\,dx.$$

(b) For the curve $y = x\tan^{-1} x$.

(i) For what values of x is $y = x\tan^{-1} x$ an increasing function? — 1

(ii) Show that $y = x\tan^{-1} x$ is concave up for all real values of x. — 2

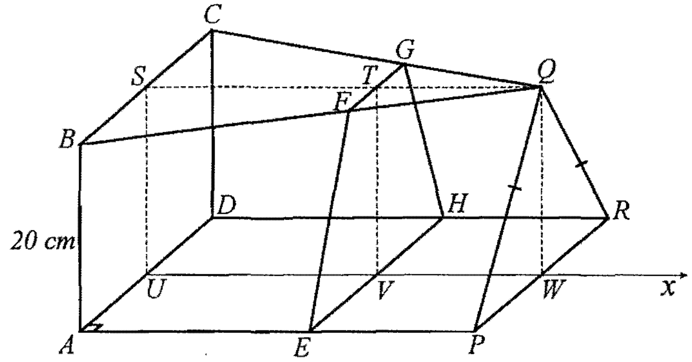
(iii) Sketch the graph of $y = x\tan^{-1} x$ for $-2.5 \le x \le 2.5$, clearly showing any stationary points. — 2

(iv) If $f(x) = x\tan^{-1} x$, $x \ge 0$, show on a new diagram the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$, and the line $y = x$. Clearly label the coordinates of their points of intersection. — 3

(v) With the aid of the result from Q3(a), find the area of the region bounded by $y = f(x)$ and $y = f^{-1}(x)$. — 3

Question 4 (15 Marks) START A NEW PAGE — Marks

(a)



$UV = h$ cm and $UW = 100$ cm

The above solid has a square face ($ABCD$) at one end and a triangular face (PQR) that is parallel to it at the other end. The solid is 100 cm long. The square face has sides of length 20 cm and is perpendicular to the base ($ADRP$) while the triangular face is isosceles with $QP = QR$. The triangular face BCQ and rectangular face $ADRP$ are parallel.

(i) If the length $UV = h$ cm, show that the area, A cm^2, of the cross-section $EFGH$ is given by $A = 400 - 2h$. — 2

(ii) Hence calculate the volume of the above solid. — 3

(b) The total surface area, A cm^2, of a cylinder is given by $A = 2\pi r(r + h)$, where r cm is the radius of the base and h cm is the height. Find the rate at which the surface area is changing if the radius of the base is increasing at 0.2 cm/minute and the height is decreasing at 0.5 cm/minute when the base radius equals 16 cm and the height equals 20 cm. — 3

(c) (i) If $y = x^k + (c - x)^k$, where $c > 0$, $k > 0$, $k \ne 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $0 < k < 1$ and a minimum if $k > 1$. — 3

(ii) Hence show that if $a > 0$, $b > 0$, $a \ne b$, then $\dfrac{a^k + b^k}{2} < \left(\dfrac{a+b}{2}\right)^k$ if $0 < k < 1$ and $\dfrac{a^k + b^k}{2} > \left(\dfrac{a+b}{2}\right)^k$ if $k > 1$. — 4

Question 5 (15 Marks) START A NEW PAGE

Marks

A toy soldier of mass 500 grams has a parachute attached to it. The toy soldier is released from rest at a position 60 m above the ground. During its fall the forces acting on the toy are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{80}v^2$ when the velocity of the toy is $v\ m/s$. After $5\ln 2$ seconds the parachute fails and from then on the only force acting on the toy is gravity. The acceleration due to gravity is taken as $10\ ms^{-2}$ and at time t seconds the toy has fallen $x\ m$ from its release point and its velocity is $v\ m/s$.

(i) With the aid of a force diagram show that while the parachute is operating, $\ddot{x} = 10 - \frac{1}{40}v^2$. 1

(ii) Show that $v = 20\left(\frac{e^t - 1}{e^t + 1}\right)$. 4

(iii) Show that $x = -20\ln\left[1 - \left(\frac{v}{20}\right)^2\right]$. 4

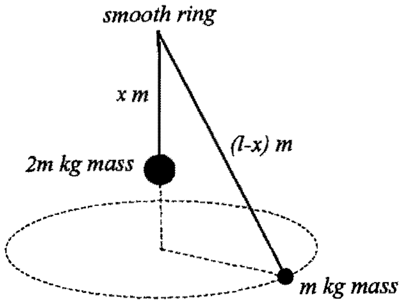
(iv) Find the exact speed of the toy and the exact distance fallen at the instant the parachute fails. 3

(v) After the parachute fails, find an expression for the acceleration $\ddot{x}$ and hence find the speed of the toy the instant it hits the ground. Give your answer correct to two significant figures. 3

Question 6 (15 Marks) START A NEW PAGE

Marks

(a) Two particles of masses $2m$ kg and m kg respectively are connected to a light inextensible string of length l m. The string passes through a fixed smooth ring. While the $2m$ kg mass hangs in equilibrium a distance x m below the ring, the other mass describes a horizontal circle with angular velocity ω rad/sec. (Take the acceleration due to gravity as g m/s^2)

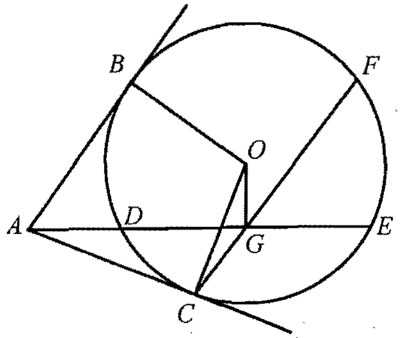


(i) Draw a diagram showing the forces acting on each mass. 2

(ii) Find the angle that the string makes with the vertical. 3

(iii) Prove that the angular velocity of the smaller mass equals $\sqrt{\frac{2g}{l-x}}$ rad/sec. 2

(b) In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.

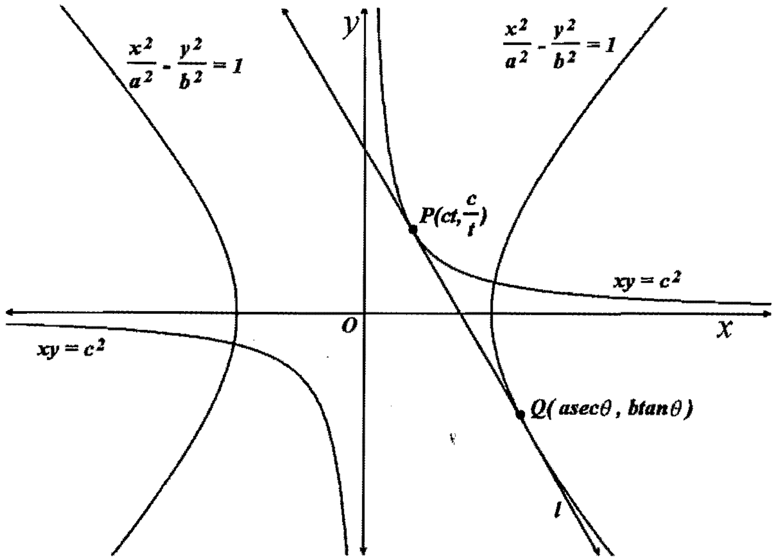


(i) Copy the diagram onto your answer sheet and prove that $ABOC$ and $AOGC$ are cyclic quadrilaterals. 2

(ii) Explain why $\angle OGF = \angle OAC$. 1

(iii) Prove that $BF \parallel ADE$. 5

Question 7 (15 Marks) START A NEW PAGE Marks



The line l is a common tangent to the hyperbola $xy=c^2$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with points of contact P and Q respectively.

(i) Considering l as a tangent to $xy=c^2$ at $P\left(ct,\frac{c}{t}\right)$, prove that l has the equation $x+t^2y=2ct$. 3

(ii) Considering l as a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $Q(a\sec\theta, b\tan\theta)$, prove that l has the equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. 3

(iii) Deduce that $\frac{\sec\theta}{a}=\frac{-\tan\theta}{bt^2}=\frac{1}{2ct}$. 1

(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4+4c^2t^2-a^2=0$. Deduce that there are exactly two such common tangents to the hyperbolas. 3

(v) Copy the diagram onto your answer sheet, and using the symmetry of the graph, draw in the second common tangent with points of contact R on $xy=c^2$ and S on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Write down the coordinates of R and S in terms of t, a, b and c. 2

(vi) Show that if $PQRS$ is a rhombus, then $b^2=a^2$. 3

Question 8 (15 Marks) Marks

(a) Six lines are drawn on a plane such that no two lines are parallel and no three lines are concurrent (i.e. pass through the same point).

(i) Show that there are 15 points of intersection. 1

(ii) If three of these points are chosen at random, show that the probability that they all lie on one of the given lines is $\frac{12}{91}$. 2

(iii) Find the probability that if three of these points are chosen at random then none of them lie on the same line. 2

(b) (i) Prove that: 2

$$(\alpha)\ \frac{{}^1C_0}{x}-\frac{{}^1C_1}{x+1}=\frac{1!}{x(x+1)}.$$

$$(\beta)\ \frac{{}^2C_0}{x}-\frac{{}^2C_1}{x+1}+\frac{{}^2C_2}{x+2}=\frac{2!}{x(x+1)(x+2)}.$$

(ii) Given $T(k,x)=\frac{k!}{x(x+1)(x+2)\ldots(x+k)}$, prove that $T(k,x)-T(k,x+1)=T(k+1,x)$. 3

(iii) Hence prove, using Mathematical Induction or otherwise, that for $n\geq 1$: 5

$$\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\frac{{}^nC_3}{x+3}+\ldots+(-1)^n\frac{{}^nC_n}{x+n}=\frac{n!}{x(x+1)(x+2)(x+3)\ldots(x+n)}.$$

[You may use the result : ${}^{k+1}C_r={}^kC_r+{}^kC_{r-1}$]

THIS IS THE END OF THE EXAMINATION PAPER

Question 1 (15 Marks) Marks

1(a) Find the modulus and principle argument of z if $z = 1 + i\sqrt{3}$. Hence find the smallest positive integer n such that z^n is a real number. 3

SOLUTION:

$|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\arg(z) = \frac{\pi}{3}$

$z^n = \left(2cis\frac{\pi}{3}\right)^n$

$= 2^n cis\frac{n\pi}{3}$

If z^n is real then $\frac{n\pi}{3} = k\pi,\ k \in Z, n \in Z$

$n = 3k,\ k = 1,2,3....$

$\therefore n = 3$

1(b) (i) Find both complex roots of $\omega^2 = 3 + 4i$. 3

SOLUTION:

Let $\omega = a + ib, \quad a, b \in R$

Then $(a+ib)^2 = 3 + 4i$

$\Rightarrow a^2 - b^2 + 2iab = 3 + 4i$

$\therefore a^2 - b^2 = 3$ and $ab = 2$

$\Rightarrow a = 2, b = 1$ or $a = -2, b = -1$ (by inspection)

$\omega = \pm(2 + i)$

(ii) Hence find the complex roots of the equation $z^2 + (4-2i)z - 8i = 0$. 3

SOLUTION:

$$z = \frac{-(4-2i) \pm \sqrt{(4-2i)^2 - 4(1)(-8i)}}{2}$$
$$= \frac{(-4+2i) \pm \sqrt{16 - 16i - 4 + 32i}}{2}$$
$$= \frac{(-4+2i) \pm \sqrt{12 + 16i}}{2}$$
$$= \frac{(-4+2i) \pm 2\sqrt{3+4i}}{2}$$
$$= \frac{(-4+2i) \pm 2(2+i)}{2}$$
$$= \frac{(-4+2i) + (4+2i)}{2} \text{ or } \frac{(-4+2i) - (4+2i)}{2}$$
$$= \frac{4i}{2} \text{ or } \frac{-8}{2}$$

$z = -4$ or $2i$

1(c) Find all the complex numbers that satisfy $|z|^2 - iz = 36 + 4i$. 4

SOLUTION:

Let $z = x + iy, \quad x, y \in R$

Then $x^2 + y^2 - i(x + iy) = 36 + 4i$

$x^2 + y^2 + y - ix = 36 + 4i$

$\Rightarrow x^2 + y^2 + y = 36$ and $x = -4$

$\therefore 16 + y^2 + y = 36$

$y^2 + y - 20 = 0$

$(y+5)(y-4) = 0$

$y = -5$ or 4

$z = -4 + 4i$ or $-4 - 5i$

(d) Find $\int \frac{1}{x^2 + 2x + 5}\,dx$. 2

$$\int \frac{1}{x^2 + 2x + 5}\,dx = \int \frac{1}{(x+1)^2 + 4}\,dx$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

Question 2 (15 Marks) START A NEW PAGE Marks

(a) Hot liquid metal is poured into a cooling vat. The initial temperature, T_1, of the metal is $900°C$ and the initial temperature, T_2, of the vat is $36°C$. The temperatures of the metal and vat satisfy Newton's law of cooling according to the formulae

$\frac{dT_1}{dt} = -k(T_1 - T_2)$ and $\frac{dT_2}{dt} = 0.8k(T_1 - T_2)$ where $k > 0$.

(a) (i) Show that $\frac{4}{5}\frac{dT_1}{dt} + \frac{dT_2}{dt} = 0$ and hence prove that $\frac{4}{5}T_1 + T_2 = 756$. 3

SOLUTION:

$$\frac{4}{5}\frac{dT_1}{dt} + \frac{dT_2}{dt} = 0.8\{-k(T_1 - T_2)\} + 0.8k(T_1 - T_2)$$
$$= -0.8kT_1 + 0.8kT_2 - 0.8kT_1 + 0.8kT_2$$
$$= 0$$

Since $0.8\frac{dT_1}{dt} + \frac{dT_2}{dt} = 0$

$0.8T_1 + T_2 = const.$

when $t = 0$, $T_1 = 900$ and $T_2 = 36$

$0.8 \times 900 + 36 = const.$

$const. = 756$

$\therefore 0.8T_1 + T_2 = 756$

(a) (ii) Find a formula for $\frac{dT_1}{dt}$ in terms of T_1 and hence show that $T_1 = 420 + Ae^{-\frac{9k}{5}t}$, where A and k are constants, satisfies this differential equation. 3

SOLUTION:

$$\frac{dT_1}{dt} = -k(T_1 - T_2)$$
$$= -k(T_1 - 756 + 0.8T_1)$$
$$= -k(1.8T_1 - 756)$$
$$= -1.8k(T_1 - 420)$$

$$T_1 = 420 + Ae^{-\frac{9k}{5}t}$$
$$\frac{dT_1}{dt} = A(-1.8k)e^{-\frac{9k}{5}t}$$
$$= -1.8k\left(Ae^{-\frac{9k}{5}t}\right)$$

$\therefore \frac{dT_1}{dt} = -1.8k(T_1 - 420)$ since $Ae^{-\frac{9k}{5}t} = T_1 - 420$

2(a) (iii) Four hours after the metal is poured into the vat, the metal has cooled to $500°C$. Find the temperature of the metal after a further 2 hours. Give your answer correct to the nearest degree. 3

SOLUTION:

when $t = 0$, $T_1 = 900$

$\therefore 900 = 420 + Ae^0$

$A = 480$

$T_1 = 420 + 480e^{-1.8kt}$

when $t = 4$, $T_1 = 500$

$\therefore 500 = 420 + 480e^{-7.2k}$

$480e^{-7.2k} = 80$

$e^{-7.2k} = \frac{1}{6}$

$-7.2k = \ln\left(\frac{1}{6}\right)$

$k = \frac{-\ln 6}{-7.2}$

$T_1 = 420 + 480e^{-1.8\left(\frac{\ln 6}{7.2}\right)t}$

$T_1 = 420 + 480e^{-\frac{1}{4}\ln 6t}$

when $t = 6$

$T_1 = 420 + 480e^{-\frac{1}{4}\ln 6 \times 6}$

$= 420 + 480e^{-\frac{3}{2}\ln 6}$

$T_1 = 452.65986...$

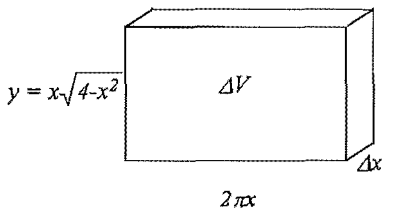
temperature $= 453°C$

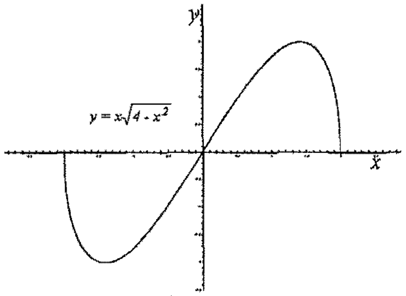
2(b) (i) The area bounded by the graph of $y = x\sqrt{4-x^2}$ and the x-axis is rotated one revolution about the y-axis to form a solid. Show, with the aid of a sketch of the curve $y = x\sqrt{4-x^2}$, and using the method of Volume by Cylindrical Shells, that the volume, $V\ u^3$, of the solid is given by: 2

$$V = 4\pi\int_0^2 x^2\sqrt{4-x^2}\ dx.$$

SOLUTION:

Volume of cylinder formed by rotating right side section of curve approximates the volume of a rectangular prism

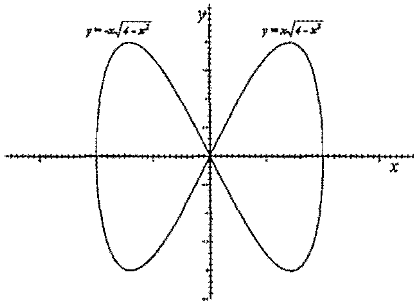




$$V_R = \lim_{\Delta x \to 0}\sum_0^2 2\pi xy\Delta x$$

$$V_R = 2\pi\int_0^2 xy\ dx$$

but graph is odd and corresponding volume is an even integral $V_{Right} = V_{Left}$



$$V = 2\times 2\pi\int_0^2 xy\ dx$$

$$= 4\pi\int_0^2 xy\ dx$$

$$= 4\pi\int_0^2 x.x\sqrt{4-x^2}\ dx$$

$$= 4\pi\int_0^2 x^2\sqrt{4-x^2}\ dx$$

Or

$height = 2y = 2x\sqrt{4-x^2}$ by symmetry of solid after rotation, then

$$V = \lim_{\Delta x \to 0}\sum_0^2 2\pi x(2y)\Delta x$$

$$V = 4\pi\int_0^2 xy\ dx$$

$$V = 4\pi\int_0^2 x.x\sqrt{4-x^2}\ dx$$

2(b) (ii) Using the substitution $x = 2\sin\theta$, find the volume of the solid that is formed. 4

SOLUTION:

$$V = 4\pi\int_0^2 x^2\sqrt{4-x^2}\ dx$$

$dx = 2\cos\theta\ d\theta$

$x = 0 \Rightarrow \theta = 0$

$x = 2 \Rightarrow \theta = \frac{\pi}{2}$

$$V = 4\pi\int_0^{\frac{\pi}{2}} 4\sin^2\theta\sqrt{4-4\sin^2\theta}\ .2\cos\theta\ d\theta$$

$$= 4\pi\int_0^{\frac{\pi}{2}} (4\sin^2\theta)(2\cos\theta)(2\cos\theta)\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} (4\sin^2\theta\cos^2\theta)\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} \sin^2 2\theta\ d\theta$$

$$= 16\pi\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$= 8\pi\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

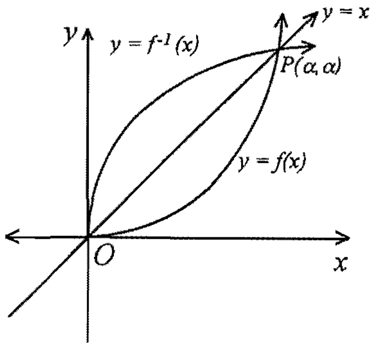
$$= 8\pi\{(\tfrac{\pi}{2} - \sin 2\pi) - (0 - \sin 0)\}$$

$$= 4\pi^2$$

volume $= 4\pi^2\ u^3$

Question 3 (15 Marks) START A NEW PAGE Marks

(a) The diagram shows the graph of an increasing function $y = f(x)$ and its inverse $y = f^{-1}(x)$ for $x \geq 0$, and the line $y = x$. The graphs intersect at $(0,0)$ and $P(\alpha, \alpha)$. 4



Use the substitution $u = f^{-1}(x)$ to show that $\int_0^{\alpha} f^{-1}(x)\,dx = \int_0^{\alpha} u f'(u)\,du$ and hence show that the area, A square units, bounded by $y = f(x)$ and $y = f^{-1}(x)$ for $x \geq 0$ is given by

$$A = \int_0^{\alpha} \{x f'(x) - f(x)\}\,dx.$$

SOLUTION:

$u = f^{-1}(x) \Rightarrow x = f(u)$
$dx = f'(u)du$
$x = 0 \Rightarrow u = f^{-1}(0)$
$\quad = 0$
$x = \alpha \Rightarrow u = f^{-1}(\alpha)$
$\quad = \alpha$

$$\int_0^{\alpha} f^{-1}(x)\,dx = \int_0^{\alpha} u\, f'(u)\,du$$
$$= \int_0^{\alpha} x\, f'(x)\,dx$$

$$A = \int_0^{\alpha} f^{-1}(x)\,dx - \int_0^{\alpha} f(x)\,dx$$
$$A = \int_0^{\alpha} \{f^{-1}(x) - f(x)\}dx$$
$$A = \int_0^{\alpha} \{x f'(x) - f(x)\}\,dx$$

3(b) For the curve $y = x\tan^{-1} x$.

3(b) (i) For what values of x is $y = x\tan^{-1} x$ an increasing function? 1

SOLUTION:

$y = x\tan^{-1} x$
$$\frac{dy}{dx} = \frac{x}{1+x^2} + \tan^{-1} x$$
$$\therefore \frac{dy}{dx} > 0 \text{ if } x > 0 \text{ since } \frac{x}{1+x^2} > 0 \text{ and } \tan^{-1} x > 0 \text{ when } x > 0$$

3(b) (ii) Show that $y = x\tan^{-1} x$ is concave up for all real values of x. 2

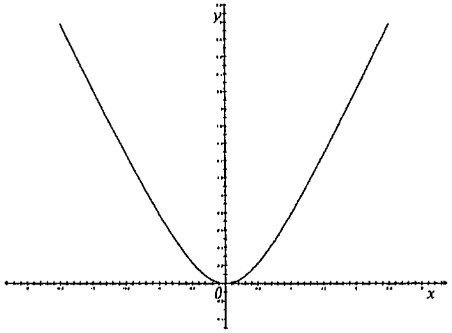
SOLUTION:

$$\frac{dy}{dx} = \frac{x}{1+x^2} + \tan^{-1} x$$
$$\frac{d^2y}{dx^2} = \frac{(1+x^2)(1) - (x)(2x)}{(1+x^2)^2} + \frac{1}{1+x^2}$$
$$= \frac{1-x^2}{(1+x^2)^2} + \frac{1}{1+x^2}$$
$$= \frac{1-x^2+1+x^2}{(1+x^2)^2}$$
$$= \frac{2}{(1+x^2)^2}$$
> 0 for all $x \in R$
$\therefore$ concave up for all $x \in R$

3(b) (iii) Sketch the graph of $y = x\tan^{-1} x$ for $-2.5 \le x \le 2.5$, clearly showing any stationary points. 2

SOLUTION:

x	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5
y	2.98	2.21	1.47	0.79	0.23	0	0.23	0.79	1.47	2.21	2.98



3(b) (iv) If $f(x) = x\tan^{-1} x$, $x \ge 0$, show on a new diagram the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$, and the line $y = x$. Clearly label the coordinates of their points of intersection. 3

SOLUTION:

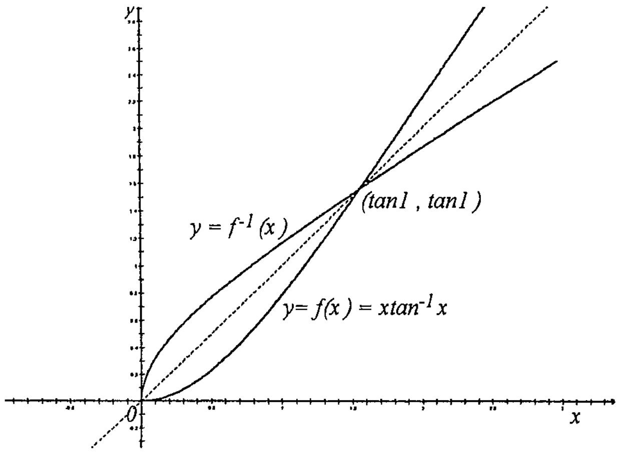
Curves meet when

$x\tan^{-1} x = x$

$x(\tan^{-1} x - 1) = 0$

$x = 0$ or $\tan^{-1} x = 1$

$x = 0$ or $x = \tan 1$



3(b) (v) With the aid of the result from Q3(a), find the area of the region bounded by $y = f(x)$ and $y = f^{-1}(x)$. 3

SOLUTION:

$$A = \int_0^{\tan 1} \{xf'(x) - f(x)\}\, dx \quad \text{where } f(x) = x\tan^{-1} x$$

$$= \int_0^{\tan 1} \left\{x\left(\tan^{-1} x + \frac{x}{1+x^2}\right) - \left(x\tan^{-1} x\right)\right\} dx$$

$$= \int_0^{\tan 1} \left\{\left(x\tan^{-1} x + \frac{x^2}{1+x^2}\right) - \left(x\tan^{-1} x\right)\right\} dx$$

$$= \int_0^{\tan 1} \left\{\frac{x^2}{1+x^2}\right\} dx$$

$$= \int_0^{\tan 1} \left\{1 - \frac{1}{1+x^2}\right\} dx$$

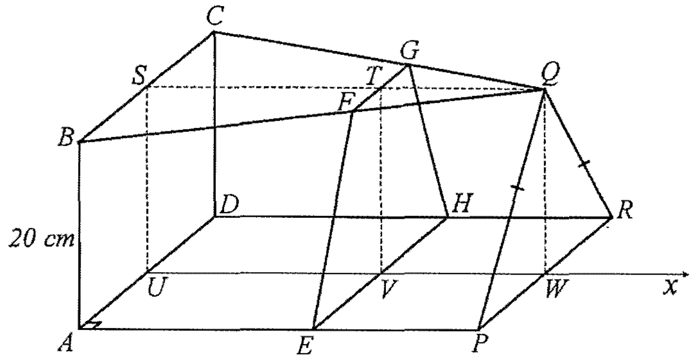
$$= \left[x - \tan^{-1} x\right]_0^{\tan 1}$$

$$= (\tan 1 - 1) - (0 - 0)$$

$$\text{Area} = (\tan 1 - 1)\ u^2$$

uestion 4 (15 Marks) START A NEW PAGE Marks

(a)



$UV = h\ cm$ and $UW = 100\ cm$

The above solid has a square face ($ABCD$) at one end and a triangular face (PQR) that is parallel to it at the other end. The solid is 100 cm long. The square face has sides of length 20 cm and is perpendicular to the base ($ADRP$) while the triangular face is isosceles with $QP = QR$. The triangular face BCQ and rectangular face $ADRP$ are also parallel.

(a) (i) If the length $UV = h$ cm, show that the area, A cm^2, of the cross-section $EFGH$ is given by $A = 400 - 2h$. 2

SOLUTION:

ΔQBC and ΔQFG are similar

$\therefore \dfrac{QT}{QS} = \dfrac{FG}{BC}$ (ratios of corresponding sides are equal)

$\dfrac{100-h}{100} = \dfrac{FG}{20}$

$FG = \dfrac{100-h}{5}$

Area of $EFGH = \dfrac{1}{2}(TV)(EH + FG)\,u^2$

$$A = \frac{1}{2}(20)\left(20 + \frac{100-h}{5}\right)$$
$$= 10\left(\frac{100 + 100 - h}{5}\right)$$
$$= 2(200-h)$$
$$A = 400 - 2h$$

4(a) (ii) Hence calculate the volume of the above solid. 3

SOLUTION:

$$V = \lim_{\Delta h \to 0} \sum_{0}^{100} A.\Delta h$$
$$= \int_0^{100} (400 - 2h)\, dh$$
$$= \left[400h - h^2\right]_0^{100}$$
$$= \left(400 \times 100 - 100^2\right) - (0 - 0)$$
$$= 30000$$

Volume $= 30000\ \text{cm}^3$

4(b) The total surface area, A cm^2, of a cylinder is given by $A = 2\pi r(r+h)$, where r cm is the radius of the base and h cm is the height. Find the rate at which the surface area is changing if the radius of the base is increasing at 0.2 cm/minute and the height is decreasing at 0.5 cm/minute when the base radius equals 16 cm and the height equals 20 cm. 3

SOLUTION:

$$A = 2\pi\left(r^2 + rh\right)$$
$$\frac{dA}{dt} = 2\pi\left\{\frac{d}{dt}\left(r^2\right) + \frac{d}{dt}(rh)\right\}$$
$$\frac{dA}{dt} = 2\pi\left\{\frac{d}{dr}\left(r^2\right)\frac{dr}{dt} + r\frac{d}{dt}(h) + h\frac{d}{dt}(r)\right\}$$
$$\frac{dA}{dt} = 2\pi\left\{2r\frac{dr}{dt} + r\frac{dh}{dt} + h\frac{dr}{dt}\right\}$$

$when\ \dfrac{dr}{dt} = 0.2,\ \dfrac{dh}{dt} = -0.5,\ r = 16,\ h = 20$

$$\frac{dA}{dt} = 2\pi\{2 \times 16 \times 0.2 + 16 \times -0.5 + 20 \times 0.2\}$$
$$= 4.8\pi$$

Surface area is increasing at $4.8\pi\ cm^2$ / min

4(c) (i) If $y = x^k + (c-x)^k$, where $c > 0$, $k > 0$, $k \neq 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $0 < k < 1$ and a minimum if $k > 1$. 3

SOLUTION:

$y = x^k + (c-x)^k$

$y' = kx^{k-1} - k(c-x)^{k-1}$

when $y' = 0$

$kx^{k-1} = k(c-x)^{k-1}$

$x^{k-1} = (c-x)^{k-1}$

$x = c - x$

$x = \frac{1}{2}c$

$y'' = k(k-1)x^{k-2} + k(k-1)(c-x)^{k-2}$

when $x = \frac{1}{2}c$

$y'' = k(k-1)\left\{\left(\frac{c}{2}\right)^{k-2} + \left(c - \frac{c}{2}\right)^{k-2}\right\}$

$= 2k(k-1)\left(\frac{c}{2}\right)^{k-2}$

If $0 < k < 1 \Rightarrow k(k-1) < 0 \Rightarrow y'' < 0$, $\therefore$ local max. t.p.

If $k > 1 \Rightarrow k(k-1) > 0 \Rightarrow y'' > 0$, $\therefore$ local min. t.p.

4(c) (ii) Hence show that if $a > 0$, $b > 0$, $a \neq b$, then $\frac{a^k + b^k}{2} < \left(\frac{a+b}{2}\right)^k$ if $0 < k < 1$ and $\frac{a^k + b^k}{2} > \left(\frac{a+b}{2}\right)^k$ if $k > 1$. 4

SOLUTION:

when $x = \frac{1}{2}c$

$y = \left(\frac{1}{2}c\right)^k + \left(c - \frac{1}{2}c\right)^k$

$= 2\left(\frac{c}{2}\right)^k$

if $0 < k < 1$, stat. pt. is a local max.

$\therefore x^k + (c-x)^k < 2\left(\frac{c}{2}\right)^k$

$\therefore \frac{x^k + (c-x)^k}{2} < \left(\frac{c}{2}\right)^k$

let $c = a + b$ and $x = a$

$\frac{a^k + (a+b-a)^k}{2} < \left(\frac{a+b}{2}\right)^k$

$\therefore \frac{a^k + b^k}{2} < \left(\frac{a+b}{2}\right)^k$

if $k > 1$, stat. pt. is a local min.

$\therefore x^k + (c-x)^k > 2\left(\frac{c}{2}\right)^k$

$\therefore \frac{x^k + (c-x)^k}{2} > \left(\frac{c}{2}\right)^k$

let $c = a + b$ and $x = a$

$\frac{a^k + (a+b-a)^k}{2} > \left(\frac{a+b}{2}\right)^k$

$\therefore \frac{a^k + b^k}{2} > \left(\frac{a+b}{2}\right)^k$

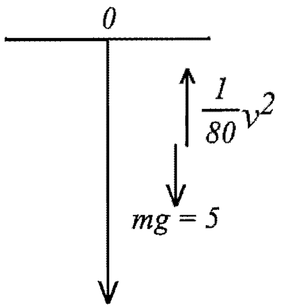
uestion 5 (15 Marks) START A NEW PAGE Marks

A toy soldier of mass 500 grams has a parachute attached to it. The toy soldier is released from rest at a position 60 m above the ground. During its fall the forces acting on the toy are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{80}v^2$ when the velocity of the toy is $v\ m/s$. After $5\ln 2$ seconds the parachute fails and from then on the only force acting on the toy is gravity. The acceleration due to gravity is taken as $10\ ms^{-2}$ and at time t seconds the toy has fallen $x\ m$ from its release point and its velocity is $v\ m/s$.

(a) (i) With the aid of a force diagram show that while the parachute is operating, $\ddot{x} = 10 - \frac{1}{40}v^2$. 1

SOLUTION:

$$m\ddot{x} = force\downarrow - force\uparrow$$
$$\frac{1}{2}\ddot{x} = 5 - \frac{1}{80}v^2$$
$$\ddot{x} = 10 - \frac{1}{40}v^2$$



(a) (ii) Show that $v = 20\left(\frac{e^t - 1}{e^t + 1}\right)$. 4

SOLUTION:

$$\ddot{x} = 10 - \frac{1}{40}v^2$$
$$\frac{dv}{dt} = \frac{400 - v^2}{40}$$
$$\frac{40}{400 - v^2}dv = dt$$
$$t = \int_0^v \frac{40}{400 - v^2}dv$$
$$= \int_0^v \left\{\frac{1}{20 - v} + \frac{1}{20 + v}\right\}dv$$
$$= \left[\ln(20 + v) - \ln(20 - v)\right]_0^v$$
$$= \ln(20 + v) - \ln(20 - v) - 0$$
$$t = \ln\left(\frac{20 + v}{20 - v}\right)$$

$$t = \ln\left(\frac{20 + v}{20 - v}\right)$$
$$\left(\frac{20 + v}{20 - v}\right) = e^t$$
$$20 + v = 20e^t - ve^t$$
$$v(e^t + 1) = 20(e^t - 1)$$
$$v = \frac{20(e^t - 1)}{(e^t + 1)}$$

5(a) (iii) Show that $x = -20\ln\left[1 - \left(\frac{v}{20}\right)^2\right]$. 4

SOLUTION:

$$v\frac{dv}{dx} = \frac{400 - v^2}{40}$$
$$dx = \frac{40v}{400 - v^2}dv$$
$$x = \int_0^v \frac{40v}{400 - v^2}dv$$
$$= \left[-20\ln(400 - v^2)\right]_0^v$$
$$= -20\ln(400 - v^2) + 20\ln(400)$$
$$= -20\ln\left(\frac{400 - v^2}{400}\right)$$
$$x = -20\ln\left(1 - \left(\frac{v}{20}\right)^2\right)$$

5(a) (iv) Find the exact speed of the toy and the exact distance fallen at the instant the parachute fails. 3

SOLUTION:

when $t = 5\ln 2$
$$v = \frac{20(e^{5\ln 2} - 1)}{(e^{5\ln 2} + 1)}$$
$$= \frac{20(32 - 1)}{32 + 1}$$
$$= \frac{620}{33}$$
$$speed = \frac{620}{33}\ m/s$$

when $v=\dfrac{620}{33}$

$$x=-20\ln\left(1-\left(\frac{\frac{620}{33}}{20}\right)^2\right)$$

$$=-20\ln\left(\frac{128}{1089}\right)$$

$$x=20\ln\left(\frac{1089}{128}\right)$$

$$dist.=20\ln\left(\frac{1089}{128}\right)\ m$$

5(a) (v) After the parachute fails, find an expression for the acceleration $\ddot{x}$ and hence find the speed of the toy the instant it hits the ground. Give your answer correct to two significant figures. 3

SOLUTION:

$$\ddot{x}=10$$

$$v\frac{dv}{dx}=10$$

$$v\,dv=10\,dx$$

$$\frac{1}{2}v^2=10x+c$$

when $x=20\ln\left(\dfrac{1089}{128}\right), v=\dfrac{620}{33}$

$$\frac{1}{2}\left(\frac{620}{33}\right)^2=200\ln\left(\frac{1089}{128}\right)+c$$

$$c=\frac{1}{2}\left(\frac{620}{33}\right)^2-200\ln\left(\frac{1089}{128}\right)$$

$$v^2=20x+\left(\frac{620}{33}\right)^2-400\ln\left(\frac{1089}{128}\right)$$

when $x=60$

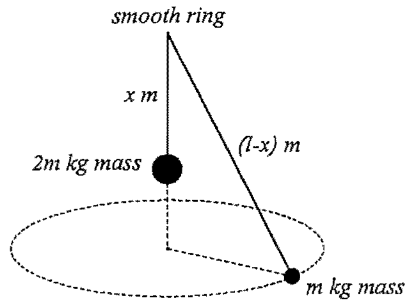
$$v^2=20(60)+\left(\frac{620}{33}\right)^2-400\ln\left(\frac{1089}{128}\right)$$

$$v\approx 26.393$$

speed $=26\,m/s$ (to 2 sig. fig.)

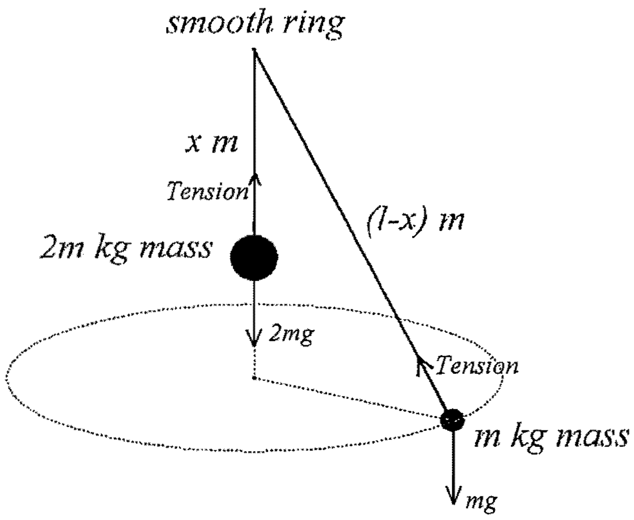
Question 6 (15 Marks) START A NEW PAGE Marks

6(a) Two particles of masses $2m$ kg and m kg respectively are connected to a light inextensible string of length l m. The string passes through a fixed smooth ring. While the $2m$ kg mass hangs in equilibrium a distance x m below the ring, the other mass describes a horizontal circle with angular velocity ω rad/sec. (Take the acceleration due to gravity as g m/s^2)



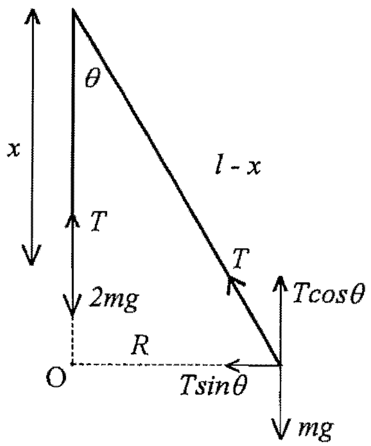
6(a) (i) Draw a diagram showing the forces acting on each mass. 2

SOLUTION:



(a) (ii) Find the angle that the string makes with the vertical. 3

SOLUTION:



at $2m$ mass

$T = 2mg$ ...... (1)

at $1m$ mass

$T\cos\theta = mg$ ...... (2)

$T\sin\theta = mR\omega^2$ ...... (3)

from (1) $T = 2mg$

sub. into (2)

$2mg\cos\theta = mg$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}$

(a) (iii) Prove that the angular velocity of the smaller mass equals $\sqrt{\dfrac{2g}{l-x}}$ rad/sec. 2

SOLUTION:

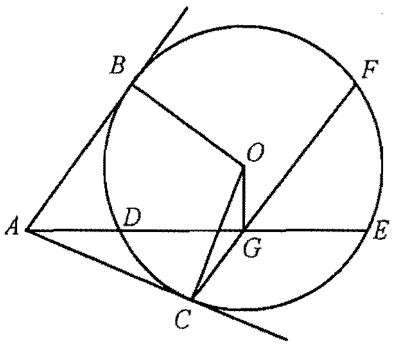
from(3)

$\omega^2 = \sqrt{\dfrac{T\sin\theta}{mR}}$

$\omega = \sqrt{\dfrac{2mg.\dfrac{R}{l-x}}{mR}}$

$\therefore$ angular velocity $= \sqrt{\dfrac{2g}{l-x}}$ *rad* / sec

(b) In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.



6(b) (i) Copy the diagram onto your answer sheet and prove that $ABOC$ and $AOGC$ are cyclic quadrilaterals. 2

SOLUTION:

$\angle ABO = \angle ACO = 90°$

$\therefore \angle ABO + \angle ACO = 180°$

$ABOC$ is a cyclic quad (one pair of opposite angles are supplementary)

$\angle AGO = 90°$(radius that bisects a chord is perpendicular to the chord)

$\angle AGO = \angle ACO = 90°$

$AOGC$ is a cyclic quad. (interval AO subtends equal angles at vertex G and C)

6(b) (ii) Explain why $\angle OGF = \angle OAC$. 1

SOLUTION:

$\angle OGF = \angle OAC$ (exterior angle of cyclic quad. equals opposite interior angle)

6(b) (iii) Prove that $BF \parallel ADE$. 5

SOLUTION:

Let $\angle OGF = \theta$

$\therefore \angle OAC = \theta$ (from part (ii))

$\Delta ABO \equiv \Delta ACO$ (SSS)

$\therefore \angle BAO = \theta$ (corresponding angles in congruent triangles are equal)

$\angle BOC = \pi - 2\theta$ (opposite angles of cyclic quad. $ABOC$ are supplementary)

$\angle BFG = \dfrac{\pi}{2} - \theta$ (angle at circumference is half angle at centre on same arc BC)

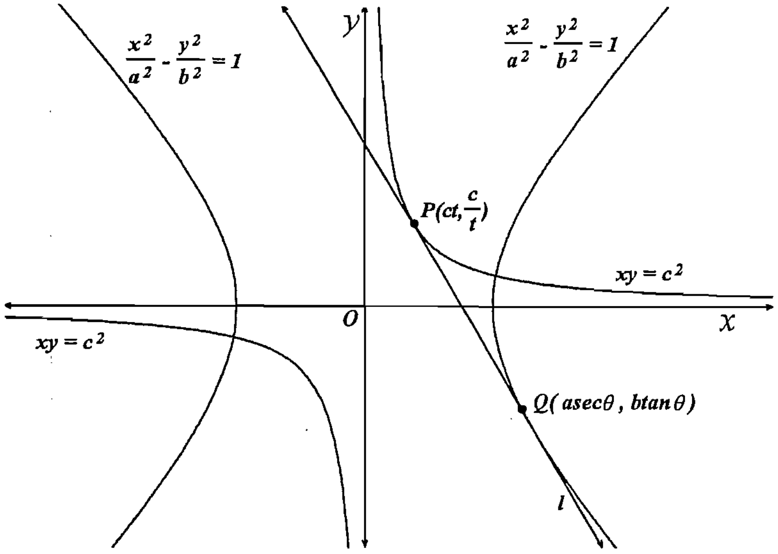
$\angle FGE + \dfrac{\pi}{2} + \theta = \pi$ (angle sum of straight angle $\angle AGE = \pi$)

$\angle FGE = \dfrac{\pi}{2} - \theta$

$\therefore \angle FGE = \angle BFG \left(\text{both} = \dfrac{\pi}{2} - \theta\right)$

$\therefore BF \parallel ADE$ (alternate angles are equal)

Question 7 (15 Marks) START A NEW PAGE Marks



The line l is a common tangent to the hyperbola $xy=c^2$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with points of contact P and Q respectively.

7(i) Considering l as a tangent to $xy=c^2$ at $P\left(ct,\frac{c}{t}\right)$, prove that l has the equation $x+t^2y=2ct$. 3

SOLUTION:

$$y=c^2x^{-1}$$
$$y'=-c^2x^{-2}$$
$$\text{at } P,\quad y'=\frac{-c^2}{(ct)^2}=-\frac{1}{t^2}$$
$$\text{tangent is}:\quad y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$$
$$t^2y-ct=-x+ct$$
$$x+t^2y=2ct$$

7(ii) Considering l as a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $Q(a\sec\theta, b\tan\theta)$, prove that l has the equation $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$. 3

SOLUTION:

$$\frac{d}{dx}\left(\frac{x^2}{a^2}\right)-\frac{d}{dx}\left(\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$
$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$
$$\text{at } Q,\quad \frac{dy}{dx}=\frac{b^2(a\sec\theta)}{a^2(b\tan\theta)}=\frac{b\sec\theta}{a\tan\theta}$$
$$\text{tangent is}:\quad y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$
$$(a\tan\theta)y-ab\tan^2\theta=(b\sec\theta)x-ab\sec^2\theta$$
$$(b\sec\theta)x-(a\tan\theta)y=ab(\sec^2\theta-\tan^2\theta)$$
$$(b\sec\theta)x-(a\tan\theta)y=ab\quad(\text{since }\sec^2\theta-\tan^2\theta=1)$$
$$\therefore\quad \frac{\sec\theta}{a}x-\frac{\tan\theta}{b}y=1$$

(iii) Deduce that $\dfrac{\sec\theta}{a} = \dfrac{-\tan\theta}{bt^2} = \dfrac{1}{2ct}$. 1

SOLUTION:

$y = c^2x^{-1}$

$y' = -c^2x^{-2}$

at P, $\quad y' = \dfrac{-c^2}{(ct)^2}$

$\quad = -\dfrac{1}{t^2}$

tangent is : $\quad x + t^2y = 2ct$

and also $\dfrac{\tan\theta}{b}x + \dfrac{\sec\theta}{a}y = 1$

on comparing ratios of coefficients and the constant

$$\frac{\left(\frac{\tan\theta}{b}\right)}{1} = \frac{\left(\frac{\sec\theta}{a}\right)}{t^2} = \frac{1}{2ct}$$

$$\Rightarrow \frac{\sec\theta}{a} = \frac{\tan\theta}{b} = \frac{1}{2ct}$$

7(iv) Write the coordinates of Q in terms of t, a, b and c, and show that $b^2t^4 + 4c^2t^2 - a^2 = 0$. 3
Deduce that there are exactly two such common tangents to the hyperbolas.

SOLUTION:

at Q, $x = a\sec\theta$

$$= a.\frac{a}{2ct}$$

$$= \frac{a^2}{2ct}$$

$$y = b\tan\theta$$

$$= b\left(\frac{-bt^2}{2ct}\right)$$

$$= -\frac{b^2t}{2c}$$

but Q lies on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$$\therefore \quad \frac{\left(\frac{a^2}{2ct}\right)^2}{a^2} - \frac{\left(-\frac{b^2t}{2c}\right)^2}{b^2} = 1$$

$$\frac{a^2}{4c^2t^2} - \frac{b^2t^2}{4c^2} = 1$$

$$b^2t^4 + 4c^2t^2 - a^2 = 0$$

$$t^2 = \frac{-4c^2 \pm \sqrt{16c^4 + 4a^2b^2}}{2b^2}$$

$$= \frac{-4c^2 \pm 2\sqrt{4c^4 + a^2b^2}}{2b^2}$$

$$= \frac{-2c^2 \pm \sqrt{4c^4 + a^2b^2}}{b^2}$$

but $t^2 > 0$

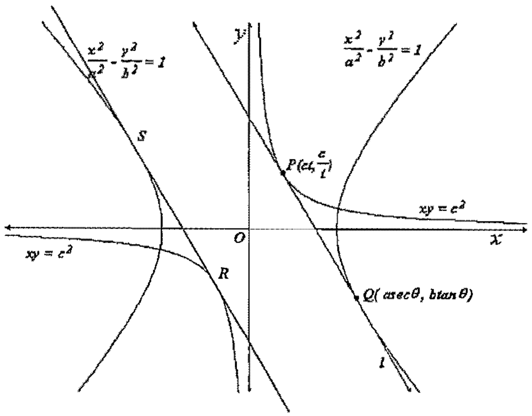
$$\therefore t^2 = \frac{-2c^2 + \sqrt{4c^4 + a^2b^2}}{b^2}$$

$$\therefore t = \pm\sqrt{\frac{-2c^2 + \sqrt{4c^4 + a^2b^2}}{b^2}}$$

$\therefore$ there are two values for t, so there exists 2 possible positions for the line ℓ

7(v) Copy the diagram onto your answer sheet, and using the symmetry of the graph, draw in the second common tangent with points of contact R on $xy=c^2$ and S on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Write down the coordinates of R and S in terms of t, a, b and c. 2

SOLUTION:



$$R\left(-ct,-\frac{c}{t}\right) \text{ and } S\left(-\frac{a}{2ct},\frac{b^2t}{2c}\right)$$

7(vi) Show that if $PQRS$ is a rhombus, then $b^2=a^2$. 3

SOLUTION:

If $PQRS$ is a rhombus then $PR \perp QS$

$$m(PR)=\frac{\frac{c}{t}+\frac{c}{t}}{2ct+2ct}$$
$$=\frac{1}{t^2}$$
$$m(SQ)=\frac{-\frac{b^2t}{2c}-\frac{b^2t}{2c}}{\frac{a^2}{2ct}+\frac{a^2}{2ct}}$$
$$=\frac{-b^2t^2}{a^2}$$
$PR \perp SQ \Rightarrow m(PR)\times m(SQ)=-1$
$$\therefore\ \frac{1}{t^2}\times\frac{-b^2t^2}{a^2}=-1$$
$$\frac{-b^2}{a^2}=-1$$
$\therefore b^2=a^2$

Question 8 (15 Marks) Marks

8(a) Six lines are drawn on a plane such that no two lines are parallel and no three lines are concurrent (i.e. pass through the same point).

8(a) (i) Show that there are 15 points of intersection. 1

SOLUTION:

No. points $={}^6C_2$
$=15$

8(a) (ii) If three of these points are chosen at random, show that the probability that they all lie on one of the given lines is $\frac{12}{91}$. 2

SOLUTION:

Sample space $={}^{15}C_3$
Favourable events $=6\times{}^5C_3$
$$\text{Prob.}=\frac{6\times{}^5C_3}{{}^{15}C_3}$$
$$=\frac{12}{91}$$

8(a) (iii) Find the probability that if three of these points are chosen at random then none of them lie on the same line. 2

SOLUTION:

Choices for 1st point = 15
Choices for 2nd point = 6
Choices for 3rd point = 1
$$\text{Prob.}=\frac{15\times6\times1}{{}^{15}C_3\times 3!}$$
$$=\cancel{\frac{18}{91}}\ \frac{3}{91}$$

8(b) (i) Prove that: 2

$(\alpha)\ \frac{{}^1C_0}{x}-\frac{{}^1C_1}{x+1}=\frac{1!}{x(x+1)}$.

$(\beta)\ \frac{{}^2C_0}{x}-\frac{{}^2C_1}{x+1}+\frac{{}^2C_2}{x+2}=\frac{2!}{x(x+1)(x+2)}$

SOLUTION:

$$(\alpha)\ \frac{{}^1C_0}{x}-\frac{{}^1C_1}{x+1}=\frac{1}{x}-\frac{1}{x+1}=\frac{x+1-x}{x(x+1)}=\frac{1}{x(x+1)}$$

$$(\beta)\ \frac{{}^2C_0}{x}-\frac{{}^2C_1}{x+1}+\frac{{}^2C_2}{x+2}=\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}$$
$$=\frac{1(x+1)(x+2)-2(x)(x+2)+(x)(x+1)}{x(x+1)(x+2)}$$
$$=\frac{(x^2+3x+2)-2(x^2+2x)+(x^2+x)}{x(x+1)(x+2)}$$
$$=\frac{x^2+3x+2-2x^2-4x+x^2+x}{x(x+1)(x+2)}$$
$$=\frac{2}{x(x+1)(x+2)}$$

8(b) (ii) Given $T(k,x)=\frac{k!}{x(x+1)(x+2)\dots(x+k)}$, prove that $T(k,x)-T(k,x+1)=T(k+1,x)$. 3

SOLUTION:

$$T(k+1,x)=\frac{(k+1)!}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$$

$$T(k,x)-T(k,x+1)=\frac{k!}{x(x+1)(x+2)\dots(x+k)}-\frac{k!}{(x+1)(x+2)\dots(x+k)(x+1+k)}$$
$$=\frac{k!(x+k+1)-k!\,x}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$$
$$=\frac{k!(x+k+1-x)}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$$
$$=\frac{k!(k+1)}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$$
$$=\frac{(k+1)!}{x(x+1)(x+2)\dots(x+k)(x+k+1)}$$
$$=T(k+1,x)$$

8(b) (iii) Hence prove, using Mathematical Induction or otherwise, that for $n\ge 1$: 5

$$\frac{{}^nC_0}{x}-\frac{{}^nC_1}{x+1}+\frac{{}^nC_2}{x+2}-\frac{{}^nC_3}{x+3}+\dots+(-1)^n\frac{{}^nC_n}{x+n}=\frac{n!}{x(x+1)(x+2)(x+3)\dots(x+n)}.$$

[You may use the result : ${}^{k+1}C_r={}^kC_r+{}^kC_{r-1}$]

SOLUTION:

True for $n=1,2$ by (i)

Assume true for $n=k$, $(n,k\in Z^+)$

$$\frac{{}^kC_0}{x}-\frac{{}^kC_1}{x+1}+\frac{{}^kC_2}{x+2}-\frac{{}^kC_3}{x+3}+\dots+(-1)^k\frac{{}^kC_k}{x+k}=\frac{k!}{x(x+1)(x+2)(x+3)\dots(x+k)}$$

To prove true for $n=k+1$, $(n,k\in Z^+)$

$$\frac{{}^{k+1}C_0}{x}-\frac{{}^{k+1}C_1}{x+1}+\frac{{}^{k+1}C_2}{x+2}-\frac{{}^{k+1}C_3}{x+3}+\dots+(-1)^k\frac{{}^kC_k}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$
$$=\frac{(k+1)!}{x(x+1)(x+2)(x+3)\dots(x+k)(x+k+1)}$$

$$LHS=\frac{{}^{k+1}C_0}{x}-\frac{{}^{k+1}C_1}{x+1}+\frac{{}^{k+1}C_2}{x+2}-\frac{{}^{k+1}C_3}{x+3}+\dots+(-1)^k\frac{{}^kC_k}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$
$$=\frac{{}^{k+1}C_0}{x}-\frac{({}^kC_1+{}^kC_0)}{x+1}+\frac{({}^kC_2+{}^kC_1)}{x+2}-\frac{({}^kC_3+{}^kC_2)}{x+3}+\dots$$
$$+(-1)^k\frac{({}^kC_k+{}^kC_{k-1})}{x+k}+(-1)^{k+1}\frac{{}^{k+1}C_{k+1}}{x+k+1}$$
$$=\left\{\frac{{}^kC_0}{x}-\frac{{}^kC_1}{x+1}+\frac{{}^kC_2}{x+2}-\frac{{}^kC_3}{x+3}+\dots+(-1)^k\frac{{}^kC_k}{x+k}\right\}$$
$$-\left\{\frac{{}^kC_0}{x+1}-\frac{{}^kC_1}{x+2}+\frac{{}^kC_2}{x+3}+\dots+(-1)^{k-1}\frac{{}^kC_{k-1}}{x+k}+(-1)^k\frac{{}^kC_k}{x+k+1}\right\}$$
(since ${}^{k+1}C_0={}^kC_0$ and ${}^{k+1}C_{k+1}={}^kC_k$)

$$=\frac{k!}{x(x+1)(x+2)\dots(x+k)}-\frac{k!}{(x+1)(x+2)\dots(x+k)(x+k+1)}$$
(by assumption for $x=x$ and $x=x+1$)

$$=\frac{(k+1)!}{x(x+1)(x+2)\dots(x+k)}\quad \text{(by (ii))}$$
$$=RHS$$

$\therefore$ true by the Principle of Mathematical Induction

THIS IS THE END OF THE EXAMINATION PAPER