JRAHS M. Ext 2 Trial, 2010

Question One (Start a new page)

Marks

a) Consider the complex numbers $z_1 = \sqrt{2}\,(1 + i\sqrt{3})$ and $z_2 = 2\sqrt{6}\,(1+i)$.

i. Express $z = \dfrac{z_1}{z_2}$ in the form of $x + iy$, where x and y are real. 2

ii. Write z_1, z_2 and z in modulus/ argument form. 5

iii. Hence find the exact value of $\cos\dfrac{\pi}{12}$. 1

b) Sketch on separate Argand diagrams the regions where

i. $\text{Re}\,(z + iz) \geq 2$ 2

ii. $1 \leq |z - 1 - i| \leq 3$ where $z = x + iy$. 2

c) By applying De Moivre's Theorem and by expanding $(\cos\theta + i\sin\theta)^5$, express $\sin 5\theta$ as a polynomial in $\sin\theta$. 3

Question Two (Start a new page)

a) Given real positive numbers a, b and c such that $a > b > c$.

i. Prove that $(a + b) > 2\sqrt{ab}$. 1

ii. Show that $b^2 - a^2 < 2(b-a)\sqrt{ab}$. 1

iii. Deduce that $(b-a)\sqrt{a} + (c-b)\sqrt{c} > \dfrac{c^2 - a^2}{2\sqrt{b}}$. 2

b) i. Sketch the graph of $f(x) = 1 - \dfrac{9}{x^2} + \dfrac{18}{x^4}$, showing all stationary points and other essential features. 5

ii. Hence find the set of values of the real numbers k such that the equations $f(x) = k$ has four distinct real roots. 1

c) An object of mass m kg is travelling around a circular banked track of radius r metres and angle of banking θ. The mass is travelling at $v\,ms^{-1}$. The forces acting on the object are the gravitational force mg newtons, a sideways friction force F newtons (acting down the road as shown) and a normal reaction N newtons to the road.

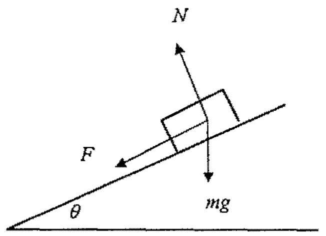


Diagram not to scale

i. By resolving forces vertically and horizontally, derive expressions for N and F. 4

ii. Given the radius of the curve is 1 km and $\tan\theta = \dfrac{1}{100}$, find the velocity which will ensure no sideways friction. (Take $g = 10ms^{-1}$) 1

Question Three (Start a new page)

a) Evaluate $\displaystyle\int_0^4 \frac{dx}{3 + \sqrt{x}}$. 3

b) Let α, β and δ be the roots of $x^3 - x^2 + 2x - 1 = 0$.

i. Find the value of $\alpha + \beta + \delta$. 1

ii. Hence, or otherwise, find the cubic equation with roots: $-(\alpha+\beta), -(\beta+\delta)$ and $-(\alpha+\delta)$. 2

c) Consider the ellipse E with equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ and its auxillary circle C with equation $\dfrac{x^2}{16} + \dfrac{y^2}{16} = 1$.

A straight line l parallel to the y axis, intersects the x axis at N and the curves E and C at the points P and Q respectively.

Given that P and Q are both in the first quadrant and the coordinates of P on E are $(4\cos\theta, 3\sin\theta)$.

i. Sketch the curves E and C showing the above information. 1
ii. Write down the coordinates of N and Q in terms of θ. 2
iii. Derive the equation of the tangent to the curve E at the point P. 2
iv. Write down the equation of the tangent to the curve C at the point Q. 1
v. The tangents at P and Q intersect at a point R. Show that R lies on the x axis. 2
vi. Prove that $ON.OR$ is independent of the positions P and Q. 1

Question Four (Start a new page) Marks

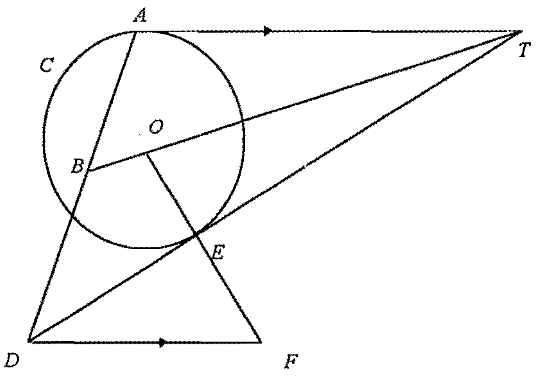
a) Using the Table of Standard Integrals, find $\int \frac{1}{\sqrt{x^2-4x+5}}\,dx$. 2

b) i. Find the constants A and B such that 2

$$\frac{1}{\cos x}=\frac{A\cos x}{1-\sin x}+\frac{B\cos x}{1+\sin x}.$$

ii. Hence find the exact value of $\int_0^{\frac{\pi}{6}} \sec x \ dx$. 2

c) *Diagram not to scale*



In the above diagram, C is a circle with exterior point T. Tangents from T are drawn to meet C at the points A and E. The point O is the centre of C. The line BT passes through O. The line AD passes through B. The line OF passes through E. AT is parallel to DF.

i. Trace or copy the diagram onto your answer book and prove $\Delta OET \equiv \Delta OAT$. 2

ii. Considering ΔOET and ΔDEF, show that $DE=\frac{DF(ET^2-OE^2)}{OT^2}$ by using double angle formula. 4

iii. Use the sine rule to show that $\frac{AB}{BD}=\frac{AT}{DT.}$ 3

Question Five (Start a new page) Marks

a) i. Prove $\int_0^a f(x)dx=\int_0^a f(a-x)dx$. 2

ii. Hence evaluate $\int_0^1 x(1-x)^n dx$, . 3

b) Let $w=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$.

i Find all the complex roots of the equation $z^{10}-1=0$ by writing them in terms of w and k, while k is a positive integer. 2

ii. Prove that $1+\ w+w^2+w^3+\cdots+w^9=0$. 1

iii. The quadratic equation $x^2+bx+c=0$, where b and c are real, has the root $w+w^4$. Find the other root in terms of w. 2

iv. Find b and express c in terms of $\sin\frac{\pi}{5}$. 5

Question Six (Start a new page)

a) Find $\int \frac{1}{e^x+e^{-x}}\,dx$. 3

b) Given that $I_n=\int_1^e (\ln y)^n dy,\ \ n\ =0,1,2,3,\cdots$

i. Prove that $I_n=e-nI_{n-1}$ 3

ii. Hence evaluate $\int_1^e\ (\ln y)^2\,dy$. 2

c) The depth of water at the entrance to a harbour can be modeled using the equation $x=b+a\cos nt$ where x metres is the depth of water and t is time measured in hours. For a certain harbour, the first low tide for the day is at 5am and the water depth is 20m. The next high tide is $6\frac{1}{2}$ hours later and the corresponding depth is 28m.

i. Taking the first low tide for the day as the origin for measuring the time, write down the values of a, b and n. 2

ii. Find the depth of water at 9am. (correct to 3 significant figures) 2

iii. Find all the times after mid-night and before mid-day when water depth is 23 m. (correct to nearest minute) 2

iv. Find the greatest rate at which the tide is rising. 1

Question Seven (Start a new page)

Marks

a) The circle $x^2 + y^2 = 9$ is rotated about the line $x = 8$ to form a torus. Using the method of cylindrical shells, find the volume of the torus. **4**

b) i. $xy = c^2$ is the result of rotating $x^2 - y^2 = a^2$ anticlockwise through an angle of 45^0. Write down the relationship between a^2 and c^2. **1**

ii. $P(x_1, y_1)$ is the point of intersection of the hyperbolas $xy = c^2$ and $x^2 - y^2 = a^2$ in the first quadrant.

Prove that the tangent to $xy = c^2$ at the point P is $xy_1 + yx_1 = 2c^2$. **2**

iii. Write down the equation of the tangent to $x^2 - y^2 = a^2$ at the point P. **1**

iv. The tangent to the hyperbola $x^2 - y^2 = a^2$ at P meets its asymptotes at A and C while the tangent to the hyperbola $xy = c^2$ at P meets its asymptotes in B and D.

v. Show that the co-ordinates of A are $(x_1 + y_1,\ x_1 + y_1)$. **2**

vi. Find the co-ordinates of B, C and D. **3**

vii. Prove that $ABCD$ is a square. **2**

Question Eight (Start a new page)

a) As shown in the diagram below, a light string of $2l$ metres long is attached to two points A and B. A mass of $3m$ kg is attached to the middle of the string and a second mass of m kg in the form of a ring is attached to the end of the string at B.

The $3m$ kg mass is rotating in circular motion at ω radians per second and the m kg mass is free to move up or down the smooth vertical rod AB. The string makes an angle θ with the vertical. (Assume the acceleration due to gravity is $g\ ms^{-2}$).

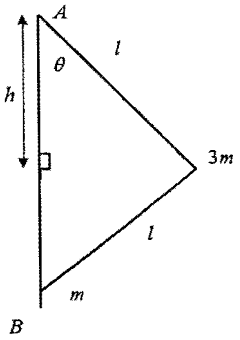


Diagram not to scale

i. Given that h is the distance between A and the centre of the circular motion, find an expression for h in terms of g and ω. **4**

ii. If the $3m$ kg mass is replaced by a mass of m kg mass and the m kg ring is replaced by a ring of $3m$ kg, the speed of the rotating mass is doubled to 2ω radians per second. Determine if h is increased or deceased and give reasons.(note that $\omega > 1$) **3**

b) Mr Dud's crystal ball rests on a solid stand which is in the shape of a square based frustum as shown.

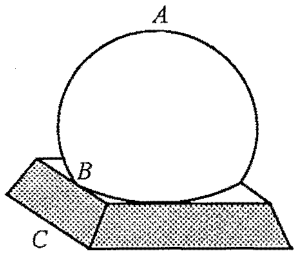
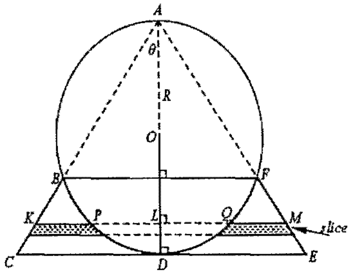


Diagram not to scale



cross-section

The stand is constructed so that the crystal ball of radius R fits snugly inside and just touches the centre of the square base. The side BC of the base slopes so that if extended it would pass through the top-most point of the ball at A and makes an angle θ with the vertical AD. Take O as the centre of the circle and let the distance OL be x units.

i. Explain why $LQ = \sqrt{R^2 - x^2}$ and $LM = (R + x)\tan\theta$. **2**

ii. Consider a slice KLM of thickness Δx as shown perpendicular to AD. **2**

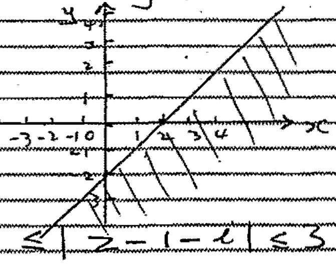
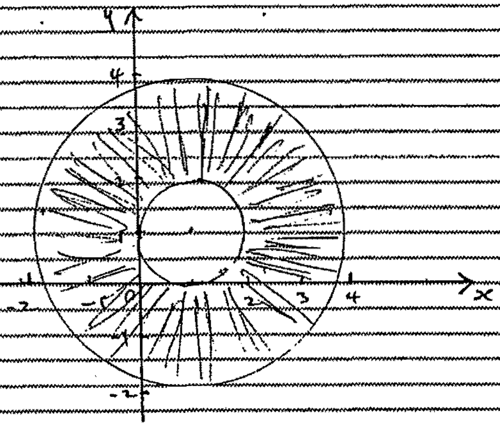
Show that it has a volume $\Delta V \approx \left\{4\tan^2\theta(R+x)^2 - \pi(R^2 - x^2)\right\}\Delta x$.

iii. Find the volume of such a solid when the angle $\theta = \frac{\pi}{6.}$ **4**

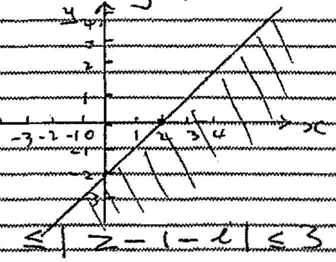
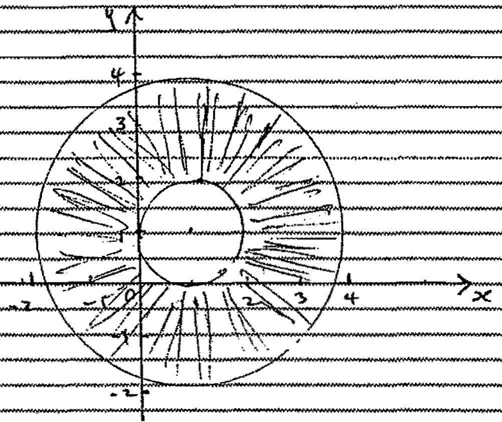
END

JRAHS M.EXT 2 TRIAL SOLNS 2010

MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
b) (i) $\text{Re}(z+iz) \geq 2$. $\text{Re}(x+iy+i(x+iy)) \geq 2$. $\therefore x-y \geq 2$ 	②	Graphs must be to scale. ① inequality $\frac{1}{2}$ line $x-y=2$ $\frac{1}{2}$ shading
(ii) $1 \leq \lvert z-1-i \rvert \leq 3$ $1 \leq \lvert z-(1+i) \rvert \leq 3$. 	②	① inside circle and shading ① outside circle Marks deducted for graphs not to scale or poorly drawn
$(\cos\theta+i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre's Theorem) $= \cos^5\theta + i5\cos^4\theta\sin\theta + i^2 10\cos^3\theta\sin^2\theta + i^3 10\cos^2\theta\sin^3\theta + i^4 5\cos^1\theta\sin^4\theta + i^5\sin^5\theta$ Equate imaginary terms. $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ $= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$ $= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta$ $+ 10\sin^5\theta + \sin^5\theta$ $= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta -$ $10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$ $= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$	③	① ① ①

MATHEMATICS Extension 2: Question 1

Suggested Solutions	Marks	Marker's Comments
b) (i) $\text{Re}(z+iz) \geq 2$. $\text{Re}(x+iy+i(x+iy)) \geq 2$. $\therefore x-y \geq 2$ 	②	Graphs must be to scale. ① inequality $\frac{1}{2}$ line $x-y=2$ $\frac{1}{2}$ shading
(ii) $1 \leq \lvert z-1-i \rvert \leq 3$ $1 \leq \lvert z-(1+i) \rvert \leq 3$. 	②	① inside circle and shading ① outside circle Marks deducted for graphs not to scale or poorly drawn
$(\cos\theta+i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre's Theorem) $= \cos^5\theta + i5\cos^4\theta\sin\theta + i^2 10\cos^3\theta\sin^2\theta + i^3 10\cos^2\theta\sin^3\theta + i^4 5\cos^1\theta\sin^4\theta + i^5\sin^5\theta$ Equate imaginary terms. $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ $= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$ $= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta$ $+ 10\sin^5\theta + \sin^5\theta$ $= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta -$ $10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$ $= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$	③	① ① ①

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Q2.

a) $a > b > c > 0 \Rightarrow \sqrt{a} > \sqrt{b} > \sqrt{c} > 0$

i) $(\sqrt{a}-\sqrt{b})^2 > 0$ (equality iff $a = b$)

$a+b-2\sqrt{ab} > 0$

$a+b > 2\sqrt{ab}$ # /

ii) $b < a \quad \therefore b-a < 0$

$(a+b)(b-a) < 2\sqrt{ab}(b-a)$

$b^2-a^2 < 2\sqrt{ab}(b-a)$ # ① /

iii) from ii since $c < b$

$\therefore c^2-b^2 < 2\sqrt{bc}(c-b)$ ② /

①+② $c^2-a^2 < 2\sqrt{ab}(b-a) + 2\sqrt{bc}(c-b)$

$c^2-a^2 < 2\sqrt{b}\left[\sqrt{a}(b-a)+\sqrt{c}(c-b)\right]$

$\therefore \sqrt{a}(b-a)+\sqrt{c}(c-b) > \frac{c^2-a^2}{2\sqrt{b}}$ # /

b i) $f(x) = 1 - \frac{9}{x^2} + \frac{18}{x^4} = \frac{x^4-9x^2+18}{x^4} = \frac{(x^2-6)(x^2-3)}{x^4}$

$f'(x) = \frac{9\times 2}{x^3} - \frac{72}{x^5} = 0$

$f(x) = 0$ when $x = \pm\sqrt{3}$, $\pm\sqrt{6}$

$\frac{18}{x^3} = \frac{72}{x^5} \quad \therefore x^2 = 4 \quad (x \neq 0)$

$x = \pm 2$ (S.P)

SP $x = 2,\ y = -\frac{1}{8}$ /

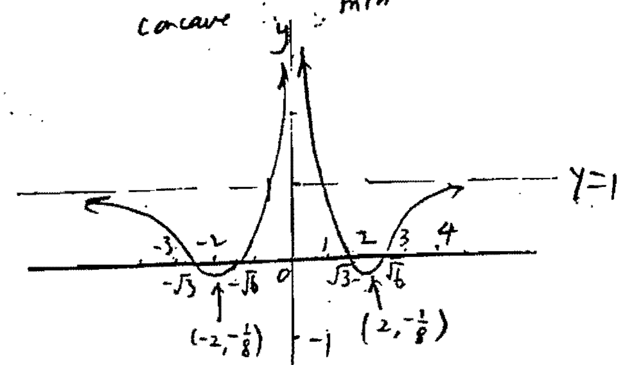
$x = -2,\ y = -\frac{1}{8}$

$f''(x) = \frac{-18\times 3}{x^4} + \frac{72\times 5}{x^6} = \frac{360}{x^6} - \frac{54}{x^4}$

/ $f''(\pm 2) = \frac{360}{64} - \frac{54}{16} = \frac{90-54}{16} = \frac{36}{16} > 0$ (+)

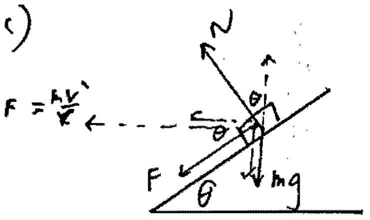
concave $\therefore$ min

2



/ when $-\frac{1}{8} < k < 1$ $f(x) = k$ has 4 distinct real roots

c)



vertically $N\cos\theta = F\sin\theta + mg$ ① /

Horizontally $N\sin\theta + F\cos\theta = \frac{mv^2}{r}$ ② /

①× $\sin\theta$ $N\cos\theta\sin\theta = F\sin^2\theta + mg\sin\theta$ ③

②× $\cos\theta$ $N\cos\theta\sin\theta = -F\cos^2\theta + \frac{mv^2}{r}\cos\theta$ ④

③ − ④ $0 = F + mg\sin\theta - \frac{mv^2}{r}$

$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$ # /

①× $\cos\theta$ $N\cos^2\theta = F\sin\theta\cos\theta + mg\cos\theta$ ⑤

②× $\sin\theta$ $N\sin^2\theta = -F\cos\theta\sin\theta + \frac{mv^2}{r}\sin\theta$ ⑥

⑤ + ⑥ $N = m\left[g\cos\theta + \frac{v^2}{r}\sin\theta\right]$ # /

since $\cos^2\theta + \sin^2\theta = 1$

when $F = 0$

$\frac{\cancel{m}v^2}{r}\cos\theta = \cancel{m}g\sin\theta$

$\frac{v^2}{\cancel{1000}r\times 10} = \tan\theta$ $(g=10)$

$\frac{v^2}{10000} = \frac{1}{100}$ $r = 1000$ m

$v^2 = \frac{10000}{100} = 100$ /

$v = 10$ m/s #

Question 3

a) $\int_0^4 \frac{dx}{3+\sqrt{x}} =$ $\sqrt{x} = u-3 \quad u > 3$

$\therefore x = (u-3)^2$

$= \int_3^5 \frac{2(u-3)\,du}{u}$ / $dx = 2(u-3)\,du$

$= \int_3^5 2\,du - \int_3^5 \frac{6}{u}\,du$ /

$= [2u]_3^5 - 6\ln u]_3^5$

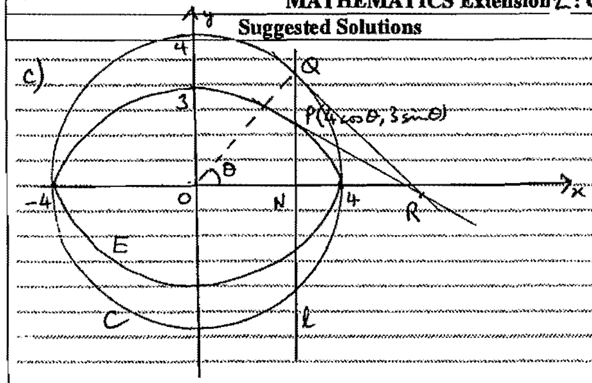
$= 2\times 2 - 6\ln\frac{5}{3}$ # /

MATHEMATICS Extension 2 : Question 3

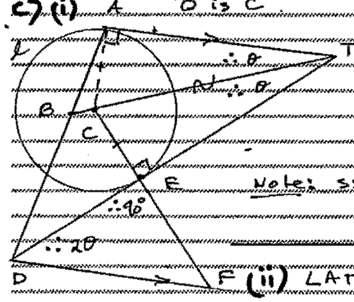
Suggested Solutions	Marks	Marker's Comments
a) If $I=\int_0^4 \frac{dx}{3+\sqrt{x}}$, substitute $u^2=x$ $(u>0)$		
"$2u\,du=dx$"		
When $x=4,\ u=2$	1	
$x=0,\ u=0$		
$\therefore I=\int_0^2 \frac{2u\,du}{3+u}$		
$=\int_0^2 \frac{2u+6-6}{u+3}\,du$		
$=\int_0^2 2-\frac{6}{u+3}\,du$	1	
$=\left[2u-6\ln(u+3)\right]_0^2$		
$=4-6\ln5+6\ln3$		
$\underline{\underline{=4-6\ln(5/3)}}$	1	
b) i) $\alpha+\beta+\gamma=$ "$-\frac{b}{a}$"		
$=\underline{\underline{1}}$	1	
ii) $-(\alpha+\beta)=\gamma-1$ (from (i))		
Roots are to be: $\alpha-1,\ \beta-1 + \gamma-1$		
ubstitute: $y=x-1 \Rightarrow x=y+1$	1	Too many used the long method and got lost in the algebra.
$(y+1)^3-(y+1)^2+2(y+1)-1=0$		
$y^3+3y^2+3y+1-y^2-2y-1+2y+2-1=0$		
$\underline{\underline{y^3+2y^2+3y-1=0}}$	1	

2010 TRIAL

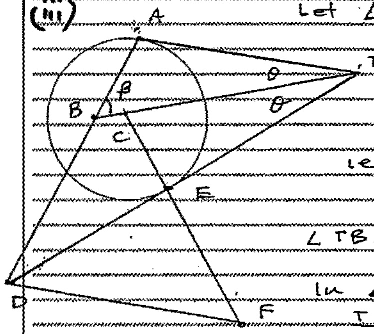
MATHEMATICS Extension 2 : Question 3 (cont)

Suggested Solutions	Marks	Marker's Comments
c) 	1	For mark, needed some defining points on E and C. Also N, Q and P(4cosθ, sinθ) needed to be marked (You should know where θ is but it was not required. Nor were the tangents for the first mark.)
ii) $\underline{N=(4\cos\theta,0)}$	1	
$\underline{\underline{Q=(4\cos\theta,\ 4\sin\theta)}}$	1	
iii) $\frac{dy}{d\theta}=3\cos\theta,\ \frac{dx}{d\theta}=-4\sin\theta$		
$\therefore \frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{-3\cos\theta}{4\sin\theta}$	1	
$\therefore y-3\sin\theta=\frac{-3\cos\theta}{4\sin\theta}(x-4\cos\theta)$		
$4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$		
$\underline{\underline{3x\cos\theta+4y\sin\theta=12}}$ Ⓐ $\left(\text{or } \frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1\right)$	1	Too many people derived tangent to the circle. Question said WRITE DOWN (for 1 mark only).
iv) $4x\cos\theta+4y\sin\theta=16$ $\left(\text{or } \frac{x\cos\theta}{4}+\frac{y\sin\theta}{4}=1\right)$	1	
$\underline{\underline{x\cos\theta+y\sin\theta=4}}$ Ⓑ		
v) Solve: Ⓐ-3B		
$y\sin\theta=0$		
$\therefore y=0$ $(0<\theta<\pi/2$ as P, Q in 1st Q) (so $\sin\theta\neq0$)		
$\therefore$ $\underline{\underline{\text{Crosses at } (4\sec\theta,0) \text{ on } x \text{ axis.}}}$		
vi) $ON.OR=4\cos\theta\ 4\sec\theta=16$	1	
This is $\underline{\text{independent of }\theta}$ and hence of P and Q.		

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments
a) $I = \int \frac{dx}{\sqrt{x^2-4x+5}} = \int \frac{dx}{\sqrt{1+(x-2)^2}}$, $(x>3)$ $= \ln\left[(x-2)+\sqrt{(x-2)^2+1}\right] + C$ $= \ln\left[(x-2)+\sqrt{x^2-4x+5}\right] + C$	$\frac{1}{2}$	$\boxed{2}$ $\frac{1}{2}$ For $(x-2)$ $\frac{1}{2}$ For $\ln[$ and $\sqrt{---}$ $\frac{1}{2}$ For C
b) (i) $\frac{1}{\cos x} = \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$ becomes $1 \equiv A(1+\sin x) + B(1-\sin x)$ Either use x-values or c/- coefficients $x=0, \pi \Rightarrow A+B=1$ $x=\frac{\pi}{2}$ $A-B=0$ $\therefore$ A and $B = \frac{1}{2}$		1 For correct method $\frac{1}{2}$ For each A, B, $\boxed{2}$
(ii) So $\int_0^{\pi/6} \sec x\, dx = \int_0^{\pi/6} \frac{\frac{1}{2}\cos x}{1-\sin x} + \frac{\frac{1}{2}\cos x}{1+\sin x}\, dx$ $= \frac{1}{2}\left[-\ln\lvert 1-\sin x\rvert + \ln\lvert 1+\sin x\rvert\right]_0^{\pi/6}$ $= \frac{1}{2}\left[-\ln(\frac{1}{2}) + \ln(1\frac{1}{2}) - (-0+0)\right]$ $= \frac{1}{2}\ln 3$ ✓	1	$\frac{1}{2}$ For $-\ln\lvert 1-\sin x\rvert$. $\frac{1}{2}$ For $\sin\frac{\pi}{6} = \frac{1}{2}$. $\ln(\frac{3}{2} \div \frac{1}{2})$ $\boxed{2}$
c) (i) O is C  In Δs CAT and CET R 1. $\angle CAT = \angle CET = 90°$ (Tangent to radius at point of contact is 90°) H 2. CT is common S 3. CA = CE equal radii $\therefore \Delta CAT \equiv \Delta CET$ (RHS) Note: SSS and SAS requires a derived result. Eg: TA = TE etc	$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\boxed{2}$
(ii) $\angle ATC = \angle ETC = \theta$ (Corresponding angles in congruent triangles equal) $\therefore \angle ATE = 2\theta$ $\therefore \angle TDF = 2\theta$ (Alternate angles equal as AT ∥ DF) ✓ $\angle DEF = 90°$ (Vertically opposite angles equal) ✓ from c(i) $\therefore \cos 2\theta = \frac{DE}{DF} = \cos^2\theta - \sin^2\theta$ $= \frac{ET^2}{CT^2} - \frac{CE^2}{CT^2}$ $\therefore DE = DF.\frac{(ET^2 - CE^2)}{CT^2}$ qed.	1 1 $\frac{1}{2}$	1 For $\cos\theta = \frac{ET}{CT}$ $\frac{1}{2}$ For $\cos 2\theta = c^2 - s^2$ $(\frac{1}{2})$ $\sin\theta = \frac{CE}{CT}$. $\boxed{4}$

MATHEMATICS Extension 2: Question 4

Suggested Solutions	Marks	Marker's Comments
(iii)  Let $\angle ABT = \beta$ In ΔABT: $\frac{AT}{\sin\beta} = \frac{AB}{\sin\theta}$ ie $\frac{AT}{AB} = \frac{\sin\beta}{\sin\theta}$ $\angle TBD = \pi - \beta$ (angle sum at B of straight angle is π) In ΔTBD $\frac{TD}{\sin(\pi-\beta)} = \frac{BD}{\sin\theta}$ $\frac{TD}{BD} = \frac{\sin(\pi-\beta)}{\sin\theta} \equiv \frac{\sin\beta}{\sin\theta}$ $\therefore \frac{\sin\beta}{\sin\theta} = \frac{AT}{AB} = \frac{TD}{BD}$ ie $\frac{TA}{TD} = \frac{AB}{BD}$ qed [this is the angle bisector theorem!]	1 $\frac{1}{2}$ $\frac{1}{2}$	$\boxed{3}$

MATH. EXT2 TRIAL SOLUTIONS 2010

MATHEMATICS Extension 2: Question 5

Suggested Solutions | **Marks** | **Marker's Comments**

(a)(i) $LHS = \int_0^a f(x)\,dx$ $u = a-x$

$\therefore x = a-u$

$dx = -du$ ($\frac{1}{2}$)

x	u
a	0
0	a

($\frac{1}{2}$)

$= \int_a^0 f(a-u) \times -du$

$= +\int_0^a f(a-u)\,du$ ($\frac{1}{2}$) ($\frac{1}{2}$)

cos u is a dummy variable

$\equiv \int_0^a f(a-x)\,dx$ qed.

$= RHS$

Can do RHS to LHS

or $\int_0^a f(x)\,dx = F(x)\Big]_0^a$

$\int_0^a f(a-x)\,dx = -\Big[F(a-x)\Big]_0^a$

$\boxed{2}$

(ii) $\int_0^1 x(1-x)^n\,dx = \int_0^1 (1-x)\left[1-(1-x)\right]^n\,dx$ using (i) with $a=1$

$\frac{1}{2}$ For showing how $(1-x)x^n$

$= \int_0^1 (1-x)x^n\,dx$

$= \int_0^1 x^n - x^{n+1}\,dx$ $\frac{1}{2}$

$= \left[\frac{1}{n+1}x^{n+1} - \frac{1}{n+2}x^{n+2}\right]_0^1$ 1

$= \frac{1}{n+1} - \frac{1}{n+2} - 0$ 1

$= \frac{1}{(n+1)(n+2)}$

NisE $\boxed{3}$

(b)(i) $z^{10} = 1$

$(r\,\text{cis}\,\theta)^{10} = 1\,\text{cis}\,0 = 1\,\text{cis}(0+2k\pi)$ $k = 0, 1, 2, \ldots, 8, 9$

$r^{10}\,\text{cis}\,10\theta = 1\,\text{cis}(2k\pi)$

$r = |z| = 1$, $\theta_k = \frac{k\pi}{5}$

OR $z = 1$ by inspection and roots separated by $\frac{2\pi}{10}$ ie $\frac{\pi}{5}$

and as $w = \text{cis}\frac{\pi}{5}$

$\therefore z_k = 1\,\text{cis}\,k\frac{\pi}{5} \equiv w^k$ (De Moivres thm)

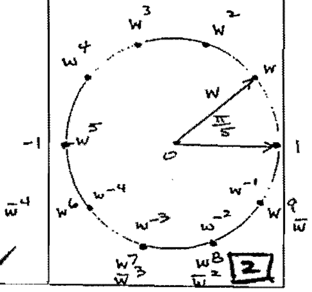
$\frac{1}{2}$ For $\frac{\pi}{5}$

$\frac{1}{2}$ For "$r=1$"

1 For w^k form $k = 1,2,3,4,6,7,8,9$

k	z_k	w^k	Note 1	Note 2
0	1	$w^0 = 1$	1	1
1	$1\,\text{cis}\,\pi/5$	w	w	w
2	$1\,\text{cis}\,2\pi/5$	w^2	w^2	w^2
3	$1\,\text{cis}\frac{3\pi}{5}$	w^3	w^3	w^3
4	$1\,\text{cis}\frac{4\pi}{5}$	w^4	w^4	w^4
5	$1\,\text{cis}\frac{5\pi}{5}$	$w^5 = -1$	-1	-1
6	$1\,\text{cis}\,6\pi/5$	w^6	w^{-4}	$\bar{w}^4$
7	$1\,\text{cis}\,7\pi/5$	w^7	w^{-3}	$\bar{w}^3$
8	$1\,\text{cis}\,8\pi/5$	w^8	w^{-2}	$\bar{w}^2$
9	$1\,\text{cis}\,9\pi/5$	w^9	w^{-1}	$\bar{w}$

$\therefore$ complex roots of $z^{10}-1=0$ are $w, w^2, w^3, w^4, w^6, w^7, w^8$ and w^9 ✓



$\boxed{2}$

TRIAL SOLUTIONS 2010

MATHEMATICS Extension 2: Question 5

Suggested Solutions | **Marks** | **Marker's Comments**

b) (ii) Method I $1+w+w^2+\ldots+w^8+w^9 = \frac{w^{10}-1}{w-1}$

$= \frac{1-1}{w-1}$

$= \frac{0}{w-1} = 0$ as $w = 1\,\text{cis}\frac{\pi}{5} \neq 1$

Geom. Series $a=1$, $r=w$

$\therefore w^{10} = 1$.

$*\frac{1}{2}$ for reason$*$

II $1+w+w^2+\ldots+w^9$ is the sum of the roots of $z^{10}-1=0$

$\therefore$ sum $= s_1 = -\frac{b}{a} = \frac{0}{1} = 0$

III $z^{10}-1 = (z-1)(1+z+z^2+\ldots+z^9) = 0$

w is a root, where $w \neq 1$

$\therefore w$ satisfies factor $1+z+z^2+\ldots+z^9$

$\therefore 1+w+w^2+\ldots+w^9 = 0$

IV write each in cis form, rearrange, use trig results and eventually get 0 ☹

V Do VECTOR ADDITIONS to show sum goes to 0. ☹

$\boxed{1}$

(iii) As the coefficients 1, b and c are real any complex roots will be in conjugate pairs 1 Can prove if wish

$\therefore$ Let $\alpha = w+w^4$ which is complex 1

$\therefore \beta = \bar{\alpha} = \bar{w}+\bar{w}^4 = \overline{w}+\overline{w^4} = w^9+w^6 = w^{-1}+w^{-4} = \ldots$

OR $\alpha = w+w^4 = \cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$

$= \cos\frac{\pi}{5}+i\sin\frac{\pi}{5}+-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$

$\cos\frac{4\pi}{5} = \cos(\pi-\frac{\pi}{5}) = -\cos\frac{\pi}{5}$

$= 2i\sin\frac{\pi}{5}$ ☺

$\therefore \beta = \bar{\alpha} = -2i\sin\frac{\pi}{5}$ ✓

$\boxed{2}$

i) Many ways !!! $x^2+bx+c=0$

$s_1 = \alpha+\beta = \alpha+\bar{\alpha} = -b$

$*$ $2i\sin\frac{\pi}{5} + -2i\sin\frac{\pi}{5} = -b$

$\therefore b = 0$

$b = -[w+w^4+w^9+w^6]$

$= -[w^4(w^5+1)+w(w^5+1)]$

$= 0$ as $w^5 = -1$

②

✓ $b = -[w+w^{-1}+w^4+w^{-4}]$

$= -2[\cos\frac{\pi}{5}+\cos\frac{4\pi}{5}]$

$= -2[\cos\frac{\pi}{5}-\cos\frac{\pi}{5}]$ $*\frac{1}{2}$

$= 0$

$*$ Very useful facts: $\alpha+\bar{\alpha} = 2\text{Re}(\alpha) = 2\cos(\ldots)$

$w^k+\bar{w}^k = w^k+w^{-k} = 2\cos(\ldots)$

$s_2 = \alpha\beta = \alpha\bar{\alpha} = c$

$\alpha\bar{\alpha} = 2i\sin\frac{\pi}{5} \times -2i\sin\frac{\pi}{5}$

$c = 4\sin^2\frac{\pi}{5}$ ///

$\alpha\bar{\alpha} = (w+w^4)(w^9+w^6)$, $(w+w^4)(\bar{w}+\bar{w}^4)$, $(w+w^4)(w^{-1}+w^{-4})$

$|w+w^4|^2 = |2i\sin\frac{\pi}{5}|^2$

$= 2+w^3+w^7$ / $2+w^3+w^{-3}$ // $2+w^3+w^{-3}$ 1

$= 2+2\text{Re}(w^3)$

$= 2+2\cos\frac{3\pi}{5}$ $\frac{1}{2}$ ③

$= 2-2\cos\frac{2\pi}{5}$

"$1-\cos 2\theta = 2\sin^2\theta$"

$= 2\left[1-\cos\frac{2\pi}{5}\right] \equiv$ $\frac{1}{2}$ $2.2\sin^2\frac{\pi}{5}$ 1

$= 4\sin^2\frac{\pi}{5}$

$\boxed{5}$

TRIAL 2010

MATHEMATICS Extension 2 : Question 6

Suggested Solutions	Marks	Marker's Comments
a) $I = \int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1}$ $(e^x>0)$	1	
Let $u=e^x$, $\therefore \frac{du}{dx}=e^x$, "$du=e^x dx$"		
$\therefore I = \int \frac{du}{u^2+1} = \tan^{-1}u + k$	1	
$= \underline{\underline{\tan^{-1}(e^x)+k}}$	1	
b) i) $I_n = \int_1^e 1(\ln y)^n\,dy$.		
$I_n = \left[y(\ln y)^n\right]_1^e - \int_1^e \frac{yn(\ln y)^{n-1}\,dy}{y}$	1	
$I_n = \left\{e(\ln e)^n - 1(\ln 1)^n\right\} - \int_1^e n(\ln y)^{n-1}\,dy$	1	Too many short cuts taken with substitution. The result is GIVEN and thus must be fully explained
$I_n = e - 0 - n\int_1^e (\ln y)^{n-1}\,dy$		
$\underline{\underline{I_n = e - nI_{n-1}}}$	1	
ii) $I_0 = \int_1^e 1\,dy = [y]_1^e = e-1$	1	
$I_2 = e-2I_1$ (from above)		
$= e-2(e-2I_0) = e-2(e-(e-1))$		
$= \underline{\underline{e-2}}$	1	
) i) This is SHM about $x=b$.		
Low tide 20 } Centre of Motion $x=24$ High tide 28 } $\therefore b=24$		
Period = 13 hrs = $\frac{2\pi}{n}$ $\therefore$ $n=\frac{2\pi}{13}$.		
Motion at low tide when $t=0$ $\therefore a=-4$		
$\therefore \underline{\underline{a=-4,\ b=24,\ n=2\pi/13}}$	½, ½, 1	

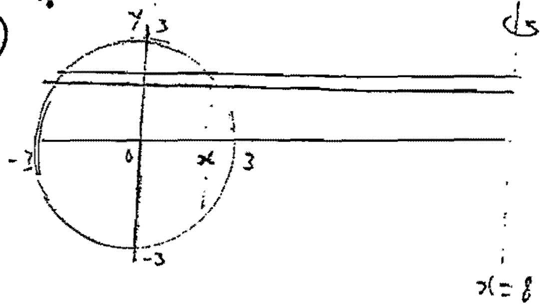
MATHEMATICS Extension 2 Question 6 (cont)

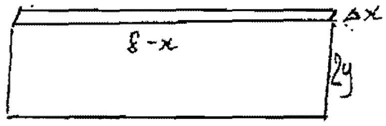
Suggested Solutions	Marks	Marker's Comments
c) ii) When $t=4$,		
$x = 24-4\cos\left(\frac{8\pi}{13}\right) = 25.4$ (3 SF)	1	Too many people stopped at 25.4.
$\therefore$ Depth is 25.4m at 9am	1	Need units and expressed answer.
iii) The day goes from $-5\le t\le 7$.		There is nothing in the question to assume tomorrow.
$\therefore x = 24-4\cos\frac{2\pi t}{13}$		
Substitute $x=23$ when $t=T$		
$23=24-4\cos\frac{2\pi T}{13}$		
$\therefore \cos\frac{2\pi T}{13}=\frac{1}{4}$		
$\frac{2\pi T}{13} = 2n\pi \pm \cos^{-1}(1/4)$	1	
$T = 13n \pm 2.7272$		
Only $n=0$ gives values in allowed range.		
$T = \pm 2.7272$ (hrs)		
$= \pm 2$ hrs 44m		
$\therefore$ Times are $\underline{\underline{2.16 \text{ am } \& \ 7.44 \text{ am}}}$	½ ½	
iv) $\frac{dx}{dt} = +\frac{8\pi}{13}\sin\frac{2\pi t}{13}$		
$\sin\left(\frac{2\pi t}{13}\right)$ has max value of 1.		
$\therefore$ Max rate of increase is $\underline{\underline{\frac{8\pi}{13} \text{ m/hr}}}$	1	

Question seven

Q7.

a)





Area of cross-section of shell

$= 2\pi r h = 2\pi(8-x)2y$

$\Delta V = 4\pi(8-x)y\,\Delta x$

$vol = 4\pi\int_{-3}^{3}(8-x)y\,dx$ $\frac{1}{2}$

$= 4\pi\int_{-3}^{3}(8-x)\sqrt{9-x^2}\,dx$ $\frac{1}{2}$

$= \pi 32\int_{-3}^{3}\sqrt{9-x^2}\,dx - 2\pi\int_{-3}^{3}2x\sqrt{9-x^2}\,dx$

$\quad \parallel$

Area of

$= \frac{32}{2}\pi^2 \cdot 3^2 + 2\pi\left[(9-x^2)^{3/2}\frac{2}{3}\right]_{-3}^{3}$ 1 1

$= 144\pi^2 + 0$

$= 144\pi^2\ u^3$ # 1

For B: $xy_1 + yx_1 = 2c^2$ at $x = 0$

$\therefore y = \frac{2c^2}{x_1}$

but $x_1y_1 = c^2$

$\therefore Y_i = 2y_1$

$\therefore B = (0, 2y_1)$ 1

Similarly for D: $D = (2x_1, 0)$ 1

METHODS to prove ABCD is a SQUARE worth 2 marks with a conclusion.

Question seven

bi) $c^2 = \frac{a^2}{2}$ 1

ii) $xy = c^2$

$xy' + y \cdot 1 = 0$

$y' = \frac{-y}{x}$

At (x_1, y_1) $y' = \frac{-y_1}{x_1}$

Eq of tangent at (x_1, y_1) is

$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$ 1

$x_1y - x_1y_1 = -xy_1 + x_1y_1$

$x_1y + xy_1 = 2x_1y_1$

Now (x_1, y_1) lies on $xy = c^2$

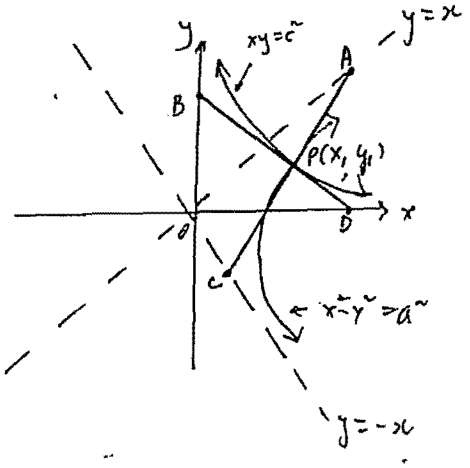
$\therefore x_1y_1 = c^2$

Eq tangent $x_1y + xy_1 = 2c^2$ # 1

iii) $x^2 - y^2 = a^2$

$\frac{xx_1}{a^2} - \frac{yy_1}{a^2} = 1$ # 1 or $xx_1 - yy_1 = a^2$ #

iv)



FOR A $xx_1 - yy_1 = a^2$ meet at $y = x$

$x(x_1 - y_1) = a^2$

1 But $x_1^2 - y_1^2 = a^2$ $\therefore (x_1+y_1)(x_1-y_1) = a^2$

$\therefore x = \frac{(x_1+y_1)\cancel{(x_1-y_1)}}{\cancel{x_1-y_1}} = x_1 + y_1$

$(x_1 \neq y_1)$

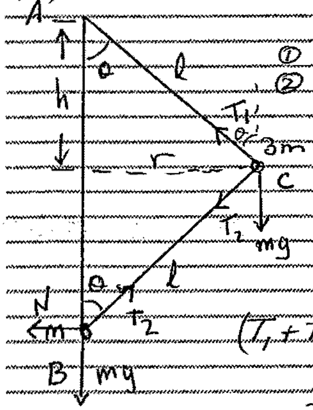
1 $\therefore A = (x_1+y_1, x_1+y_1)$

Similarly for C: $xx_1 - yy_1 = a^2$ meet at $y = -x$

$\therefore C = (x_1 - y_1, y_1 - x_1)$ 1

(a)

(i)

Suggested Solutions	Marks	Marker's Comments
	(4)	
Forces at C		
① $T_1\cos\theta = T_2\cos\theta + 3mg$		½
② $T_1\sin\theta + T_2\sin\theta = 3m\omega^2 r$		½
Forces at B		
③ $T_2\cos\theta = mg$		½
④ $T_2\sin\theta = N$ (not required)		
From ① & ③		
$T_1\cos\theta = mg + 3mg$		
$T_1\cos\theta = 4mg$		½
$(T_1+T_2)\sin\theta = 3m\omega^2 l\sin\theta$		
$T_1+T_2 = 3m\omega^2 l$		①
$\frac{4mg}{\cos\theta} + \frac{mg}{\cos\theta} = \frac{3m\omega^2 h}{\cos\theta}$		
$\therefore h_1 = \frac{5g}{3\omega^2}$		①
(ii) Reverse masses	(3)	
Forces at C $T_1\cos\theta = T_2\cos\theta + mg$ $T_1\sin\theta + T_2\sin\theta = m(2\omega)^2 r$		1½
Forces at B $T_2\cos\theta = 3mg$		
$T_1\cos\theta = 3mg + mg$		
$T_1 = \frac{4mg}{\cos\theta}$		
$T_1 + T_2 = 4M\omega^2 r$		
$\frac{4mg}{\cos\theta} + \frac{3mg}{\cos\theta} = 4m\omega^2\frac{h}{\cos\theta}$		1
$h_2 = \frac{7g}{4\omega^2}$		
$\therefore \frac{h_2}{h_1} = \frac{7g}{4\omega^2} \times \frac{3\omega^2}{5g} = \frac{21}{20}$		
$\therefore$ h increases		½

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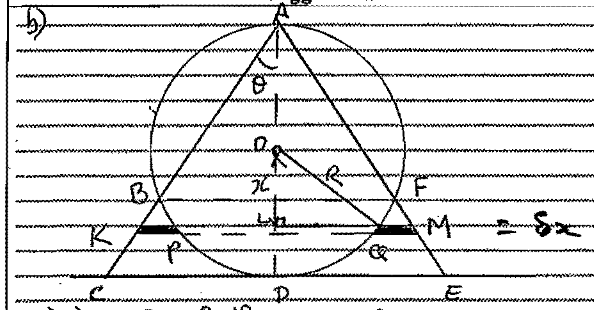
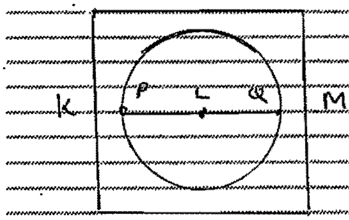
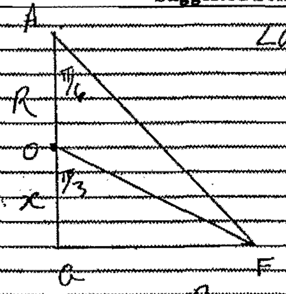
b)

Suggested Solutions	Marks	Marker's Comments
	(2)	
(i) By Pythagoras.		½ Pythagoras statement & $LQ>0$
$OQ^2 = OL^2 + LQ^2$		
$R^2 = x^2 + LQ^2$		½ Answer
$LQ^2 = R^2 - x^2$		
$LQ = \sqrt{R^2-x^2}$ $\quad LQ>0$		
(ii) By symmetry $\angle CAD = \angle EAD = \theta$		½ $\angle EAD = \theta$.
in Δ LAM		
$\tan\theta = \frac{LM}{AL}$ $\quad AL = x+R$		½ tan ratio
$\therefore LM = (R+x)\tan\theta$		
 Area of slice = Area of Square − Area of circle $= (KM)^2 - \pi(LQ)^2$ $= (2LM)^2 - \pi(LQ)^2$	(2)	½ ½ area.
$A = (2(R+x)\tan\theta)^2 - \pi(\sqrt{R^2-x^2})^2$		
$= 4(R+x)\tan\theta - \pi(R^2-x^2)$		
$\therefore \Delta V = A\Delta x.$		½ + ½ each part of answer.
$= [(4(R+x)\tan\theta) - \pi(R^2-x^2)]\Delta x$		
	PTO	

MATHEMATICS Extension 2: Question 8

Suggested Solutions	Marks	Marker's Comments
	④	
$\angle GOF = 2 \times \pi/6 = \pi/3$ $RG = R\cos \pi/3$ $= R/2$		
		$(\frac{1}{2}) + (\frac{1}{2})$ for limit values.
$\text{Volume} = \int_{R/2}^{R} 4\tan^2\theta(R+x)^2 - \pi(R^2-x^2)\,dx$		
$\tan\theta = \frac{1}{\sqrt{3}}$ when $\theta = \frac{\pi}{6}$		
$V = \int_{R/2}^{R} \left[\frac{4}{3}(R+x)^2 - \pi(R^2-x^2)\right]dx$		
$= \left[\frac{4}{3}\frac{(R+x)^3}{3} \quad \pi\left[R^2x - \frac{x^3}{3}\right]\right]_{R/2}^{R}$		① integration ① sub $R/2$.
$= \frac{4}{9}\left[(2R)^3\right] - \frac{4}{9}\left[\left(\frac{3R}{2}\right)^3\right] - \pi\left(\left[R^3 - \frac{R^3}{3}\right] - \left(\frac{R^3}{2} - \frac{R^3}{24}\right)\right]$		
$= \frac{4}{9}\left(8R^3 - \frac{27}{8}R^3\right) - \pi\left(\frac{2R^3}{3} - \frac{R^3}{2} + \frac{R^3}{24}\right)]$		
$= \frac{37}{18}R^3 - \frac{5}{24}\pi R^3$		Simplified ① answer.