Student Number:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2017

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 3 hours.
- Write in black pen.

- Board-approved calculators & templates may be used.
- A Reference Sheet is provided.

- In Questions 11-16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: **10 marks**
- Attempt Questions 1-10.
- Answer on the Multiple Choice answer sheet provided.

- Allow about 15 minutes for this section.

Section II: **90 Marks**
- Attempt Question 11-16.
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate, *stapled* bundles, clearly labeled as Question 11, Question 12, etc.

Each question must show your Candidate Number.

QUESTION ONE

Which of the following describes the conic section given by

$$4x^2 + 7y^2 + 32x - 56y + 148 = 0\,?$$

(A) Ellipse with centre $(-4, 4)$ and foci at $(4 \pm \sqrt{3}, -4)$.

(B) Hyperbola with centre $(-4, 4)$ and foci at $(4, -4 \pm \sqrt{3})$.

(C) Ellipse with centre $(-4, 4)$ and foci at $(-4 \pm \sqrt{3}, 4)$.

(D) Hyperbola with centre $(-4, 4)$ and foci at $(-4, 4 \pm \sqrt{3})$.

QUESTION TWO

If $\sum_{n=0}^{\infty} \cos^{2n}\theta = 5$, what is the value of $\cos 2\theta$?

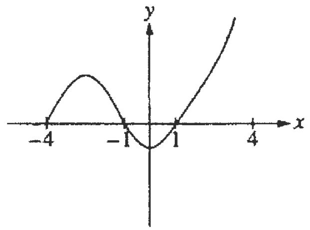
(A) $\frac{4}{5}$

(B) $\frac{3}{5}$

(C) $\frac{2}{5}$

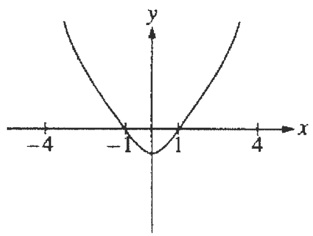
(D) $\frac{1}{5}$

QUESTION THREE

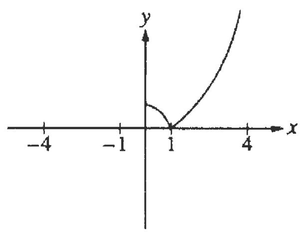


The graph of $y = f(x)$ is shown above. Which of the following is a possible graph of $y = f(|x|)$?

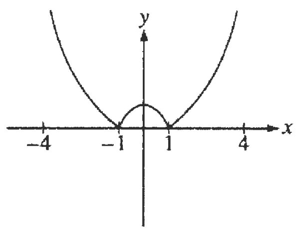
(A)



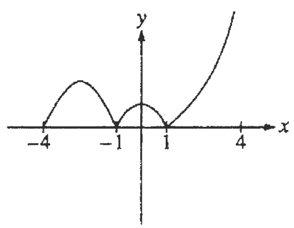
(B)



(C)



(D)



Examination continues on next page ...

QUESTION FOUR

Without evaluating directly, which of the following integrals is positive?

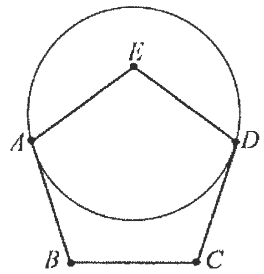
(A) $\displaystyle\int_{-1}^{1} \frac{\sin^{-1} x}{1+x^2}\,dx$

(B) $\displaystyle\int_{-1}^{1} \frac{\cos^{-1} x}{1+x^2}\,dx$

(C) $\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x}{\cos x}\,dx$

(D) $\displaystyle\int_{-1}^{1} (x^2-1)e^{-x^2}\,dx$

QUESTION FIVE



The diagram above shows a regular pentagon $ABCDE$, and a circle of unit radius that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at A. What is the length of the arc AD that is contained within the pentagon?

(A) $\dfrac{2\pi}{5}$

(B) $\dfrac{3\pi}{5}$

(C) $\dfrac{4\pi}{5}$

(D) $\dfrac{6\pi}{5}$

Examination continues on next page ...

QUESTION SIX

Let $f(x) = ax^4 - bx^2 + x + 5$ with $f(-3) = 2$. What is the value of $f(3)$?

(A) 8

(B) 1

(C) −2

(D) −5

QUESTION SEVEN

If the complex number z satisfies $z + |z| = 2(1 + 4i)$, which of the following is $|z|^2$?

(A) 68

(B) 100

(C) 208

(D) 289

Examination continues on next page ...

QUESTION EIGHT

For $xy = e^{xy}$, what is $\dfrac{dy}{dx}$ where y is an implicit function of x?

(A) $\dfrac{x}{y}$

(B) $-\dfrac{x}{y}$

(C) $\dfrac{y}{x}$

(D) $-\dfrac{y}{x}$

QUESTION NINE

A funfair game has the following setup: a player may toss two balls into any of k chutes, and if both balls are returned via a single chute, the player wins. What is the probability that both balls enter via different chutes, but are returned via the same chute, where that chute is neither of the chutes by which any ball entered?

(A) $\dfrac{k^2 - 3k + 2}{k^3}$

(B) $\dfrac{k^3 - 4k^2 + 5k - 2}{k^3}$

(C) $\dfrac{k^2 - k}{k^2 - 5k + 6}$

(D) $\dfrac{k^2 - 2k + 2}{k^4}$

Examination continues on next page ...

QUESTION TEN

Suppose f and g are differentiable functions such that $g(x) > 0$ for all x and $f(0) = 1$.

If $\frac{d}{dx}(f(x)g(x)) = f(x)g'(x)$, which of the following is $f(x)$?

(A) 0

(B) 1

(C) e^x

(D) $g(x)$

QUESTION ELEVEN (15 Marks) Use a SEPARATE writing booklet. Marks

(a) Define $I_n = \int_0^1 x^n e^x \, dx$.

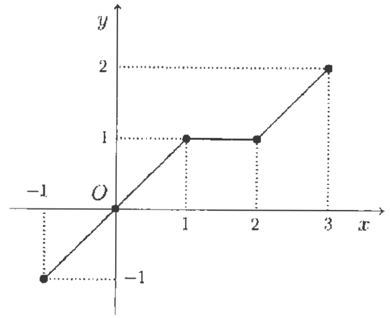
(i) Show that $I_n = e - nI_{n-1}$. 2

(ii) Hence evaluate I_2. 2

(b) (i) Show that $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$. 2

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$, for n a positive integer. 3

(c)



The diagram above shows a piecewise function $f(x)$ comprised of linear segments for all real x over $[-1, 3]$. Sketch, over the real numbers, the graphs of:

(i) $y = \log_e f(x)$ 2

(ii) $y = \frac{1}{f(x)}$ 2

(iii) $y^2 = f(x)$ 2

QUESTION TWELVE (15 Marks) Use a SEPARATE writing booklet. Marks

(a) Solve $z^2 + iz + 1 = 0$. 2

(b) Sketch the region in the Argand plane where 2

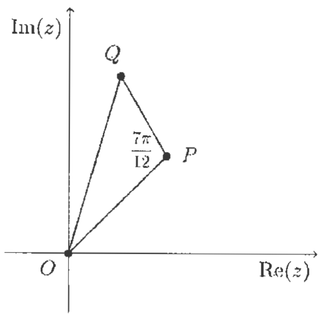
$$1 \leq |z - 2i| \leq 2$$

and

$$-\frac{\pi}{4} \leq \arg(z - 2i) \leq \frac{\pi}{4}$$

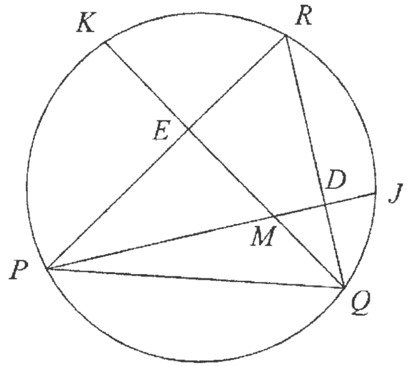
hold simultaneously.

(c) 3



The diagram above shows triangle OPQ in the Argand plane with the point P being represented by the complex number $\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$. If $\angle OPQ = \frac{7\pi}{12}$ and $|PQ| = 2$, find, in Cartesian form, the complex number representing the point Q.

(d)



In the diagram above, C is the circumcircle of triangle PQR. The altitude PD is produced to meet C at J, and the altitude QE is produced to meet C at K. Also, let the altitudes intersect at the point M.

(i) Copy the diagram and state why quadrilaterals $PQDE$ and $REMD$ are cyclic. 2

(ii) Prove that PR bisects $\angle KRM$. 2

(iii) Hence show that $KR = JR$. 2

(e) In how many ways can seven identical cats be put into three identical pens so that all of the pens are occupied? You must state reasoning. 2

QUESTION THIRTEEN (15 Marks) Use a SEPARATE writing booklet. Marks

(a) If $P(x) = x^4 - 8x^3 + 18x^2 - 27$, show that $P(x)$ has a multiple zero and determine its multiplicity. 2

(b) Let $P = (a\sec\theta, b\tan\theta)$ be a point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(i) Show that the equation of the tangent to the hyperbola at the point P is given by 2

$$bx\sec\theta - ay\tan\theta = ab.$$

(ii) Show that the tangent intersects the asymptotes of the hyperbola at the points Q and R where 3

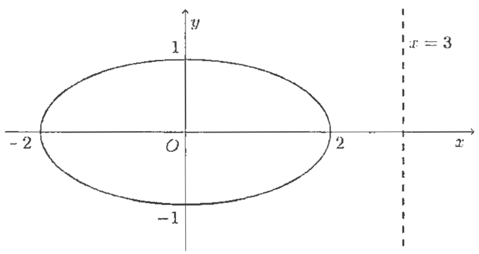
$$Q = (a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta))$$

and

$$R = (a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)).$$

(iii) Hence, for O the origin, show that $|OQ| \cdot |OR|$ is a constant. 2

(c)



The area bounded by the ellipse $x^2 + 4y^2 = 4$ is rotated about the line $x = 3$ to form an oblate torus.

(i) Using the method of cylindrical shells, show that the volume V of the torus is given by the integral 3

$$V = 2\pi \int_{-2}^{2} (3 - x)\sqrt{4 - x^2}\, dx$$

(ii) Hence show that $V = 12\pi^2$. 3

Examination continues on next page ...

QUESTION FOURTEEN (15 Marks) Use a SEPARATE writing booklet. Marks

(a) Define $P(x) = x^3 - 2x^2 + 3x + 2$ and let α, β, γ be the roots of $P(x) = 0$.

(i) By considering $\alpha^2 + \beta^2 + \gamma^2$, explain why $P(x) = 0$ has only one real solution. 1

(ii) Find the monic polynomial having roots $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$. 3

(b) An object of mass m kg is launched vertically upward into a resistive medium. It reaches a maximum height H metres and then descends vertically downward. In both the upward and downward motions, the object experiences a resistive force of magnitude mkv^2 where v m/s is the velocity of the object and $k > 0$ is a constant. The only other force acting on the object is that due to gravity, which exerts an acceleration of magnitude g m/s^2.

(i) If the object is launched with an initial velocity of u m/s, show that the maximum height reached is 3

$$H = \frac{1}{2k}\log_e\left(1 + \frac{k}{g}u^2\right)$$

(ii) After reaching its maximum height, the object descends.

(α) Explain why the terminal velocity of the object is $\sqrt{\dfrac{g}{k}}$. 1

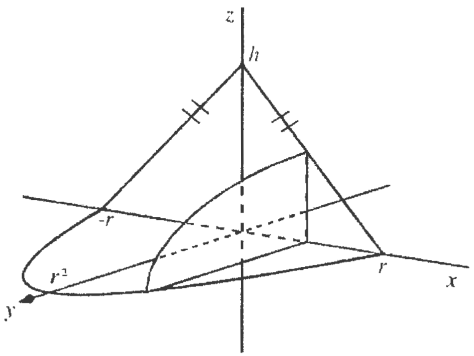
(β) Show that the falling time t is given by 3

$$t = \frac{w}{2g}\log_e\left(\frac{w + v}{w - v}\right)$$

where w is the terminal velocity.

Examination continues on next page ...

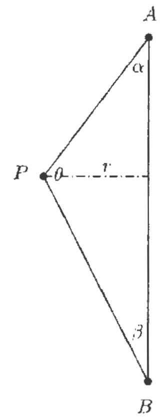
(c) 4



A solid is formed with a parabolic base having equation $y = r^2 - x^2$. A typical vertical cross-section, taken perpendicular to the x–axis, is that of a quarter of an ellipse with one semi-axis bounded by the parabola and the other semi-axis bounded by the side of an isosceles triangle having height h and base length $2r$. Find the volume of the solid given that the area of an ellipse is πab, where a and b are the lengths of the semi-major and semi-minor axes.

QUESTION FIFTEEN (15 Marks) Use a SEPARATE writing booklet. Marks

(a)



The diagram above shows an object of mass m kg fixed at a point P between two light, taut strings AP and BP. The length AB corresponds to a fixed rod in line with the vertical. The mass rotates in a horizontal circle at uniform speed. The construction forms a triangle with $\angle PAB = \alpha$, $\angle PBA = \beta$ and $\angle APB = \theta$. Let the distance from P to AB be r metres, and let the tensions supplied by strings AP and BP be T and U respectively. Also, let ω rad/s be the angular speed and g m/s^2 the magnitude of the acceleration due to gravity.

(i) Show that the tensions supplied by the strings are: 3

$$T = \frac{m(r\omega^2 \cos\beta + g \sin\beta)}{\sin\theta}$$

and

$$U = \frac{m(r\omega^2 \cos\alpha - g \sin\alpha)}{\sin\theta}$$

(ii) For the case where $\theta = \dfrac{\pi}{2}$, show that the system must have 3

$$\frac{\omega^2}{g} \geq \frac{AB}{AP^2}$$

for both strings to remain taut.

(b) Define $P(z) = z^5 - 1$ where z is complex.

(i) Use de Moivre's theorem to find the roots of $P(z) = 0$. 1

(ii) Express $P(z)$:

(α) as a product of real linear and irreducible factors; 2

(β) in the form $P(z) = (z-1)Q(z)$ where $Q(z)$ is a polynomial in z. 1

(iii) Hence show that 2

$$\left(1+\cos\left(\tfrac{2\pi}{5}\right)\right)\left(1+\cos\left(\tfrac{4\pi}{5}\right)\right) = \frac{1}{4}$$

(c) (i) Given 1

$$\frac{1}{3}(x+y+z) \geq (xyz)^{1/3}$$

for all x, y, z non-negative real numbers, show that

$$xy + yz + zx \geq 3(xyz)^{2/3}$$

(ii) Hence show that 2

$$xyz \leq (a-1)^3$$

if $1 + (x+y+z) + (xy+yz+zx) + xyz = a^3$ where $a \geq 2$ is a real number.

Examination continues on next page ...

QUESTION SIXTEEN (15 Marks) Use a SEPARATE writing booklet. Marks

(a) Define the integral I_n by

$$I_n = \int_0^{\pi} f_n(x)\sin x\,dx$$

where n is a non-negative integer and where $f_n(x)$ is an n-times differentiable function on the real numbers such that $f_n^{(k)}(0) = f_n^{(k)}(\pi) = 0$ for all integers $k = 0, 1, 2, \ldots, n$ where $f_n^{(k)}(x)$ is the k^{th} derivative of $f_n(x)$.

(i) Using integration by parts, show that $I_n = -\int_0^{\pi} f_n^{(2)}(x)\sin x\,dx$. 2

(ii) If $f_n(x) = \dfrac{(\pi x - x^2)^n}{n!}$, show that 3

$$I_n = (4n-2)I_{n-1} - \pi^2 I_{n-2}$$

(iii) Use strong induction to prove that for all integers $n \geq 2$, the recursion formula in (ii) generates polynomials in π of the form, 3

$$I_n = \sum_{k=0}^{n} c_k \pi^k$$

having all integer coefficients c_k and with degree at most n. You may assume that $I_0 = 2$ and $I_1 = 4$, and that each iteration generates a polynomial in π that may have differing coefficients to the last. Hence, if I_n is as given above, then I_{n+1} would be of the form $I_{n+1} = \sum\limits_{k=0}^{n+1} d_k \pi^k$ where c_k and d_k are not necessarily equal for $k = 0, 1, 2, \ldots, n$.

(b) This section will make use of the results determined in part (a).
Assume that $\pi = \dfrac{a}{b}$ where a and b are both positive integers, and construct a sequence of numbers z_n where $z_n = b^n I_n$ for n a non-negative integer.

(i) By considering I_n in its polynomial form, explain why z_n must be an integer. 1

(ii) By considering I_n in its integral form, explain why z_n must be positive. 2

Examination continues on next page ...

(iii) It can be easily shown that the integrand of I_n obtains its maximum at $\frac{\pi}{2}$. **2**

Given that if $f(x) \le g(x)$ over an interval $[\alpha, \beta]$, then $\int_\alpha^\beta f(x)\,dx \le \int_\alpha^\beta g(x)\,dx$, show that

$$0 < z_n \le \left(\frac{b\pi^2}{4}\right)^n \frac{\pi}{n!}$$

(iv) Hence given $\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for c a constant, show that π is irrational. **2**

END OF EXAMINATION

MCQ

1	C
2	B
3	A
4	B
5	C
6	A
7	D
8	D
9	A
10	B

MATHEMATICS X2 2017 trial : Question 11

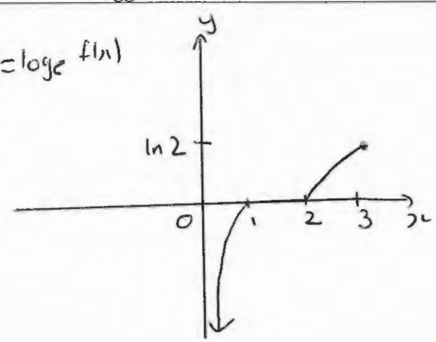
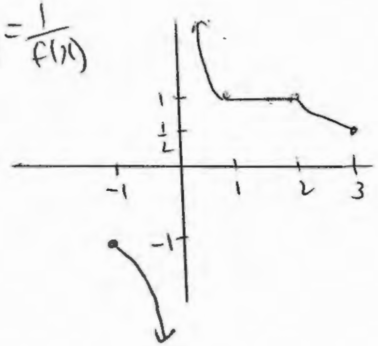
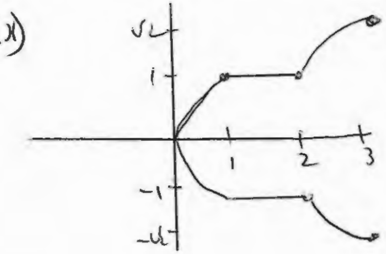
Suggested Solutions	Marks	Marker's Comments
$\int_0^a f(a-x)\,dx = \int_a^0 f(u)\,-du$		
$= \int_0^a f(u)\,du$		
$= \int_0^a f(x)\,dx$ (dummy variable)	1	Fairly well done.
(ii) Let $I = \int_0^{\pi/2} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx$		
$= \int_0^{\pi/2} \frac{\cos^n(\frac{\pi}{2}-x)}{\sin^n(\frac{\pi}{2}-x) + \cos^n(\frac{\pi}{2}-x)}\,dx$		
$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx$	1	
$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx + \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx$		students equated integrals instead of adding them
$= \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\,dx$	1	
$= \int_0^{\pi/2} dx$		
$= \pi/2$		
$\therefore I = \pi/4$.	1	A number of students failed to divide by 2

MATHEMATICS X2 2017 trial : Question 11

Suggested Solutions	Marks	Marker's Comments
$I_n = \int_0^1 x^n e^x\,dx$		
(i) $u = x^n \quad dv = e^x dx$		
$du = nx^{n-1}dx \quad v = e^x$	1	
$I_n = [x^n e^x]_0^1 - n\int_0^1 e^x x^{n-1}\,dx$		
$= (e-0) - nI_{n-1}$		
$= \underline{e - nI_{n-1}}$	1	
(ii) $I_0 = \int_0^1 x^0 e^x\,dx$		
$= \int_0^1 e^x$		
$= e-1$	1	poor algebraic skills cost students easy marks.
$I_1 = e - I_0$		
$= e - e + 1$		
$= 1$		
$I_2 = e - nI_1$		some students left n instead of substituting.
$= e - 2(1)$		
$= e - 2$	1	
b) (i) $\int_0^a f(a-x)\,dx$		
let $u = a-x$ then $x=0 \rightarrow u=a$		
$x = a \rightarrow u = 0$		
$du = -dx$	1	

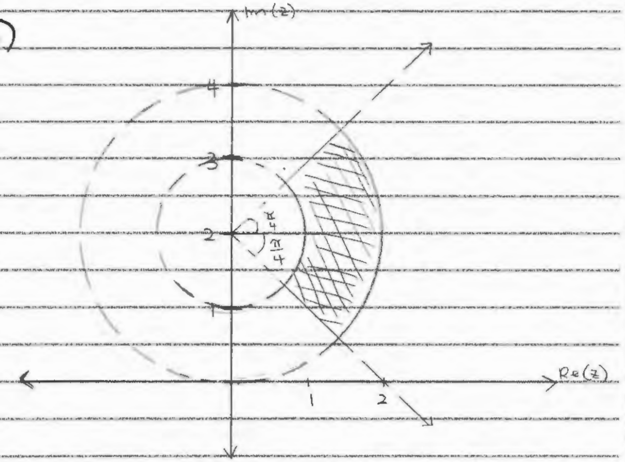
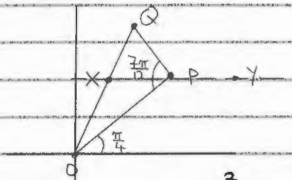
3 of 3

MATHEMATICS Extension 2 : Question 11

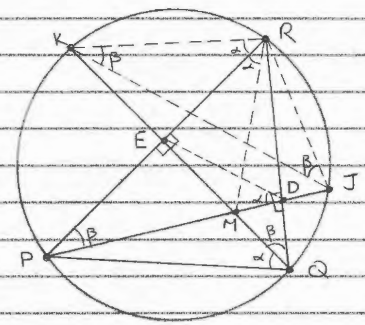
Suggested Solutions	Marks	Marker's Comments
c (i) $y = \log_e f(x)$ 	2	1 for two correct parts 2 for correct with ln 2 and correct concavity from 2 to 3.
(ii) $y = \frac{1}{f(x)}$ 	2	1 for two correct parts 2 for correct with labelled points correct concavity
(iii) $y^2 = f(x)$ 	2	As above ~~some~~ students failed to do reflection concavity from 2 to 3 was 1-correct.

1

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments
a) $z^2 + iz + 1 = 0$ $z = \frac{-i \pm \sqrt{i^2 - 4}}{2}$	1	
$= \frac{-i \pm \sqrt{-5}}{2}$ $= \frac{-i \pm i\sqrt{5}}{2}$ $\therefore z = \frac{-i(1+\sqrt{5})}{2}$ or $\frac{-i(1-\sqrt{5})}{2}$	1	
b) 		1 correct region 1 correct circles & lines with correct labels & scale
c)  $\arg \vec{OP} = \tan^{-1}\left(\frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}}\right)$ $= \frac{\pi}{4}$		

Suggested Solutions	Marks	Marker's Comments
$\therefore \angle XPO = \frac{\pi}{4}$ (alternate angles, parallel lines)		
$\therefore \angle QPX = \frac{7\pi}{12} - \frac{\pi}{4}$		
$= \frac{\pi}{3}$		
$\therefore \angle QPY = \frac{2\pi}{3}$ (angle sum of $\angle XPY$ is π)	1	
$\therefore \overrightarrow{PQ} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$		
$= 2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$		
$= -1 + i\sqrt{3}$	1	
$\therefore \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$		
$= \frac{3}{\sqrt{2}} + \frac{i3}{\sqrt{2}} + (-1 + i\sqrt{3})$		
$= \left(\frac{3}{\sqrt{2}} - 1\right) + i\left(\frac{3}{\sqrt{2}} + \sqrt{3}\right)$	1	
$= \frac{3\sqrt{2}-2}{2} + \frac{3\sqrt{2}+2\sqrt{3}}{2}i$		
OR		
$\lvert\overrightarrow{OP}\rvert = 3$		
$\arg \overrightarrow{OP} = \frac{\pi}{4}$		
$\therefore \overrightarrow{OP} = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$	1	
$\overrightarrow{PQ}$ is $\frac{2}{3}\overrightarrow{PO}$ rotated clockwise by $\frac{7\pi}{12}$		
$\therefore \overrightarrow{PQ} = \frac{2}{3} \times \left(-3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)$		
$\times\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$		
$= -2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$		
$= -1 + i\sqrt{3}$	1	
$\therefore \overrightarrow{OQ} = \overrightarrow{PQ} + \overrightarrow{OP}$		
= as above	1	

Suggested Solutions	Marks	Marker's Comments
d)		
		
i) $\angle PEM = \angle MER = 90°$ (EQ is altitude of ΔPRQ)		
$\angle RDM = \angle QDM = 90°$ (PD is altitude of ΔPQR)		
$\therefore \angle PEQ = \angle PDQ = 90°$		
$\therefore$ PQDE is cyclic (PQ subtends equal angles on the same side at E and D)	1	
$\angle REM + \angle RDM = 180°$		
$\therefore$ REMD is cyclic (opposite angles are supplementary)	1	
ii) let $\angle KRP = \alpha$		
$\angle KRP = \angle KQP = \alpha$ (angles at the circumference standing on the same arc)	1	
Similarly in cyclic quad PQDE,		
$\angle PQE = \angle PDE = \alpha$		
Similarly in cyclic quad REMD,		
$\angle MDE = \angle MRE = \alpha$		
$\therefore \angle KRP = \angle MRE = \alpha$	1	
$\therefore$ PR bisects $\angle KRM$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
iii) let $\angle RKJ = \beta$		Another common proof:
$\angle RKJ = \angle RPJ = \beta$ (angles at the circumference standing on the same arc)	1	$\angle RPJ = \angle KQR$
similarly in cyclic quad PQDE,		$\therefore$ arc KR = arc RJ
$\angle RPJ = \angle EQD = \beta$		(equal angles at the circumference subtend equal arcs) NOT CHORD
similarly in cyclic quad KRQP,		$\therefore KR = RJ$ (equal arcs subtend equal chords)
$\angle EQD = \angle KJR = \beta$		
$\therefore \angle RKJ = \angle RJK = \beta$		
$\therefore KR = JR$ (equal sides are opposite equal angles in ΔKRJ	1	
e) since they are all identical then the only possible ways are:		
1, 1, 5		
1, 2, 4		
1, 3, 3		
2, 2, 3	2	
$\therefore$ 4 ways		

Mathematics Ext2 Question 13.

Suggested Solutions	Marks	Marker's Comments
a) $P(x) = x^4 - 8x^3 + 18x^2 - 27$		
$P'(x) = 4x^3 - 24x^2 + 36x$		
$P''(x) = 12x^2 - 48x + 36$		
$P'''(x) = 24x - 48$		
$P'''(x) = 0 \rightarrow x = 2$		
$P''(x) = 0 \rightarrow 12x^2 - 48x + 36 = 0$		
$x^2 - 4x + 3 = 0$		
$(x-3)(x-1) = 0$		
$\underline{\underline{x=3}}$ or 1		Any time you get $x=3$ as a solution to $P(x)$, $P'(x)$ or $P''(x)$ 1
check $P'(3) = 4(3)^3 - 24(3)^2 + 36(3)$		
$= 108 - 216 + 108$		
$= 0$		
$P(3) = (3)^4 - 8(3)^3 + 18(3)^2 - 27$		
$= 81 - 216 + 162 - 27$		
$= 0$		
$\therefore x = 3$ is the multiple root with multiplicity $\underline{\underline{3}}$		1

Mathematics Ext 2 Question 13

Suggested Solutions	Marks	Marker's Comments

b) i) $x = a\sec\theta, \quad y = b\tan\theta$

$$\frac{dx}{d\theta} = a(\sec\theta\tan\theta) \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$ — 1

$\therefore$ Eqn of tangent:

$$(y - b\tan\theta) = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab\sec^2\theta - ab\tan^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$ — 1

$$bx\sec\theta - ay\tan\theta = ab$$

Mathematics Ext 2 Question 13

Suggested Solutions	Marks	Marker's Comments

ii) Asymptotes at $y = \pm\frac{bx}{a}$ - Eqn 1

Sub Eqn 1 into tangent:

$$\therefore bx\sec\theta - a\left(\pm\frac{b}{a}x\right)\tan\theta = ab$$

$$bx\sec\theta \mp bx\tan\theta = ab$$

$$x\sec\theta \mp x\tan\theta = a$$

$$x = \frac{a}{\sec\theta \mp \tan\theta}$$ — 1

$$= \frac{a(\sec\theta \pm \tan\theta)}{(\sec\theta \mp \tan\theta)(\sec\theta \pm \tan\theta)}$$

$$= a(\sec\theta \pm \tan\theta)$$

When $x = a(\sec\theta + \tan\theta)$

$$y = \frac{b}{a} \times a(\sec\theta + \tan\theta)$$

$$= b(\sec\theta + \tan\theta)$$

$\therefore Q\,[a(\sec\theta + \tan\theta),\ b(\sec\theta + \tan\theta)]$ — 1

When $x = a(\sec\theta - \tan\theta)$

$$y = \frac{-b}{a}(a\sec\theta - \tan\theta)$$

$$= -b(\sec\theta - \tan\theta)$$

$\therefore R\,[a(\sec\theta - \tan\theta),\ -b(\sec\theta - \tan\theta)]$ — 1

Mathematics E2 Question 13

Suggested Solutions	Marks	Marker's Comments

iii) $|OQ| = \sqrt{[a(\sec\theta + \tan\theta) - 0]^2 + [b(\sec\theta + \tan\theta) - 0]^2}$

$= \sqrt{(\sec\theta + \tan\theta)^2 (a^2 + b^2)}$

$= \sqrt{a^2 + b^2}\,|\sec\theta + \tan\theta|$ — 1

$|OR| = \sqrt{[a(\sec\theta - \tan\theta) - 0]^2 + [-b(\sec\theta - b\tan\theta) - 0]^2}$

$= \sqrt{(\sec\theta - \tan\theta)^2 (a^2 + b^2)}$

$= \sqrt{a^2 + b^2}\,|\sec\theta - \tan\theta|$

$\therefore |OQ| \cdot |OR| = (a^2 + b^2)\,|\sec^2\theta - \tan^2\theta|$

$= a^2 + b^2$ — 1

which is a constant

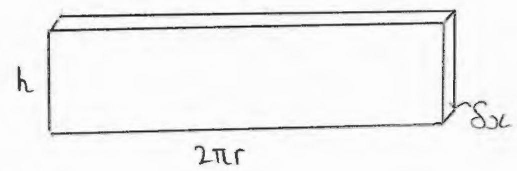
Note: If you found $|OQ|^2 \cdot |OR|^2$ without giving any explanation / conclusion you miss the last mark.

Mathematics E2 Question 13

Suggested Solutions	Marks	Marker's Comments

c)

i) $r = 3 - x$
$h = 2|y|$ — 1



$\therefore V_{Slice} = 2\pi r h \delta x$

$= 2\pi(3-x)(2|y|)\delta x$

But $x^2 + 4y^2 = 4$

$4y^2 = 4 - x^2$

$y^2 = \dfrac{4 - x^2}{4}$

$y = \dfrac{\pm\sqrt{4 - x^2}}{2}$ — 1

$2|y| = \sqrt{4 - x^2}$

$\therefore V_{Slice} = 2\pi(3-x)\sqrt{4-x^2}\,\delta x$

$\therefore V_{Solid} = \lim_{\delta x \to 0} \sum_{x=-2}^{2} 2\pi(3-x)\sqrt{4-x^2}\,\delta x$ — 1

$= 2\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\,dx$

Mathematics EL Question 13		
Suggested Solutions	Marks	Marker's Comments

ii) $V = 2\pi\int_{-2}^{2}(3-x)\sqrt{4-x^2}\,dx$

$= 2\pi\int_{-2}^{2}3\sqrt{4-x^2}\,dx - 2\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$ — 1

$= 6\pi \times \underbrace{\frac{\pi(2)^2}{2}}_{\text{Area of Semi-circle with radius 2}} - 2\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$ — 1

$= 12\pi^2 - 2\pi\int_{-2}^{2}x\sqrt{4-x^2}\,dx$ — 1

$= 12\pi^2 - 0$ Odd function with symmetrical limits

$= 12\pi^2$

MATHEMATICS Extension 2: Question 14		1/5
Suggested Solutions	Marks	Marker's Comments

(a) $P(x) = x^3 - 2x^2 + 3x + 2$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= 2^2 - 2(3)$

$= -2$

If α, β, γ are all real, then $\alpha^2+\beta^2+\gamma^2 > 0$ but this is not true so at least one root must be complex.

The coefficients of $P(x)$ are all rational so the complex roots must occur in conjugate pairs, so there are 2 complex roots and 1 real root. — 1

needed the full explaination to get the mark.

(ii) roots are $\alpha\beta, \beta\gamma, \gamma\alpha$

$\alpha\beta\gamma = -d/a = -2/1 = -2$

$\therefore \alpha\beta = \frac{-2}{\gamma}, \beta\gamma = \frac{-2}{\alpha}, \gamma\alpha = \frac{-2}{\beta}$

sub in $y = -\frac{2}{x}$ ie $x = -\frac{2}{y}$ — 1

$P(-\frac{2}{y}) = (-\frac{2}{y})^3 - 2(-\frac{2}{y})^2 + 3(-\frac{2}{y}) + 2$

$\therefore 0 = -\frac{8}{y^3} - \frac{8}{y^2} - \frac{6}{y} + 2$

$0 = -8 - 8y - 6y^2 + 2y^3$ — 1

$\therefore 0 = -4 - 4y - 3y^2 + y^3$

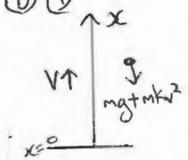
$\therefore P(x) = x^3 - 3x^2 - 4x - 4$ — 1 (dummy variable)

*some students lost the final mark because their polynomial wasn't monic

MATHEMATICS Extension 2: Question 14 cont. 2/5

Suggested Solutions | **Marks** | **Marker's Comments**

(b)(i)



$m\ddot{x} = -mg - mkv^2$ (Newton's 2nd Law)

$\ddot{x} = -g - kv^2$

$v\frac{dv}{dx} = -(g + kv^2)$

$\int_u^0 \frac{v\,dv}{g+kv^2} = \int_0^H -dx$ — (1)

$-\frac{1}{2k}\left[\ln(g+kv^2)\right]_u^0 = \left[x\right]_0^H$

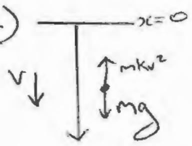
$H - 0 = -\frac{1}{2k}\left[\ln g - \ln(g+ku^2)\right]$ — (1)

$\therefore H = -\frac{1}{2k}\ln\left(\frac{g}{g+ku^2}\right)$

$= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right)$ — (1)

$= \frac{1}{2k}\ln\left(1 + \frac{k}{g}u^2\right)$

(ii)(α)



$m\ddot{x} = mg - mkv^2$

at terminal velocity $\ddot{x} = 0$

$\therefore 0 = mg - mkv^2$

$\cancel{m}g = \cancel{m}kv^2$

$\frac{g}{k} = v^2$

$V_T = \sqrt{\frac{g}{k}}$ — (1)

(β) we need $t = f(v)$ so use $\ddot{x} = \frac{dv}{dt}$

$\therefore \frac{dv}{dt} = g - kv^2$

$\int_0^V \frac{dv}{g-kv^2} = \int_0^t dt$ — (1)

Marker's comment: very badly done and terrible setting out !!!! If students started with $v\frac{dv}{dx}$ or $\frac{d(\frac{1}{2}v^2)}{dx}$ then they scored zero mks.

MATHEMATICS Extension 2: Question 14 ... 3/5

Suggested Solutions | **Marks** | **Marker's Comments**

now $\frac{1}{g-kv^2} = \frac{A}{\sqrt{g}+\sqrt{k}v} + \frac{B}{\sqrt{g}-\sqrt{k}v}$

$1 = A\sqrt{g} + B\sqrt{g}$ ... (1)

$1 = A(\sqrt{g}-\sqrt{k}) + B(\sqrt{g}+\sqrt{k})$ ... (2)

from (1) $1 - B\sqrt{g} = A\sqrt{g}$

$\frac{1}{\sqrt{g}} - B = A$

sub into (2)

$1 = \left(\frac{1}{\sqrt{g}} - B\right)(\sqrt{g}-\sqrt{k}) + B(\sqrt{g}+\sqrt{k})$

$1 = 1 - \frac{\sqrt{k}}{\sqrt{g}} - B\sqrt{g} + B\sqrt{k} + B\sqrt{g} + B\sqrt{k}$

$0 = -\frac{\sqrt{k}}{\sqrt{g}} + 2B\sqrt{k}$

$B = \frac{1}{2\sqrt{g}}$

$A = \frac{1}{\sqrt{g}} - \frac{1}{2\sqrt{g}} = \frac{1}{2\sqrt{g}}$

$\therefore \int_0^V \frac{dv}{g-kv^2} = \frac{1}{2\sqrt{g}}\int_0^V\left(\frac{1}{\sqrt{g}-\sqrt{k}v} + \frac{1}{\sqrt{g}+\sqrt{k}v}\right)dv$ — (1)

Marker's comment: * students who lost their constants lost this mark

$\therefore \frac{1}{2\sqrt{g}}\int_0^V\left(\frac{1}{\sqrt{g}-\sqrt{k}v} + \frac{1}{\sqrt{g}+\sqrt{k}v}\right)dv = \int_0^t dt$

$\frac{1}{2\sqrt{g}\sqrt{k}}\left[-\ln(\sqrt{g}-\sqrt{k}v) + \ln(\sqrt{g}+\sqrt{k}v)\right]_0^V = t - 0$

$\therefore t = \frac{1}{2\sqrt{g}\sqrt{k}}\left[\ln\left(\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right)\right]_0^V$

$t = \frac{1}{2\sqrt{g}\sqrt{k}}\left[\ln\left(\frac{\sqrt{g}+\sqrt{k}V}{\sqrt{g}-\sqrt{k}V}\right) - \ln\left(\frac{\sqrt{g}}{\sqrt{g}}\right)\right]$

$t = \frac{1}{2}\times\frac{\sqrt{g}}{\sqrt{k}}\cdot\frac{1}{g}\times\ln\left(\frac{\sqrt{g}+\sqrt{k}V}{\sqrt{g}-\sqrt{k}V}\right)$

$= \frac{w}{2g}\ln\left(\frac{\frac{\sqrt{g}}{\sqrt{k}}+v}{\frac{\sqrt{g}}{\sqrt{k}}-v}\right)$

$= \frac{w}{2g}\ln\left(\frac{w+v}{w-v}\right)$

Marker's comment: * If you fudged the answer then (1) you lost another mark

(b) $\frac{dv}{dt} = g - kv^2$

4/5

$= k\left(\frac{g}{k} - v^2\right)$

$= k(w^2 - v^2)$ where $w = \sqrt{\frac{g}{k}}$.

$w^2 = \frac{g}{k}$

$k = \frac{g}{w^2}$ $\sqrt{k} = \frac{\sqrt{g}}{w}$

$$\int_0^v \frac{dv}{w^2 - v^2} = \int_0^t k\,dt$$ (1)

Let $\frac{1}{w^2 - v^2} = \frac{A}{w-v} + \frac{B}{w+v}$

$A(w+v) + B(w-v) \equiv 1$

$v = w \Rightarrow A = \frac{1}{2w}$

$v = -w \Rightarrow B = +\frac{1}{2w}$

$$\frac{1}{2w}\int_0^v \frac{dv}{w-v} - \frac{1}{2w}\int_0^v \frac{dw}{w+v} = \int_0^t k\,dt$$ (1)

$$-\frac{1}{2w}\Big[\ln(w-v)\Big]_0^v + \frac{1}{2w}\Big[\ln(w+v)\Big]_0^v = kt$$

$$\frac{1}{2w}\left\{-\ln(w-v) + \ln w + \ln(w+v) - \ln w\right\} = kt$$

$$\frac{1}{2w}\ln\left(\frac{w+v}{w-v}\right) = kt$$ (1)

$$t = \frac{1}{2wk}\ln\left(\frac{w+v}{w-v}\right)$$

$$= \frac{w}{2g}\ln\left(\frac{w+v}{w-v}\right)$$

$\left(\frac{1}{2wk} = \frac{\sqrt{k}}{2\sqrt{g}\,k}\right)$

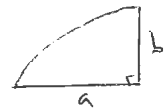
$= \frac{1}{2g}\frac{\sqrt{g}}{\sqrt{k}}$

$= \frac{w}{2g}$

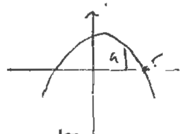
Suggested Solutions	Marks	Marker's Comments

Many methods used.

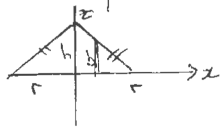
(c)



$V_{slice} = \frac{1}{4}\pi ab\,\Delta x$



$a = r^2 - x^2$ ... (1)



$z = -mx + h$ (m is gradient)

when $z = 0, x = r$

$0 = -mr + h$

$mr = h$ $m = h/r$

$\therefore z = -\frac{h}{r}x + h$

when $z = b$

$\therefore b = -\frac{h}{r}x + h$ ... (2)

sub into V_{slice}

ie $V_{slice} = \frac{\pi}{4}(r^2 - x^2)\left(-\frac{h}{r}x + h\right)\Delta x$ (1)

$= \frac{h\pi}{4}\left(-\frac{x}{r} + 1\right)(r^2 - x^2)\Delta x$

$= \frac{h\pi}{4}\left(r^2 - x^2 - xr + \frac{x^3}{r}\right)\Delta x$

$$V_{solid} = 2 \times \lim_{\Delta x \to 0} \sum_0^r \frac{\pi h}{4}\left(r^2 - x^2 - xr + \frac{x^3}{r}\right)\Delta x$$

*The solid is symmetrical, so total V is twice volume of one half

$$V = \frac{\pi h}{2}\int_0^r \left(r^2 - x^2 - xr + \frac{x^3}{r}\right)dx$$

$= \frac{\pi h}{2}\left[r^2x - \frac{1}{3}x^3 - \frac{1}{2}x^2r + \frac{1}{4r}x^4\right]_0^r$ (1)

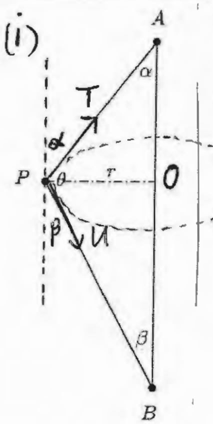
$= \frac{\pi h}{2r}\left[r^4 - \frac{1}{3}r^4 - \frac{1}{2}r^4 + \frac{1}{4}r^4 - 0\right]$

$= \frac{\pi h}{2r} \times \frac{5}{12}r^4 = \frac{5hr^3\pi}{24}$ (1)

JRAHS TRIAL MATHEMATICS Extension 2: Question 15 2017

Suggested Solutions	Marks	Marker's Comments

a) (i)



Horizontally: $\Sigma F_x = ma$

$T\sin\alpha + U\sin\beta = m\omega^2 r$ --- ① 1

Vertically: $\Sigma F_y = 0$

$T\cos\alpha - U\cos\beta - mg = 0$ --- ② 1

① $\times \cos\alpha$ − ② $\times \sin\alpha$:

$$U(\sin\beta\cos\alpha + \cos\beta\sin\alpha) = m\omega^2 r\cos\alpha - mg\sin\alpha$$

$$\therefore U = \frac{m(\omega^2 r\cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$$

$$= \frac{m(\omega^2 r\cos\alpha - g\sin\alpha)}{\sin(\pi-\theta)}$$ 1 Correct Algebraic manipulation

$$= \frac{m(\omega^2 r\cos\alpha - g\sin\alpha)}{\sin\theta}$$

Similarly ① $\times \cos\beta$ + ② $\times \sin\beta$: yields

Substituting ① into ② was the other method:

$$T = \frac{m(\omega^2 r\cos\beta + g\sin\beta)}{\sin\theta}$$

JRAHS TRIAL MATHEMATICS Extension 2: Question 15 2017

Suggested Solutions	Marks	Marker's Comments

(ii) $T > 0$ and $U \geq 0$ must hold

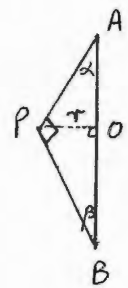
$$\Rightarrow \frac{m(\omega^2 r\cos\alpha - g\sin\alpha)}{\sin\theta} \geq 0$$

But $\theta = \pi/2 \Rightarrow \sin\theta = 1$

1 For all this!

$$\therefore \omega^2 r\cos\alpha - g\sin\alpha \geq 0$$

$$\Rightarrow \frac{\omega^2}{g} \geq \frac{\sin\alpha}{r\cos\alpha}$$ 1 or $\frac{\tan\alpha}{r}$



But $\sin\alpha = \frac{r}{AP}$

and $\cos\alpha = \frac{AP}{AB}$

1 For correctly expressing $\sin\alpha$ and $\cos\alpha$

$$\therefore \frac{\omega^2}{g} \geq \frac{r/AP}{r\ AP/AB}$$

$$= \frac{AB}{AP^2}$$

Suggested Solutions	Marks	Marker's Comments
b) (i) $P(z)=0 \Rightarrow z^5=1$		
$\Rightarrow \cos 5\theta + i\sin 5\theta = 1$ ---- de Moivre's Thm.		
So $\theta = \dfrac{2\pi k}{5}$		
If $k=0,1,2,3,4$: $z_1=1$, $z_2=\text{cis}(2\pi/5)$, $z_3=\text{cis}\left(\frac{4\pi}{5}\right)$, $z_4=\text{cis}(6\pi/5)$, $z_5=\text{cis}(8\pi/5)$ If $k=0,\pm1,\pm2$: $z_1=1$, $z_2=\text{cis}(2\pi/5)$, $z_3=\text{cis}(-2\pi/5)$, $z_4=\text{cis}(4\pi/5)$, $z_5=\text{cis}(-4\pi/5)$	1	For all this!
Note: Conjugate Pairs z_2 and z_5 z_3 and z_4 (for $k=0,\pm1,\pm2$:) z_2 and z_3 z_4 and z_5		Note: $(k=0,\pm1,\pm2)$ $z_2+z_3=2\cos\frac{2\pi}{5}$ $z_2\times z_3=1$ If z_2 and z_3 are conjugate pairs
Check: $\overline{z_2}=\cos 2\pi/5 - i\sin 2\pi/5$ $z_5=\cos(-2\pi/5)+i\sin(-2\pi/5)$ $=\cos 2\pi/5 - i\sin 2\pi/5$ $=\overline{z_2}$		

Suggested Solutions	Marks	Marker's Comments
b ii) α)		
$P(z)=(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$	1	First mark
$=(z-1)\left(z-\text{cis}\frac{2\pi}{5}\right)\left(z-\text{cis}\left(\frac{-2\pi}{5}\right)\right)\left(z-\text{cis}\left(\frac{4\pi}{5}\right)\right)\left(z-\text{cis}\left(\frac{-4\pi}{5}\right)\right)$		
$=(z-1)\left(z^2-2\cos\frac{2\pi}{5}\cdot z+1\right)\left(z^2-2\cos\frac{4\pi}{5}z+1\right)$ irreducible factors	1	For combining conjugate pairs into irreducible factors i.e $(z-w)(z-\overline{w})$ $=z^2-2\text{Re}(w)z+1$ $w\cdot\overline{w}=1$ OR just multiplying out the factors
β)		
$P(z)=(z-1)(z^4+z^3+z^2+z+1)$ $Q(z)\equiv$ polynomial in z	1	

Suggested Solutions	Marks	Marker's Comments

b iii) Hence $\Rightarrow$ use earlier results

Using (α) and (β) : Method I :

$$(z-1)(z^4+z^3+z^2+z+1) = (z-1)\left(z^2-2z\cos\tfrac{2\pi}{5}+1\right)\left(z^2-2z\cos\tfrac{4\pi}{5}+1\right)$$

Choose $z=-1$:

$$(-2)(1-1+1-1+1) = (-2)\left(1+2\cos\tfrac{2\pi}{5}+1\right)\left(1+2\cos\tfrac{4\pi}{5}+1\right)$$

$$1 = \left(2+2\cos\tfrac{2\pi}{5}\right)\left(2+2\cos\tfrac{4\pi}{5}\right)$$

$$= 4\left(1+\cos\tfrac{2\pi}{5}\right)\left(1+\cos\tfrac{4\pi}{5}\right)$$

$$\Rightarrow \left(1+\cos\tfrac{2\pi}{5}\right)\left(1+\cos\tfrac{4\pi}{5}\right) = \frac{1}{4}$$

Method II : (Sum and Product of Roots)

$$z^5-1=0$$

$$z^5+0z^4+0z^3+0z^2+0z^1-1=0$$

$$\Rightarrow a=1,\ b=c=d=e=0,\ f=-1$$

Suggested Solutions	Marks	Marker's Comments

Sum of Roots (one at a time) $= \frac{-b}{a} = 0$

$$\Rightarrow \operatorname{cis}\tfrac{2\pi}{5}+\operatorname{cis}\left(-\tfrac{2\pi}{5}\right)+\operatorname{cis}\left(\tfrac{4\pi}{5}\right)+\operatorname{cis}\left(-\tfrac{4\pi}{5}\right)+1=0$$

$$\Rightarrow \cos\tfrac{2\pi}{5}+\cos\tfrac{2\pi}{5}+\cos\tfrac{4\pi}{5}+\cos\tfrac{4\pi}{5}=-1$$

$$\Rightarrow \boxed{\cos\tfrac{2\pi}{5}+\cos\tfrac{4\pi}{5}=-\tfrac{1}{2}} \quad \text{— } 1$$

Sum of Roots (taken 2 at a time) $= \frac{c}{a} = 0$

Let $\alpha = 2\pi/5$, $\beta = 4\pi/5$

$$\Rightarrow \operatorname{cis}\alpha\operatorname{cis}(-\alpha)+\operatorname{cis}\alpha\operatorname{cis}\beta+\operatorname{cis}\alpha\operatorname{cis}(-\beta)+\operatorname{cis}(-\alpha)\operatorname{cis}(-\beta)$$
$$+$$
$$\operatorname{cis}(-\alpha)\operatorname{cis}\beta+\operatorname{cis}\beta\operatorname{cis}(-\beta)+\operatorname{cis}\alpha\cdot 1+\operatorname{cis}(-\alpha)\cdot 1$$
$$+\operatorname{cis}\beta\cdot 1+\operatorname{cis}(-\beta)\cdot 1 = 0$$

Now $\operatorname{cis}(-\alpha)\cdot\operatorname{cis}(\alpha)=1$

$\operatorname{cis}(-\beta)\cdot\operatorname{cis}(\beta)=1$

and $\operatorname{cis}\alpha+\operatorname{cis}(-\alpha)=2\cos\alpha$

$\operatorname{cis}\beta+\operatorname{cis}(-\beta)=2\cos\beta$

$$\Rightarrow \operatorname{cis}\alpha+\operatorname{cis}(-\alpha)+\operatorname{cis}\beta+\operatorname{cis}(-\beta) = 2(\cos\alpha+\cos\beta) = -1$$

as $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$

Suggested Solutions	Marks	Marker's Comments
Hence		
$1 + \text{cis}\,\beta(\text{cis}\,\alpha + \text{cis}(-\alpha)) + \text{cis}(-\beta)(\text{cis}\,\alpha + \text{cis}(-\alpha)) + 1 - 1 = 0$		
$\Rightarrow [\text{cis}\,\alpha + \text{cis}(-\alpha)][\text{cis}\,\beta + \text{cis}(-\beta)] = -1$		
or $\cos\alpha \cdot \cos\beta = -\frac{1}{4}$		
i.e $\boxed{\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} = -\frac{1}{4}}$ — (2)		
$\therefore \left(1+\cos\frac{2\pi}{5}\right)\left(1+\cos\frac{4\pi}{5}\right)$		
$= 1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5}$		
$= 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{4}\right)$ using (1) & (2) :		
$= \frac{1}{4}$ as required		
Note: Product of all 5 roots will not be useful! Why?		

Suggested Solutions	Marks	Marker's Comments
Method III :		
Put $z = i$ into		
$(z^4 + z^3 + z^2 + z + 1) = \left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$		
i.e. $(1 - i - 1 + i + 1) = \left(-1 - 2i\cos\frac{2\pi}{5} + 1\right)\left(-1 - 2i\cos\frac{4\pi}{5} + 1\right)$		
$1 = -4\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5}$		
giving $\cos\frac{2\pi}{5} \cdot \cos\frac{4\pi}{5} = \frac{-1}{4}$ ---		similar to sum of roots taken 2 at a time
Now use sum of roots taken once which gives $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$		
So it is then easy to show		
$\left(1+\cos\frac{2\pi}{5}\right)\left(1+\cos\frac{4\pi}{5}\right) = \frac{1}{4}$!		

Suggested Solutions	Marks	Marker's Comments
(c) (i) $\frac{1}{3}(x+y+z) \geq (xyz)^{1/3}$ ... given $x, y, z \in \mathbb{R}^+$		
$\Rightarrow \quad x+y+z \geq 3(xyz)^{\frac{1}{3}}$		
Consider groups $x \equiv xy$, $y \equiv yz$, $z \equiv xz$		
$\therefore \quad xy+yz+xz \geq 3(xy.yz.xz)^{\frac{1}{3}}$		
$= 3(x^2.y^2.z^2)^{\frac{1}{3}}$		
$= 3(xyz)^{2/3}$	1	
Hence		
$xy+yz+xz \geq 3(xyz)^{2/3}$		

Suggested Solutions	Marks	Marker's Comments
c (ii)		
$a^3 = (x+y+z)+(xy+yz+xz)+xyz+1$		
$\geq 3(xyz)^{1/3}+3(xyz)^{2/3}+xyz+1$		using (i)
$= 1+3(xyz)^{\frac{1}{3}}+3(xyz)^{2/3}+xyz$		1st mark
$= \left[1+(xyz)^{1/3}\right]^3$		2nd mark
$\Rightarrow \quad a \geq 1+(xyz)^{1/3}$		
or $\quad (xyz)^{1/3} \leq a-1$		
$\Rightarrow \quad xyz \leq (a-1)^3$		
Method II: Consider $xyz-(a-1)^3$		
Method III: By Contradiction i.e Assume $xyz > (a-1)^3$?		

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments

(i) $I_n = \int_0^{\pi} f_n(x)\sin x\,dx$

$= \left[f_n(x)\,(-\cos x)\right]_0^{\pi} + \int_0^{\pi} f_n'(x)\cos x\,dx$

$= f_n(\pi)\times 1 + 1\times f_n(0) + \int_0^{\pi} f_n'(x)\cos x\,dx$ — Marks: 1 — Marker's comments: had to clearly indicate why $\left[f_n(x)\,(-\cos x)\right]_0^{\pi}$ disappeared.

Now $f_n(\pi) = f_n(0) = 0$

$= \int_0^{\pi} f_n'(x)\cos x$

$= \left[f_n'(x)\sin x\right]_0^{\pi} - \int_0^{\pi} f_n''(x)\sin x\,dx$ — Marks: 1

$= 0$

since $f_n'(\pi) = f_n'(0) = 0$

$\therefore I_n = -\int_0^{\pi} f_n''(x)\sin x$

(ii) $f_n(x) = \dfrac{(\pi x - x^2)^n}{n!}$

$f_n'(x) = \dfrac{(\pi x - x^2)^{n-1}}{(n-1)!}(\pi - 2x)$ — Marks: 1

$f_n''(x) = \dfrac{(n-1)(\pi x - x^2)^{n-2}(\pi - 2x)^2}{(n-1)!} - \dfrac{2(\pi x - x^2)^{n-1}}{(n-1)!}$ — Marks: 1 — Marker's comments: A significant number of students lost the $(\pi - 2x)$ term.

MATHEMATICS Extension 2 : Question 16

Suggested Solutions	Marks	Marker's Comments

$f_n''(x) = \dfrac{(\pi x - x^2)^{n-2}}{(n-2)!}(\pi^2 - 4\pi x + 4x^2) - \dfrac{2(\pi x - x^2)^{n-1}}{(n-1)!}$

$= \dfrac{\pi^2(\pi x - x^2)^{n-2}}{(n-2)!} - \dfrac{4(\pi x - x^2)(\pi x - x^2)^{n-2}}{(n-2)!} - \dfrac{2(\pi x - x^2)^{n-1}}{(n-1)!}$

$= \dfrac{\pi^2(\pi x - x^2)^{n-2}}{(n-2)!} - \dfrac{4(\pi x - x^2)^{n-1}(n-1)}{(n-2)!\,(n-1)} - \dfrac{2(\pi x - x^2)^{n-1}}{(n-1)!}$

$= \pi^2 f_{n-2}(x) - 4(n-1)f_{n-1}(x) - 2f_{n-1}(x)$

$= \pi^2 f_{n-2}(x) - (4n-2)f_{n-1}(x)$

so

$I_n = -\int_0^{\pi} f_n''(x)\sin x\,dx$

$= -\int_0^{\pi} \pi^2 f_{n-2}(x)\sin x - (4n-2)f_{n-1}\sin x\,dx$

$= (4n-2)I_{n-1} - \pi^2 I_{n-2}$ — Marks: 1

3

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments
(iii) Base case $I_0 = 2$ $I_1 = 4$ are both polynomials in π $I_2 = (4\times 2 - 2)I_1 - \pi^2 I_0$ $= 24 - 2\pi^2$ which is a polynomial in π.	1	~~Many~~ Some students tried to find I_0 and I_1 but it was given. Students got $24-2\pi^2$ didn't explain its significance
Assume true for $n=k$, $n=k-1$ (1) $I_n = \sum_{k=0}^{n} c_j \pi^k$ $I_{k-1} = \sum_{k=0}^{k-1} d_k \pi^k$		Had to assume true for two cases
$I_{k+1} = (4n-2)I_k - \pi^2 I_{k-1}$ $= (4n-2)\sum_{k=0}^{n} c_k \pi^k - \pi^2 \sum_{k=0}^{n-1} d_k \pi^k$ $= \sum_{k=0}^{n} c_k (4n-2)\pi^k - \sum_{k=0}^{n-1} d_k \pi^{k+2}$	1	Incorrect limit on sigma a common error
All coefficients in summation are (2) integers c_k $d_k \in \mathbb{Z}$ $(4n-2)\in\mathbb{Z}$ Highest possible power on π is $n-1+2 = n+1$ So I_{k+1} is a polynomial in π having all integer coefficients and deg $\le n+1$	1	Needed (1) and (2) for third mark.

4

MATHEMATICS X2 2017 trial : Question 16

Suggested Solutions	Marks	Marker's Comments
(b) $z_n = b^n I_n$ and $\pi = \frac{a}{b} \in \mathbb{Q}^+$ (i) If $I_n = \sum_{k=0}^{n} c_k \pi^k$ is a polynomial in π over $\mathbb{Z}$ then for $\pi = \frac{a}{b}$ $b^n I_n = \sum_{k=0}^{n} c_k b^n \frac{a^k}{b^k}$ $= \sum_{k=0}^{n} c_k a^k b^{n-k}$. Every term is the sum of an integer $c_k, a^k, b^{n-k} \in \mathbb{Z}$ $b^{n-k} \in \mathbb{Z}$ since $n \geq k$. $\therefore z_n$ is an integer	1	needed to state $n \geq k$
(ii) $z_n = b^n I_n$ $= b^n \int_0^{\pi} \left(\frac{\pi x - x^2}{n!}\right)^n \sin x\, dx$.		

MATHEMATICS Extension 2 : Question 16

Suggested Solutions | **Marks** | **Marker's Comments**

Now

$\pi x - x^2 > 0$ for $0 < x < \pi$

$\sin x > 0$ for $0 < x < \pi$ [1]

$\therefore \dfrac{\pi x - x^2}{n!} \sin x > 0$

so $\int_0^{\pi}$ will be positive. [1]

(iii) $f_n(x) = \dfrac{(\pi x - x^2)^n}{n!} \sin x$

$$\le \frac{\left(\pi \times \frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2\right)^n \sin\frac{\pi}{2}}{n!}$$

$$\therefore \int_0^{\pi} \left(\frac{\pi x - x^2}{n!}\right) \sin x \, dx \le \int_0^{\pi} \frac{\left(\frac{\pi^2}{4}\right)^n}{n!} dx$$

$$= \left[\frac{\left(\frac{\pi^2}{4}\right)^n}{n!} \times x\right]_0^{\pi}$$ [1]

$$= \frac{\left(\frac{\pi^2}{4}\right)^n}{n!} \times \pi$$

MATHEMATICS X2 2017 trial : Question 16

Suggested Solutions | **Marks** | **Marker's Comments**

$$z_n = \frac{\left(\frac{b\pi^2}{4}\right)^n \pi}{n!}$$

$$\therefore 0 < z_n \le \left(\frac{b\pi^2}{4}\right) \frac{\pi}{n!}$$ [1]

(iv) Now $\dfrac{\left(\frac{b\pi^2}{4}\right)^n}{n!}$ is of form $\dfrac{e_n^n}{n!}$ from given information

$$\therefore \lim_{n\to\infty} \frac{\left(\frac{b\pi^2}{4}\right)^n}{n!} = 0$$ [1]

Now there exists a value for $n = N$ such that $\dfrac{\left(\frac{b\pi^2}{4}\right)^N}{N!} \pi < \dfrac{1}{2}$

so we have

$$0 < z_n < \frac{\left(\frac{b\pi^2}{4}\right)^N}{N!} < \frac{1}{2}.$$

But z_n is always an integer so contradiction as it can't be an integer. So original assumption π is ~~rat~~ rational is incorrect [1]