Student Number:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2016

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.
- Board approved calculators & templates may be used
- A Reference Sheet is provided.
- In Questions 11-16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: **10 marks**

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: **90 Marks**

- Attempt Question 11-16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I
10 Marks
Attempt Questions 1 - 10. Mark your answers on the sheet provided.
Allow about 15 minutes for this section.

1. The complex equation $|z - 3 + 2i| = |z + 3|$ can be expressed in Cartesian form as:

(A) $y = 3x - 1$ (B) $y = x^2 + 1$ (C) $y = 3 - x$ (D) $y = 1 - 3x$

2. A particle moves along a straight line with an acceleration of $\frac{2}{v}$ m/s^2, where v m/s is the velocity at any instant. The initial velocity of the particle is -1 m/s.
The particle will move to the

(A) Left increasing in speed
(B) Left, stop, then move to the right
(C) Right increasing in speed
(D) Right, stop, then move to the left

3. Using an appropriate substitution, then $\int e^{2x}\sqrt{e^x - 1}\,dx$ is equivalent to:

(A) $\int(u^3 - u)du$ (B) $\int(u^3 + u)du$ (C) $\int\left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$ (D) $\int\left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$

4. When factorised, $x^2 + 6x + 1$ is equal to:

(A) $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$
(B) $(x - 3 - 2\sqrt{2})(x - 3 - 2\sqrt{2})$
(C) $(x + 3 + 2\sqrt{2}\,i)(x + 3 - 2\sqrt{2}\,i)$
(D) $(x - 3 - 2\sqrt{2}\,i)(x - 3 - 2\sqrt{2}\,i)$

5. The range of the function $f(x) = e^{tanx}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is:

(A) $y \in R$ (B) $y \geq 0$ (C) $y > 0$ (D) $0 < y \leq 1$

6. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when:

(A) $y = 0$ (B) $y = \pm 3$ (C) $y = \pm\sqrt{3}$ (D) $y = \frac{1}{2}$

7. The number of non zero positive integer solutions of the equation $w + x + y + z = 20$ is:

(A) 969 (B) 1140 (C) 1771 (D) 4845

8. The position of a moving object is given by the Cartesian coordinates $(3t, e^t)$. Its acceleration is:

(A) constant in both magnitude and direction
(B) constant in magnitude only
(C) constant in direction only
(D) constant in neither magnitude nor direction

9. The region bounded by the parabolas $y = x^2$ and $y = 6x - x^2$ is rotated around the x-axis so that a vertical line segment cut off by the curves generates a ring. The value of x for which the ring of largest area is obtained is:

(A) 4 (B) 3 (C) 2.5 (D) 2

10. The value of $\lim_{h\to 0}\frac{1}{h}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{4}+h}\frac{\sin x}{x}\,dx\right)$ is:

(A) 0 (B) 1 (C) $\frac{1}{\pi\sqrt{2}}$ (D) $\frac{2\sqrt{2}}{\pi}$

END OF SECTION I

Section II 90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Question 11 (15 marks)

(a) Let $z = 1 + i$ and $w = 3 + 2i$

(i) Express $\frac{z}{w}$ in simplest form with a real denominator. **1**

(ii) Write $\bar{z}$ in modulus-argument form. **1**

(b) Simplify $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2)$, where ω is a complex cube root of unity. **3**

(c) (i) Prove for any two real numbers a and b that $a^2 + b^2 > 2ab$, where $a \neq b$. **1**

(ii) Hence, or otherwise, prove if $a + b + c = 1$ then $(1 - a)(1 - b)(1 - c) > 8abc$ where a, b and c are positive numbers and $a \neq b \neq c$. **2**

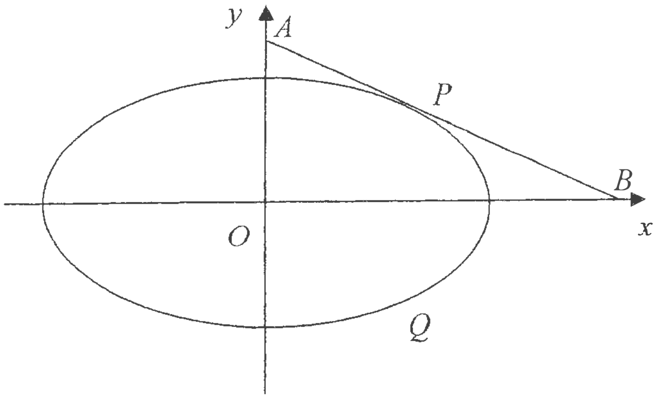
(d) Evaluate $\int_0^1 \frac{dx}{(x+2)(x+3)}$. **3**

(e) $P(3cos\theta,\ 2sin\theta)$ is a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ for $0 < \theta < \frac{\pi}{2}$.

The tangent at P meets the y-axis at A and the x-axis at B, as shown in the diagram below.

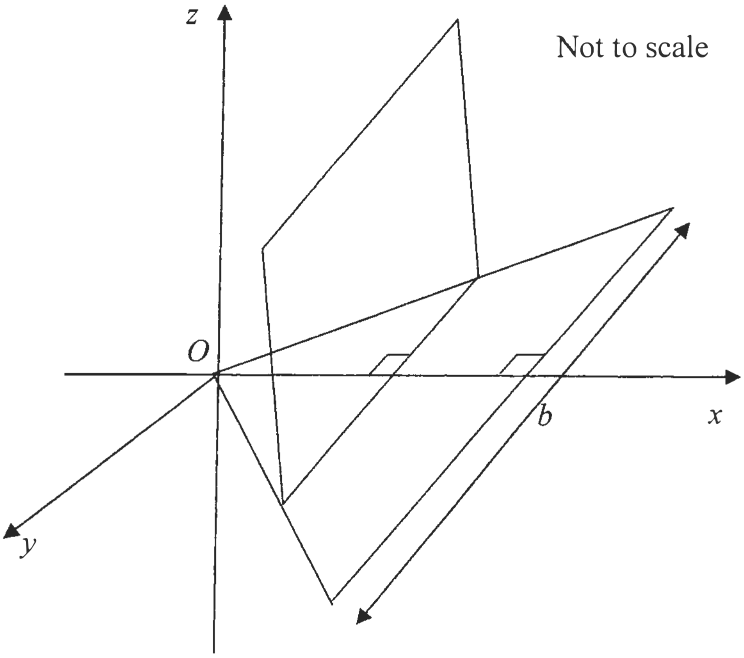
Find the minimum length of AB. **4**

You may assume the equation of the tangent at P is $\frac{xcos\theta}{3} + \frac{ysin\theta}{2} = 1$



Question 12 (15 marks) Start a new page

(a) The base of a solid is in the shape of an equilateral triangle of side b, with one vertex at the origin and its altitude along the positive x-axis as shown in the diagram below. Each cross-sectional plane perpendicular to the x-axis is a square with one side in the base of the solid .



(i) Calculate the side length of a square cross-section x units from the origin. **1**

(ii) Find the volume of the solid. **3**

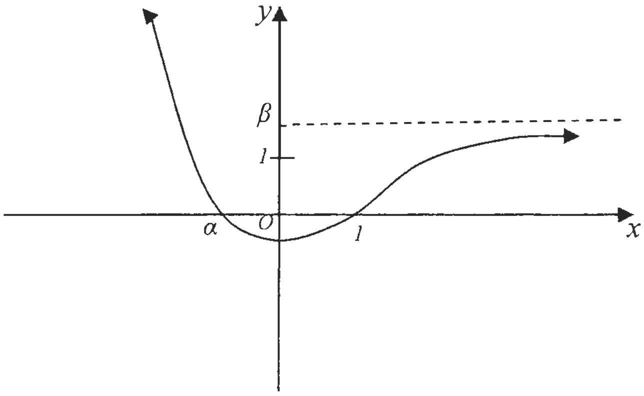
(b) Consider the function $y = \frac{3}{5}x^{\frac{5}{3}} - 3x^{\frac{2}{3}}$.

(i) Find the x-coordinate(s) of any stationary points and describe their nature. **3**

(ii) Describe the concavity of the function for $x < 0$. **2**

(iii) Sketch the graph, showing all important features. **3**

(c) The diagram below shows the graph of the function $y = f(x)$.



Copy the diagram and neatly sketch the graph of $y^2 = f(x)$ on the same set of axes, showing all important features. **3**

Question 13 (15 marks) Start a new page

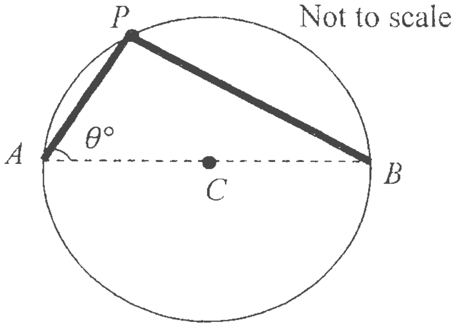
(a) A sequence is defined by the formula:

$U_0 = 0$ and $U_n = \sqrt{U_{n-1} + 2}$ for $n = 1, 2, 3 \ldots$

Prove by mathematical induction that $U_n = 2\cos\left[\frac{\pi}{2^{n+1}}\right]$ for $n = 0, 1, 2, 3 \ldots$ **4**

(b) A 50 kg boy (P) is sitting on the edge of a smooth horizontal circular plate, 3 m in diameter, which is spinning at a constant rate about a vertical axis through its centre C at 20 revolutions per minute. The boy holds onto two separate ropes which are attached to the ends of the diameter (AB) of the plate.

The ropes are such that $AP = 1.8\ m$ and $BP = 2.4\ m$. (Given $\angle APB = 90°$ and $\angle PAB = \theta°$)



(i) Write motion equations in the tangential and radial directions for the boy. **3**

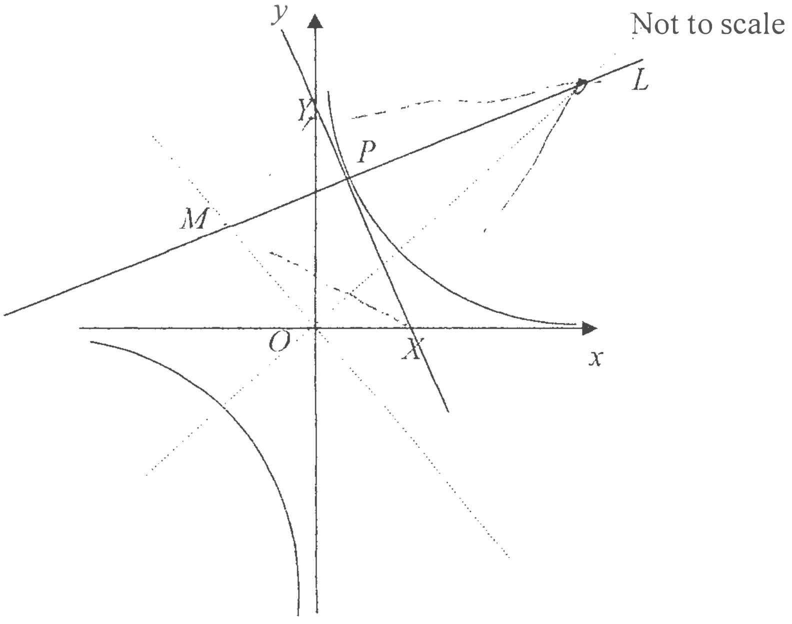
(ii) Calculate the tension in the shorter rope. **2**

(c) (i) Show that the equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is $x + p^2 y = 2cp$. **2**

(ii) The tangent at P meets the x-axis at X and the y-axis at Y and the normal at P meets the lines $y = x$ and $y = -x$ at L and M respectively, as shown in the diagram below.

Prove that $LYMX$ is a rhombus provided that $p \neq 1$. **4**

You may assume that the equation of the normal at P is $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$.



Question 14 (15 marks) Start a new page

(a) (i) Show that $\int_{-a}^{a} \frac{x^4}{1+e^x} dx = \int_{-a}^{a} \frac{x^4 e^x}{1+e^x} dx$ **2**

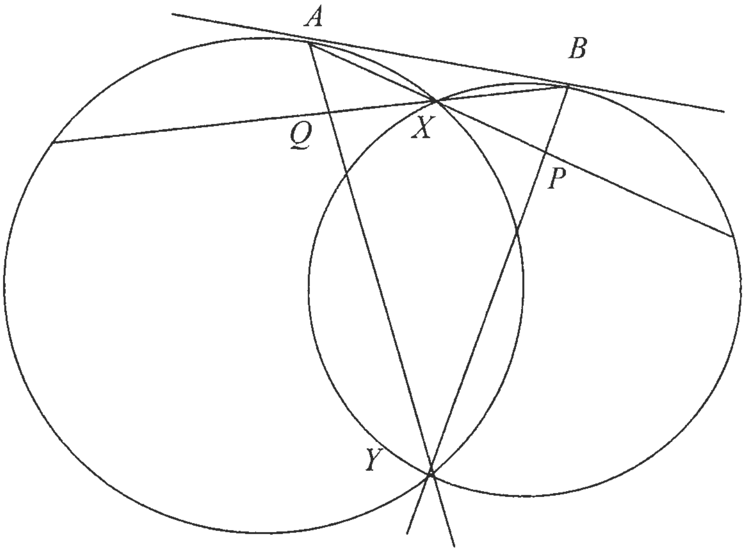
(ii) Hence, or otherwise, evaluate $\int_{-2}^{2} \frac{x^4}{e^x+1} dx$ **2**

(b) The equation $z^2 + (a + ib)z + m + in = 0$ has one real root, where a, b, m and $n \in R$.

Show that $n^2 - abn + mb^2 = 0$. **3**

(c) Two boys each throw two dice. Each person adds the two numbers uppermost on his dice. What is the probability that they gain an equal score? **2**

(d) Two circles intersect at X and Y and AB is a common tangent.
The lines AX and BY meet at P. The lines AY and BX meet at Q as shown in the diagram below.



Not to scale

(i) Copy the diagram and show that $XPYQ$ is a cyclic quadrilateral, giving reasons. **2**

(ii) Prove that $QP \parallel AB$. **1**

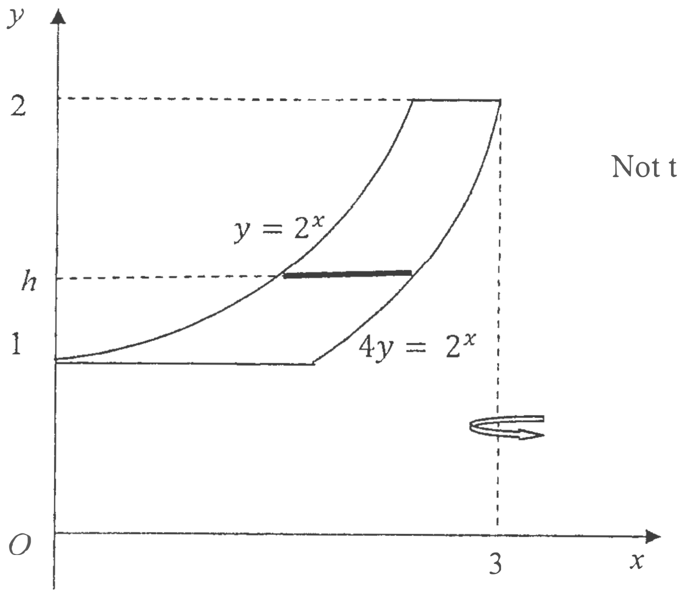
(ii) Hence, or otherwise, prove that XY bisects PQ. **3**

Question 15 (15 marks) Start a new page

(a) The equation $x^3 - x - 1 = 0$ has roots $\alpha,\ \beta$ and γ . **3**

Find an equation whose roots are $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}$ and $\frac{1+\gamma}{1-\gamma}$ in the form $Ax^3 + Bx^2 + Cx + D = 0$.

(b) A model of part of the roof of a new art gallery is generated by rotating the curves $y = 2^x$ and $4y = 2^x$ from $1 \leq y \leq 2$ around the line $x = 3$ as shown in the diagram below.



(i) Show that the area (A) of a cross-sectional slice of the roof taken at $y = h$, for $1 \leq h \leq 2$, is given by the formula $A = 4\pi[2 - log_2 h]$. **3**

(ii) Calculate the volume of the material required to build the model of this part of the roof. **3**

(c) (i) Given $z^n = (z + 1)^n$.

Show that: $Z = \dfrac{1}{\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right) - 1}$ for $k = 1,\ 2 \ldots$ **2**

(ii) Hence, or otherwise, find the roots of $z^n = (z + 1)^n$ in Cartesian form. **3**

(iii) Find the Cartesian equation of the line on which the roots lie. **1**

Question 16 (15 marks) Start a new page

(a) Given $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}} dx,$ $n = 0, 1, 2, 3, \dots\dots\dots$

(i) Show that $(2n + 1)I_n = 2\sqrt{2} - 2nI_{n-1}$ $n = 1, 2, 3, \dots\dots\dots$ **4**

(ii) Evaluate $\int_0^1 \frac{x^3}{\sqrt{x+1}} dx$ **3**

(b) A projectile of mass m kg is launched from the origin O at an angle $\theta°$ above the horizontal where the air resistance has a magnitude mkv N, v m/s is the speed of the particle and k is a positive constant.

Initially the horizontal and vertical components of its velocity are U_0 m/s^2 and V_0 m/s^2 respectively.

(i) Show that the vertical displacement (y m), whilst the projectile is rising, at time t seconds after the launch is given by: **4**

$$y = \left(\frac{g}{k^2} + \frac{V_0}{k}\right)(1 - e^{-kt}) - \frac{gt}{k}$$

where g m/s^2 is the acceleration due to gravity.

(ii) Find the Cartesian equation of the trajectory of the projectile before it reaches its maximum height. . **4**

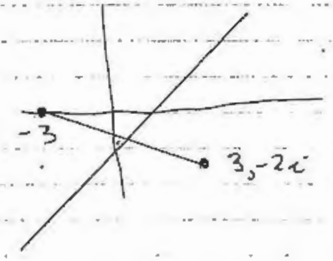
END OF EXAMINATION

Multiple choice

1. $|z-3+2i| = |z+3|$ A

$|z-(3-2i)| = |z-(-3)|$

line is perp bisector



midpt $=(0,-1)$

grad $= -\left(\frac{2}{-6}\right)^{-1}$

$= +\frac{1}{3}$.

2. Acceleration is in direction of motion ∴ not stop A

3. $\int e^{2x}\sqrt{e^x-1}\,dx$ $u = e^x - 1$

$\frac{du}{dx} = e^x$

$\int e^x \sqrt{e^x-1}\; e^x dx$ $du = e^x dx$

$\int (u+1)\, u^{\frac{1}{2}}\, du$

$\int \left(u^{3/2} + u^{1/2}\right) du$ D

④ $x^2+6x+1=0$

$x = \frac{-6 \pm \sqrt{36-4}}{2}$

$= -3 \pm 2\sqrt{2}$

$\therefore (x+3-2\sqrt{2})(x+3+2\sqrt{2})=0$ A

⑤ $f(x) = e^{\tan x}$ $-\pi/2 < x < \pi/2$

$\tan x \in R$.

$0 < f(x)$ $y > 0$. C

⑥ $y^2 - xy + 9 = 0$.

$2y\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$.

$\frac{dy}{dx}\left[2y - x\right] = y$.

~~$2y$~~ $\frac{dy}{dx} = \frac{y}{2y-x}$.

vertical when $x = 2y$.

$y^2 - 2y^2 + 9 = 0$.

$-y^2 + 9 = 0$ $\therefore y = \pm 3$. B

⑦ - - - - - - - - 20 nos

3 dividers in spaces.

${}^{19}C_3 = 969$. A

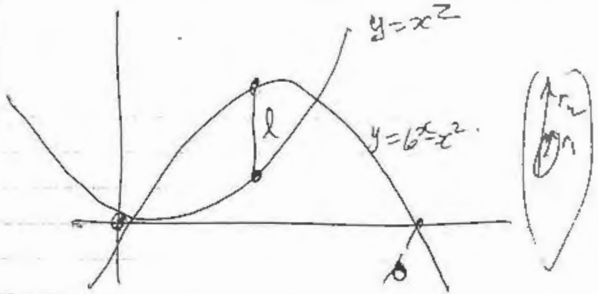
⑧ $x = 3t$ $y = e^t$ C

$\dot{x} = 3$ $\dot{y} = e^t$

$\ddot{x} = 0$ $\ddot{y} = e^t$

No x acceleration any y acceleration ∴ constant in direction

⑨



$$A = \pi\left(r_2^2-r_1^2\right)$$
$$=\pi\left((6x-x^2)^2-(x^2)^2\right)$$
$$=\pi\left(6x-x^2-x^2\right)\left(6x-x^2+x^2\right)$$
$$=\pi\left(6x-2x^2\right)6x$$
$$=\pi\left(36x^2-12x^3\right)$$
$$\frac{dA}{dx}=\pi\left(72x-36x^2\right)=0.$$
$$72x=36x^2 \qquad x\neq 0.$$
$$x=2.$$

D.

⑩

$$\lim_{h\to 0}\frac{1}{h}\int_{\pi/4}^{\pi/4+h}\frac{\sin x}{x}\,dx.$$

$$=\lim_{h\to 0}\frac{1}{h}\left[F\left(\tfrac{\pi}{4}+h\right)-F\left(\pi/4\right)\right]$$

where $F'(x)=\frac{\sin x}{x}$.

D

[illegible]

$$=\frac{\sin \pi/4}{\pi/4}=\frac{\frac{1}{\sqrt{2}}}{\pi/4}$$
$$=\frac{4}{\pi\sqrt{2}}$$
$$=\frac{2\sqrt{2}}{\pi}$$

Q11 Ext 2 Yr 12 TRIAL 2016

a i) $\frac{z}{w} = \frac{1+i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{5+i}{13}$ 1m

ii) $\bar{z} = 1-i = \sqrt{2}\cos\left(-\frac{\pi}{4} + i\sin -\frac{\pi}{4}\right)$ 1m

b) $(1+2w+3w^2)(1+3w+2w^2)$

$= 1+3w+2w^2+2w+6w^2+\overset{4}{\cancel{4w^3}}+3w^2+\overset{9}{\cancel{9w^3}}+\overset{6w}{\cancel{6w^4}}$

$= 14+11(w+w^2)$ $(w^3=1)$ 1m

$= 14+11(-1)$ $(w+w^2=-1)$ 1m

$= \underline{\underline{3}}$ 1m

c i) $(a-b)^2 > 0$ $(a \neq b)$

$a^2+b^2 > 2ab$ 1m

ii) $(1-a)(1-b)(1-c)$

$= (b+c)(a+c)(a+b) > (2\sqrt{bc})(2\sqrt{ac})(2\sqrt{ab})$ 1m

$> 8\sqrt{a^2b^2c^2}$

$> \underline{8abc}$ 1m

d) $\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$ 1m

$\int_0^1 \frac{dx}{(x+2)(x+3)} = \int_0^1 \frac{1}{x+2} - \frac{1}{x+3}\,dx = \ln\left(\frac{x+2}{x+3}\right)\Big]_0^1$ 1m

$= \underline{\ln 9/8}$ 1m

e) $A = \left(0, \frac{2}{\sin\theta}\right)$ $B = \left(\frac{3}{\cos\theta}, 0\right)$ 1m

$L^2 = \frac{4}{(\sin\theta)^2} + \frac{9}{(\cos\theta)^2}$ ∴ $2L\frac{dL}{d\theta} = \frac{-8\cos\theta}{\sin^3\theta} + \frac{18\sin\theta}{\cos^3\theta}$

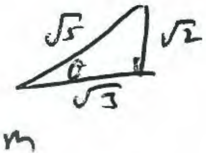
SP when $\frac{dL}{d\theta}$ $8\cos^4\theta = 18\sin^4\theta$ 1m

∴ $\tan^4\theta = \frac{4}{9}$ ∴ $\tan\theta = \sqrt{\frac{2}{3}}$

$\left(\begin{matrix} 0<\theta<90 \\ \tan\theta>0 \end{matrix}\right)$

Check max/min 1m

$\sin\theta = \sqrt{\frac{2}{5}}$, $\cos\theta = \sqrt{\frac{3}{5}}$



∴ $L^2 = \frac{9}{\frac{3}{5}} + \frac{4}{\frac{2}{5}} = 25$ ∴ min $\underline{L = 5\ (L>0)}$ 1m

Do Not accept $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ nor

non principal argument

Do Not get any marks
just expanding

$w^3=1$ or $1+w+w^2=0$ 1

2nd mark given when
simplied and very clo
to final answer = 3
like one step away

Some students wrote ≥
only 1m

Alternatively $(b+c)(a+c)(b+a)$

$= a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2$

$> 2ab + 2ab + 2ab + 2abc$ 1

$> 8abc$ 1m

1m for A & B

1m for SP $\tan\theta = \sqrt{\frac{2}{3}}$

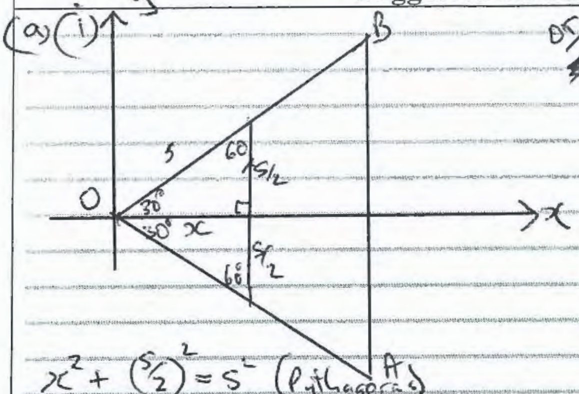
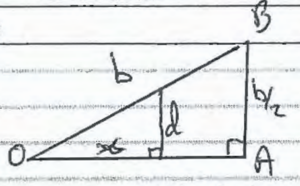
1m for check max/min

1m for correct L = 5

To get the 3rd & 4th mark, $0<\theta<\frac{\pi}{2}$ an
not a random guess

1/3

MATHEMATICS: Question 12
Extension 2

Suggested Solutions	Marks	Marker's Comments
(a)(i)  OR 		
$OA = \frac{\sqrt{3}b}{2}$ (Pythagoras) using similar triangles: $\frac{d}{b/2} = \frac{x}{\sqrt{3}/2\,b}$ $d = \frac{x}{\sqrt{3}}$ $x^2 + \left(\frac{s}{2}\right)^2 = s^2$ (Pythagoras) $\frac{s^2}{4} - s^2 = x^2$ $\frac{3s^2}{4} = x^2$ $s^2 = \frac{4x^2}{3}$ $s = \frac{2x}{\sqrt{3}}$ ∴ sidelength = 2d $= \frac{2x}{\sqrt{3}}$	①	side = $\frac{2x}{\sqrt{3}}$
(ii) Area of square = $\left(\frac{2x}{\sqrt{3}}\right)^2 = \frac{4x^2}{3}$ volume of slice: $\delta V = \frac{4x^2}{3}\delta x$ volume of solid = $\lim_{\delta x \to 0} \sum_{0}^{\sqrt{3}b/2} \frac{4x^2}{3}\delta x$	①	If the $\lim_{\delta x \to 0}$ statement is missing then students lost one mark
$= \frac{4}{3}\int_0^{\sqrt{3}b/2} x^2\,dx$ $= \frac{4}{3}\left[\frac{x^3}{2}\right]_0^{\sqrt{3}b/2}$	①	
$= \frac{4}{9}\left[\left(\frac{\sqrt{3}b}{2}\right)^3 - 0\right]$ $= \frac{4}{9} \times \frac{3\sqrt{3}b^3}{8}$ $= \frac{\sqrt{3}b^3}{6}$ units^3	①	$\frac{4h^3}{9}$ or $\frac{4b^3}{9}$ = 2mks Maximum

2/3

MATHEMATICS: Question 12
Extension 2

Suggested Solutions	Marks	Marker's Comments
(b) $y=\frac{3}{5}x^{5/3}-3x^{2/3}$		
$\frac{dy}{dx}=x^{2/3}-2x^{-1/3}$		
poss. stat. points when $\frac{dy}{dx}=0$		Students who went on to "prove" that $x=0$ was a horizontal pt of inflexion or a maximum – lost one mark
ie $0=x^{2/3}-2x^{-1/3}$	①	
$0=x^{-1/3}(x-2)$		
$x^{-1/3}\neq 0 \quad \therefore x=2$ only	①	
$\frac{d^2y}{dx^2}=\frac{2}{3}x^{-1/3}+\frac{2}{3}x^{-4/3}$		
when $x=2$, $\frac{d^2y}{dx^2}=0.52913+0.2645668$ $=0.793696$		
$>0 \quad \therefore$ concave up $\therefore$ rel. min at $x=2$	①	
(ii) when $x<0$, what happens to $\frac{d^2y}{dx^2}$??		
$\frac{d^2y}{dx^2}=\frac{2}{3}x^{-1/3}+\frac{2}{3}x^{-4/3}$ $=\frac{2}{3}x^{-1/3}(1+x^{-1})$		
poss pts of inflexion when $\frac{d^2y}{dx^2}=0$		
ie $0=\frac{2}{3}x^{-1/3}\left(1+\frac{1}{x}\right)$		
$x^{-1/3}\neq 0$, $1+\frac{1}{x}=0$ $x\neq 0$ $\quad \frac{1}{x}=-1$ $x=-1$	①	A lot of students just quoted that $\frac{d^2y}{dx^2}<0$ without substituting in values → max. 1 mark

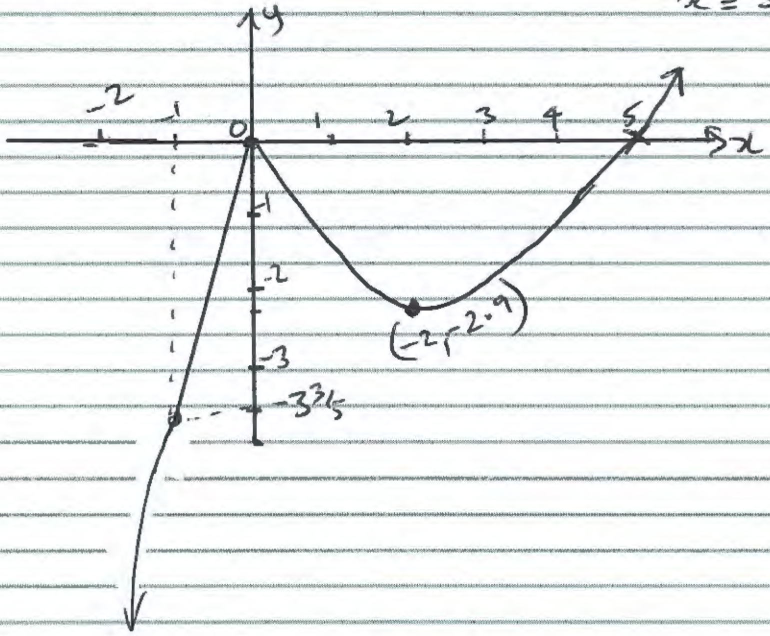
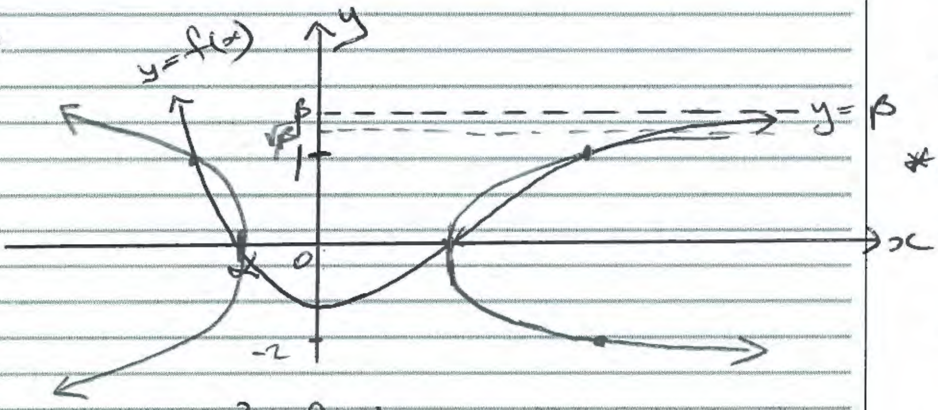
x	-1.1	-1	-0.9
$\frac{d^2y}{dx^2}$	-0.059	0	0.0767

Suggested Solutions	Marks	Marker's Comments
$\therefore$ change in concavity at $x=-1$ $\therefore$ pt of inflexion at $(-1,-3\frac{3}{5})$		
$\therefore$ concave up for $-1<x<0$ $\therefore$ concave down for $x<-1$	①	

3/3

MATHEMATICS: Question 12

Extension 2

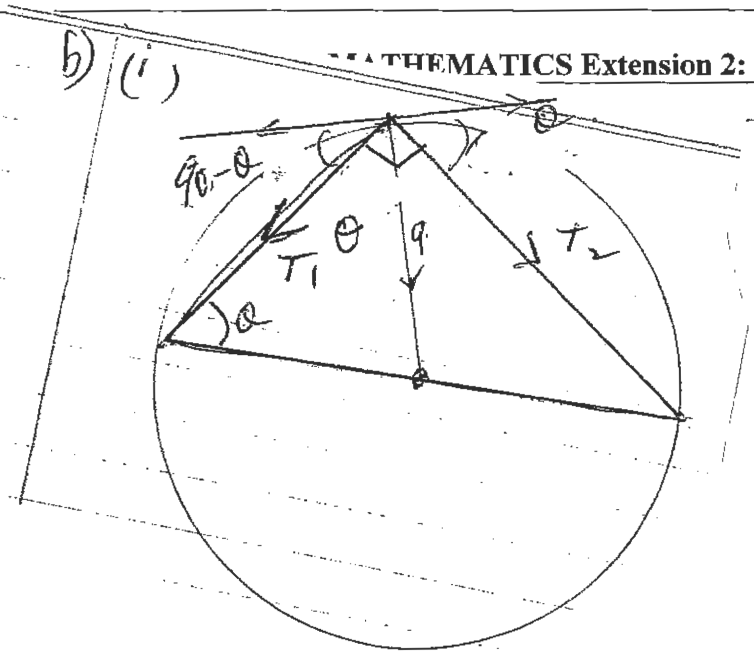
Suggested Solutions	Marks	Marker's Comments
(iii) $y = \frac{3}{5}x^{5/3} - 3x^{2/3}$ $= x^{2/3}\left(\frac{3}{5}x - 3\right)$ $y=0$ when $x=0$ or $\frac{3}{5}x = 3$ $x = 5$	1	for x intercepts
	1 1	cusp at $x=0$ for concavity change at $x=-1$ and minimum point * If the graph did NOT represent what the students found in (i) and (ii) they lost a mark.
(c)  $y^2 = f(x)$ ie $y = \pm\sqrt{f(x)}$ * A lot of students forgot to change the asymptote to $y=\sqrt{\beta}$ or they put the asymptote $y=\sqrt{\beta}$ in the wrong place.		* no graph $\alpha < x < 1$ * $y=f(x)$ and $y^2=f(x)$ intersect at $y=1$ ① * vertical tangents at $x=\alpha$ and $x=1$ ① * horizontal asymptote is lower than β but higher than $y=1$. * symmetrical about $y=0$. ① * If they didn't draw the original graph, they lost one mark.

MATHEMATICS Extension 2 : Question 13

Suggested Solutions	Marks Awarded	Marker's Comments
d) If $u_0=0$ $u_n=\sqrt{u_{n-1}+2}$ $n=1,2,3-$		Did not need to test for 1 but needed to test for 0
Test $u_0=0$		
$u_0 = 2\cos\frac{\pi}{2}$		
$= 0.$		
$\therefore$ true for $n=0$	1	
Assume true for $k=0,1,2,3\cdot\ \cdot$		
$u_k = 2\cos\left(\frac{\pi}{2^{k+1}}\right)$		
Prove true for $n=k+1$		
$u_{k+1} = 2\cos\left(\frac{\pi}{2^{k+2}}\right)$		
Now		
$u_{k+1} = \sqrt{u_k+2}$		
$= \sqrt{2\cos\left(\frac{\pi}{2^{k+1}}\right)+2}$	1	
$= \sqrt{2\left[2\cos^2\frac{\pi}{2^{k+2}}-1\right]+2}$		Needed some explanation or clear setting out.
Since $\cos 2\theta = 1-2\cos^2\theta$ so $\cos\theta = 1-2\cos^2\frac{\theta}{2}$	1	
$= \sqrt{4\cos^2\frac{\pi}{2^{k+2}}-2+2}$		
$= 2\cos\frac{\pi}{2^{k+2}}$	1	(do-cture PMI)
$= RHS$ $\quad 0 < \frac{\pi}{2^{k+2}} < \frac{\pi}{2}.$		
$\therefore$ by process of mathematical induction true for $n \geq 0$ $n \in \mathbb{Z}$		

MATHEMATICS Extension 2: Question 13....

b) (i)



	Marks Awarded	**Marker's Comments**
(i) Let tensions be T_1 and T_2 Tangential (no acceleration) $T_2\cos\theta - T_1\sin\theta = 0$	1	• did not get both marks if not clear what was radial and tangential
Radial $T_2\sin\theta + T_1\cos\theta = m\omega^2 r$	1	
$m = 50$ $\quad \omega = \frac{2\pi}{T} =$ $= 2\pi \times \frac{20}{60}$ $= \frac{2\pi}{3}$ rad/s	1	• Students did not clearly indicate which tension went with which string.
(ii) $\cos\theta = \frac{1.8}{3}$ $\quad \sin\theta = \frac{2.4}{3}$ $= 0.6$ $\quad = 0.8$ Tangential $0.6T_2 = 0.8T_1 \therefore T_2 = \frac{4}{3}T_1$	1	Equations needed to make some sense to get carry forward errors.
Radial $0.8 \times \frac{4}{3}T_1 + 0.6T_1 = \frac{100\pi^2}{3}$ $\therefore \frac{5}{3}T_1 = \frac{100\pi^2}{3}$ $T_1 = 20\pi^2$ Newtons	1	

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments

c) (i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

when $x = cp$

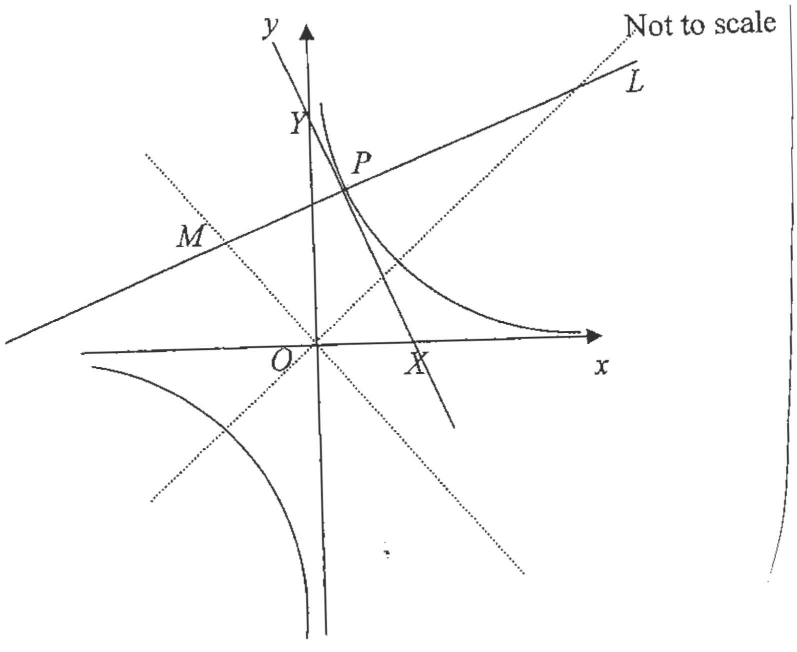
$\frac{dy}{dx} = -\frac{1}{p^2}$ — 1 — well done.

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$\underline{x + p^2y = 2cp}$ — 1

(ii)



For rhombus diagonals bisect at right angles

coords $X = (2cp, 0)$ midpoint

$Y = (0, \frac{2c}{p})$ $= P(cp, \frac{c}{p})$

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments
Coordinates L $y = x$ Normal $px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right)$ $x\left(p - \frac{1}{p}\right) = c\left(p - \frac{1}{p}\right)\left(p + \frac{1}{p}\right)$ $x = c\left(p + \frac{1}{p}\right)$ $y = c\left(p + \frac{1}{p}\right)$ $L\left(c\left(p + \frac{1}{p}\right), c\left(p + \frac{1}{p}\right)\right)$		• Students didn't simplify and ended up with messy expressions
For M $y = -x$ $M: x\left(p + \frac{1}{p}\right) = c\left(p^2 - \frac{1}{p^2}\right)$	1	• Students didn't know sufficiency conditions for rhombus
$x = c\left(p - \frac{1}{p}\right)$ $y = -c\left(p - \frac{1}{p}\right)$ $M\left(c\left(p - \frac{1}{p}\right); c\left(p \mp \frac{1}{p}\right)\right)$ $L\left(c\left(p + \frac{1}{p}\right), c\left(p + \frac{1}{p}\right)\right)$ Midpoint $\left(cp, \frac{c}{p}\right)$ which is P	1	• could also prove parallelogram with adjacent sides equal using gradients and distance. or all 4 sides equal
∴ diagonals are normal and tangent ∴ perpendicular.	1	
∴ Rhombus since diagonals bisect at right angles	1	

Q14 Ext 2 Yr 12 2016

a i) Let $u = -x$ $\quad du = -dx$

$$\int_{-a}^{a} \frac{x^4\,dx}{1+e^x} = \int_{-a}^{a} (-u)^4 \, -du = \int_{-a}^{a} \frac{u^4}{1+e^{-u}}du = \int_{-a}^{a}\frac{e^u u^4}{1+e^u}du$$

1m

$$\therefore \int_{-a}^{a} \frac{x^4\,dx}{1+e^x} = \int_{-a}^{a} \frac{e^x x^4}{1+e^x}dx \text{ (change of variable)}$$

1m

ii) Let $I = \int_{-2}^{2} \frac{x^4}{1+e^x}dx = \int_{-2}^{2}\frac{e^x x^4}{1+e^x}dx$

$$2I = \int_{-2}^{2} \frac{(1+e^x)x^4}{1+e^x}dx = \int_{-2}^{2} x^4\,dx = \left.\frac{x^5}{5}\right]_{-2}^{2}$$

1m

$$2I = \frac{1}{5}\left[32 - -32\right] = \frac{64}{5}$$

$$\therefore I = \underline{\underline{\frac{32}{5}}}$$ 1m

b) Let x be the real root, then

$x^2 + x(a+ib) + m + in = 0 + 0i$ $\quad$ 1m

$\therefore$ Real part $\quad x^2 + ax + m = 0$ ①

Imaginary part $\quad bx + n = 0 \Rightarrow x = \frac{-n}{b}$ $\quad$ 1m

Sub $x = \frac{-n}{b}$ into ① $\left(\frac{-n}{b}\right)^2 + a\left[\frac{-n}{b}\right] + m = 0$ $\quad$ 1m

$$n^2 - anb + mb^2 = 0$$

Alternatively let α is the real root and β be the complex root

$\therefore$ sum roots $\quad \alpha + \beta = -(a+ib)$
product roots $\quad \alpha\beta = m + in$ $\quad \Big\}$ 1m

$\beta = (-\alpha - a) - ib$ $\quad \alpha\beta = \alpha\left([-\alpha - a] - (iab\right)$

$\alpha\beta = m + in$

Alternatively

Let $f(x) = \frac{(1-e^x)x^4}{1+e^x}$

$$-f(-x) = \frac{(-x)^4(e^{-x}-1)}{1+e^{-x}}\cdot\frac{e^x}{e^x}$$

$$-f(-x) = \frac{(1-e^x)x^4}{1+e^x} = f(x)$$

$\therefore f(x)$ is odd $\quad$ 1m

$$\therefore \int_{-a}^{a} f(x)\,dx = 0$$

$$\therefore \int_{-a}^{a}\frac{x^4\,dx}{1+e^x} = \int_{-a}^{a}\frac{e^x x^4}{1+e^x}dx$$

1m

1m @ for equating real part / Imaginary part

1m for final proof

1m for sum & product of roots

Note A lot of students think only 1 real root

$\therefore \Delta = 0$

This only apply to real polynomial.

0m

$$\therefore \underbrace{-\alpha[\alpha + a] = m \quad \text{and} \quad n = -\alpha b}_{1m}$$

$$(\text{ie } \alpha = -\frac{n}{b})$$

1 m for Equating real & imaginary pts

Solving simultaneously

$$-\frac{-n}{b}\left[-\frac{n}{b} + a\right] = m$$

$$\frac{-n^2}{b^2} + \frac{an}{b} = m \qquad 1m$$

1m final line

$$n^2 - abn + mb^2 = 0$$

c)

f	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	10	12

$P(x=2) = \frac{1}{36}$

$P(x=3) = \frac{2}{36}$

$P(x=4) = \frac{3}{36}$

$P(x=5) = \frac{4}{36}$

$P(x=6) = 5/36$

$P(x=7) = \frac{6}{36}$

$P(x=8) = \frac{5}{36}$

$P(x=9) = 4/36$

$P(x=10) = 3/36$

$P(x=11) = \frac{2}{36}$

$P(x=12) = \frac{1}{36}$

$$P(\text{equal scores}) = \frac{1}{36^2}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2\right] \qquad 1m$$

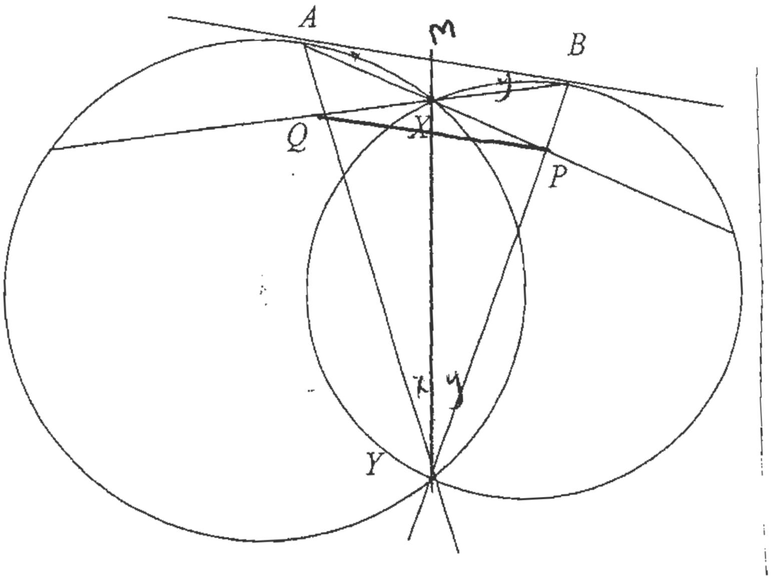
Getting the right formula

$$= \frac{73}{648} \qquad 1m$$

final answer

Well done overall

Q14 a



(i) Join XY Let $\angle BAX = x$, $\angle ABX = y$

$\angle BXP = x+y$ (exterior angle of $\triangle ABX$.

$\angle BAX = \angle AYX = x$ (angle between tangent and chord at point of contact equals angle in alternate segment) 1m

Similarly

$\angle ABX = \angle BYX = y$

$\therefore \angle BXP = \angle QYP = x+y$ 1m

$\therefore$ XPYQ is a cyclic quadrilateral

(exterior angle equals opposite interior angle)

Some students do not know th exact theorem.

(ii) Join PQ $\angle XPQ = \angle QYX = x$ (angle at circumference standing on same arc as XPYQ is cyclic

$\therefore \angle BAP = \angle XPQ = x$ 1m

$\therefore AB \parallel QP$ (alternate angles equal.)

must mention names a alten angles

iii) Extend YX to meet AB at M, QP at C

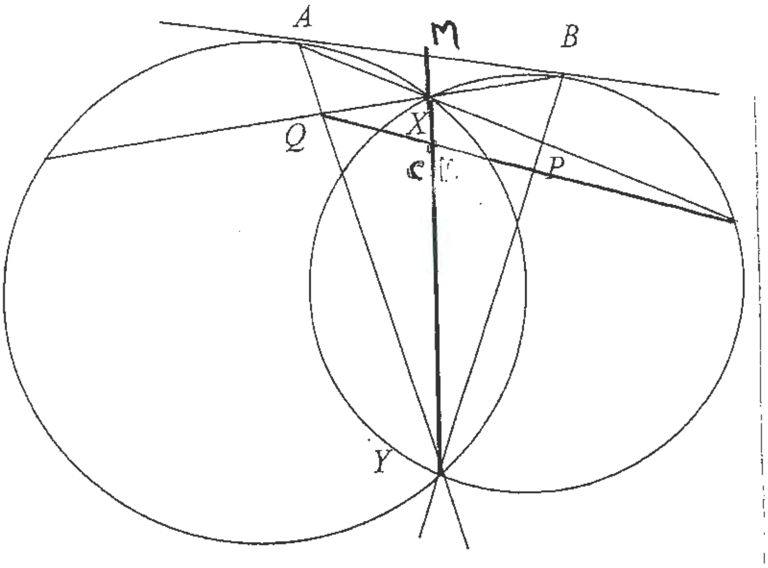
$AM^2 = MX \cdot MY$ (product of intercepts on secants equal to tangents squared)

Similarly $BM^2 = MX \cdot MY$

$\therefore AM^2 = BM^2$ 1m

$\therefore AM = BM$ ($AM > 0$, $BM > 0$)

Most students do not realize this theorem or give up.



In ΔAMY, ΔQCY

$AB \parallel QP$ (proved)

$\angle AMY = \angle QCY$ (corresponding angles on parallel lines)

$\angle AYM$ is common

$\Delta AMY \mathbin{|||} \Delta QCY$ (equiangular)

$\therefore \frac{AM}{QC} = \frac{MY}{CY}$ (ratio of corresponding sides of similar triangles) 1m

Similarly $\frac{BM}{PC} > \frac{MY}{CY}$

1m

$\therefore \frac{AM}{QC} = \frac{BM}{PC}$

As $AM = BM$ $\therefore QC = PC$

$\therefore XY$ bisects PQ.

Students must show the ratio of corresponding sides of similar triangles to get 1m not stopping at equiangular.

Note Students did not prove $AM = BM$ max 1m

2016 Trial E2 **MATHEMATICS: Question** 15

Extension 2

Suggested Solutions | **Marks** | **Marker's Comments**

(a) let $m = \frac{1+\alpha}{1-\alpha}$

$m(1-\alpha) = 1+\alpha$

$m-1 = \alpha(1+m)$

$\frac{m-1}{m+1} = \alpha$ 1

α is a root $\Rightarrow \alpha^3-\alpha-1=0$

$\therefore \left(\frac{m-1}{m+1}\right)^3 - \left(\frac{m-1}{m+1}\right) - 1 = 0$

$(m-1)^3 - (m-1)(m+1)^2 - (m+1)^3 = 0$ 1

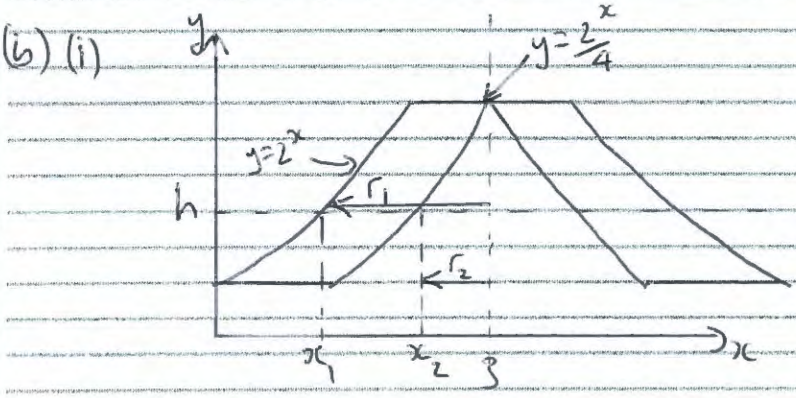
$m^3-3m^2+3m-1+m^2+m^3-1-m-m^3-3m^2$
$-3m-1 = 0$

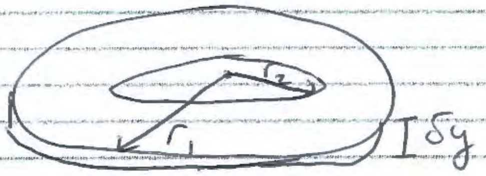
$-m^3-7m^2+m-1 = 0$ 1

$\therefore$ Equation in x is

$x^3+7x^2-x+1=0$

(b) (i)





$r_1 = 3-x_1$

$r_2 = 3-x_2$ 1

A diagram with dimension was necessary to earn the mark

Suggested Solutions	Marks	Marker's Comments
Note		
$y = 2^x \Rightarrow h = 2^{x_1}$		
$\log_2 h = x_1 \Rightarrow r_1 = 3 - \log_2 h$		Many algebraic
$4y = 2^x \Rightarrow 4h = 2^{x_2}$		methods were possible, including noting
$\log_2 4h = x_2 \Rightarrow r_2 = -(\log 4 + \log_2 h) + 3$		$A = \pi(r_1 - r_2)(r_1 + r_2)$
$= 1 - \log_2 h$		
$A = \pi\left(r_1^2 - r_2^2\right)$	1	
$= \pi\left((3-\log_2 h)^2 - (1 - \log_2 h)^2\right)$		Faulty algebra or
$= \pi\left(9 - 6\log_2 h + (\log_2 h)^2 - 1 + 2\log_2 h - (\log_2 h)^2\right)$		misused log laws
$= \pi(8 - 4\log_2 h)$		resulted in a carried
$= 4\pi(2 - \log_2 h)$ as required	1	error IF subsequent logic was sound.
(ii) $\Delta V = A\,\delta y$		
$V \doteq \lim\limits_{\delta y \to 0} \sum\limits_{1}^{2} 4\pi(2 - \log_2 y)\,dy$		
$= 4\pi \int_1^2 2 - \log_2 y\,dy$	1	
$= 4\pi\left(\int_1^2 2\,dy - \int_1^2 1 \times \log_2 y\,dy\right)$		
$= 4\pi\left(\left[2y\right]_1^2 - \left[y\log_2 y\right]_1^2 + \int_1^2 \frac{y \times \frac{1}{y}}{\ln 2}\,dy\right)$	1	

MATHEMATICS: Question........

Extension 2

Suggested Solutions	Marks	Marker's Comments
$= 4\pi\left(\cancel{\ln} 2 - 2\log_2 2 + 0 + \int_1^2 \frac{1}{\ln 2}\,dy\right)$ $= 4\pi\left(0 + \left[\frac{y}{\ln 2}\right]_1^2\right)$ $= \frac{4\pi}{\ln 2}$	1	If students made an error in integration by parts, a carried error was awarded only if the process showed progress towards the end goal and did not simplify the computation
(c) $z^n = (z+1)^n$ $\frac{(z+1)^n}{z^n} = 1$ $= \text{cis}\left(2k\pi\right)$ $k \neq 0,\ k = 1, 2, 3 \ldots$	1	
$\frac{(z+1)}{z} = \text{cis}\left(\frac{2k\pi}{n}\right)$ By De Moivre's theorem $z+1 = z\,\text{cis}\left(\frac{2k\pi}{n}\right)$ $1 = z\left(\text{cis}\left(\frac{2k\pi}{n}\right) - 1\right)$ $z = \frac{1}{\text{cis}\left(\frac{2k\pi}{n}\right) - 1}$ $k \neq 0,\ k = 1, 2, 3 \ldots$	1	Quoting De Moivre's theorem was necessary to earn the second mark Simply substituting the result into the given expressions is circular reasoning and earned no marks

MATHEMATICS: Question……..

Extension 2

Suggested Solutions	Marks	Marker's Comments
(ii) $z = \dfrac{1}{\cos\frac{2\pi k}{n} - 1 + i\sin\frac{2\pi k}{n}}$ $= \dfrac{1}{1 - 2\sin^2\frac{\pi k}{n} - 1 + 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}}$	1	A clear computational path needed to be shown for mark to be awarded.
$= \dfrac{1}{2i\sin\frac{\pi k}{n}\left(\cos\frac{\pi k}{n} + i\sin\frac{\pi k}{n}\right)}$ $= \dfrac{1}{2i\sin\frac{\pi k}{n}\left(\cos\frac{\pi k}{n} + i\sin\frac{\pi k}{n}\right)} \times \dfrac{\text{cis}\frac{\pi k}{n}}{\text{cis}\frac{\pi k}{n}}$	1	
$= \dfrac{1}{2}\left(\dfrac{-i\cos\frac{\pi k}{n}}{\sin\frac{\pi k}{n}} - \dfrac{i\sin\frac{\pi k}{n}}{i\sin\frac{\pi k}{n}}\right)$	1	
$= -\dfrac{1}{2}\left(1 - i\cot\frac{k\pi}{n}\right)$ Possible alternatives include: • $z = \dfrac{\cos\left(\frac{2k\pi}{n}\right) - 1}{2\left(1 - \cos\left(\frac{2k\pi}{n}\right)\right)} + \dfrac{i\sin\left(\frac{2\pi k}{n}\right)}{2\left(1 - \cos\left(\frac{2k\pi}{n}\right)\right)}$ • $-\dfrac{1}{2} - \dfrac{i\sin\left(\frac{2k\pi}{n}\right)}{2\left(1 - \cos\left(\frac{2k\pi}{n}\right)\right)}$ (iii) $z = x + iy$ where x is constant $\therefore z$ lies on the line $x = -\frac{1}{2}$	1	

16 a) i) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$, $n \in \mathbb{Z}$, $n \geq 0$

$$= \left[2(x+1)^{\frac{1}{2}} \times x^n\right]_0^1 - n\int_0^1 x^{n-1} \times 2\sqrt{x+1}\,dx$$ ①

① $$= 2\sqrt{2} - 2n\int_0^1 x^{n-1}\sqrt{x+1}\,dx$$

$$= 2\sqrt{2} - 2n\int_0^1 x^{n-1} \cdot \frac{(x+1)}{\sqrt{x+1}}\,dx$$

$$= 2\sqrt{2} - 2n\int_0^1 \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}\,dx$$ ①

$$= 2\sqrt{2} - 2nI_n - 2nI_{n-1}$$

$$\therefore (2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$$ ①

ii) $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx = I_3$

$$I_3 = \frac{2\sqrt{2} - 6I_2}{7}$$ ①

$$= \frac{2\sqrt{2}}{7} - \frac{6}{7}\left[\frac{2\sqrt{2}}{5} - \frac{4I_1}{5}\right]$$

$$= \frac{2\sqrt{2}}{7} - \frac{12\sqrt{2}}{35} + \frac{24}{35}\left[\frac{2\sqrt{2}}{3} - \frac{2}{3}I_0\right]$$

$$= \frac{-2\sqrt{2}}{35} + \frac{48\sqrt{2}}{105} - \frac{48}{105}I_0$$

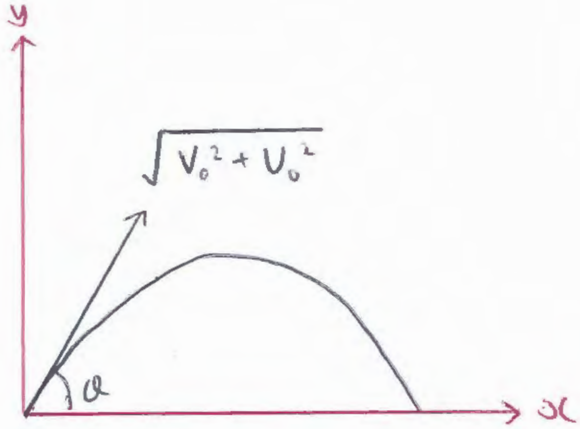
$$= \frac{42\sqrt{2}}{105} - \frac{48}{105}\int_0^1 \frac{1}{\sqrt{x+1}}\,dx$$

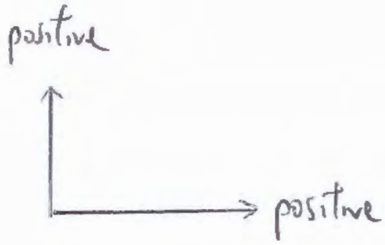
$$= \frac{42\sqrt{2}}{105} - \frac{48}{105}\left[2\sqrt{x+1}\right]_0^1$$ ①

$$= \frac{42\sqrt{2}}{105} - \frac{48}{105}\left(2\sqrt{2} - 2\right)$$

$$= \frac{96}{105} - \frac{54\sqrt{2}}{105} = \frac{32}{105} - \frac{18\sqrt{2}}{105}$$ ①

b)





vertically - $m\ddot{y} = -mg - mkv$ ——— ①

$$\frac{dv}{dt} = -g - kv$$

$$\frac{dt}{dv} = \frac{-1}{g+kv}$$

$$\int_0^t dt = -\int_{v_0}^{v} \frac{1}{g+kv}\, dv$$

$$t = \frac{-1}{k}\Big[Ln(g+kv)\Big]_{v_0}^{v}$$

$$= \frac{-1}{k} Ln \frac{g+kv}{g+kv_0}$$ ——— ①

$$-kt = Ln\left(\frac{g+kv}{g+kv_0}\right)$$

$$\frac{g+kv}{g+kv_0} = e^{-kt}$$

$$g+kv = (g+kv_0)e^{-kt}$$

$$kv = (g+kv_0)e^{-kt} - g$$

$$v = \frac{(g+kv_0)e^{-kt}}{k} - \frac{g}{k}$$

1st Approach

$$\frac{dy}{dt} = \frac{(g+kv_0)e^{-kt} - g}{k}$$

$$\int_0^y dy = \int_0^t \frac{(g+kv_0)e^{-kt} - g}{k}$$ ①

$$y = \left[\frac{-(g+kv_0)e^{-kt}}{k^2} - \frac{gt}{k}\right]_0^t$$ ①

$$= \frac{(g+kv_0)}{k^2} - \frac{(g+kv_0)e^{-kt}}{k^2} - \frac{gt}{k} = \left(\frac{g}{k^2} + \frac{v_0}{k}\right)(1-e^{-kt}) - \frac{gt}{k}$$

Alternatively:

$$\left[\ \frac{v}{g+kv} = \frac{g+kv-g}{g+kv} \times \frac{1}{k} = \frac{1}{k}\left(1 - \frac{g}{g+kv}\right)\ \right]$$

$$v\frac{dv}{dy} = -g-kv$$

$$\frac{dv}{dy} = \frac{-g-kv}{v}$$

$$\frac{dy}{dv} = \frac{-v}{g+kv}$$

$$\int_0^y dy = -\int_{v_0}^{v} \frac{v}{g+kv}\,dv$$

$$y = \frac{-1}{k}\int_{v_0}^{v}\left(1 - \frac{g}{g+kv}\right)dv$$

$$= \frac{-1}{k}\left[v - \frac{g}{k}\mathrm{Ln}(g+kv)\right]_{v_0}^{v}$$

$$= \frac{-1}{k}\left[v - \frac{g}{k}\mathrm{Ln}(g+kv) - v_0 + \frac{g}{k}\mathrm{Ln}(g+kv_0)\right]$$

$$= \frac{-1}{k}\left[v - v_0 + \frac{g}{k}\mathrm{Ln}\left(\frac{g+kv_0}{g+kv}\right)\right] \quad ——— ①$$

Sub $v = \dfrac{(g+kv_0)e^{-kt} - g}{k}$ into y

$$\therefore y = \frac{-1}{k}\left[\frac{(g+kv_0)e^{-kt} - g}{k} - v_0 + \frac{g}{k}\mathrm{Ln}\left(\frac{g+kv_0}{g+(g+kv_0)e^{-kt} - g}\right)\right]$$

$$= \frac{-1}{k}\left[\frac{(g+kv_0)e^{-kt}}{k} - \frac{g}{k} - v_0 + \frac{g}{k}\mathrm{Ln}\left(\frac{1}{e^{-kt}}\right)\right]$$

$$= \frac{-1}{k}\left[\frac{(g+kv_0)e^{-kt}}{k} - \frac{g}{k} - v_0 + gt\right]$$

$$= \frac{v_0}{k} - \frac{(g+kv_0)e^{-kt}}{k^2} + \frac{g}{k^2} - \frac{gt}{k}$$

$$= \frac{v_0}{k}(1-e^{-kt}) + \frac{g}{k^2}(1-e^{-kt}) - \frac{gt}{k}$$

$$= \left(\frac{g}{k^2} + \frac{v_0}{k}\right)(1-e^{-kt}) - \frac{gt}{k} \quad ——— ①$$

ii) Horizontally:

$$m\ddot{x} = -mku$$

$$\ddot{x} = -ku$$

$$\frac{du}{dt} = -ku$$

$$\int_0^t dt = \frac{-1}{k}\int_{u_0}^{u} \frac{1}{u}\, du$$

$$t = \frac{1}{k}\mathrm{Ln}\left(\frac{u_0}{u}\right)$$ —— ①

$$e^{kt} = \frac{u_0}{u}$$

$$u = u_0 e^{-kt}$$

$$\frac{dx}{dt} = u_0 e^{-kt}$$

$$x = u_0 \int_0^t e^{-kt}$$

$$= \frac{-u_0}{k}\left(e^{-kt} - 1\right)$$ —— ①

$$\frac{-kx}{u_0} = e^{-kt} - 1$$

$$e^{kt} = \frac{-kx}{u_0} + 1$$

$$e^{kt} = \frac{u_0 - kx}{u_0}$$

$$t = \frac{1}{k}\mathrm{Ln}\left(\frac{u_0 - kx}{u_0}\right)$$ —— ①

Sub into y from i)

$$y = \left(\frac{g}{k^2} + \frac{V_0}{k}\right)\left(1 - e^{\mathrm{Ln}\left(\frac{u_0 - kx}{u_0}\right)}\right) - \frac{g}{k^2}\mathrm{Ln}\left(\frac{u_0}{u_0 - kx}\right)$$

$$= \left(\frac{g}{k^2} + \frac{V_0}{k}\right)\left(1 - \frac{u_0 - kx}{u_0}\right) - \frac{g}{k^2}\mathrm{Ln}\left(\frac{u_0}{u_0 - kx}\right)$$

$$= \left(\frac{g}{k^2} + \frac{V_0}{k}\right)\left(\frac{kx}{u_0}\right) - \frac{g}{k^2}\mathrm{Ln}\left(\frac{u_0}{u_0 - kx}\right)$$ —— ①