Student Name	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2019

MATHEMATICS EXTENSION 2

General Instructions:

- Reading Time: 5 minutes.
- Working time: 3 hours
- Write in black pen.
- NESA approved calculators & templates may be used
- A reference sheet is supplied
- In Question 11 – 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 100

Section I: **10 marks**

- Attempt Question 1 – 10
- Answer on the Multiple-Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II: 90 marks

- Attempt Question 11 - 16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section 1

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1. The distance between the two points z and $-\bar{z}$ in the complex plane is given by

   (A) 2Re(z) (B) 2Im(z) (C) 2Re(z)+ 2Im(z) (D) $2|z|$

2. An object rotates at 40rpm and is moving at 30m/s. The radius of the motion is

   (A) 1.33m (B) 6.37m (C) 7.16m (D) 20m

3. The slope of the curve $2x^3 - y^2 = 7$ at the point where y=-3 is

   (A) -4 (B) -2 (C) 2 (D) 4

4. The equation $x^4 + px + q = 0$ where $p \neq 0$ and $q \neq 0$ has roots α, β, γ and δ . What is the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$?

   (A) -4q (B) $p^2 - 2q$ (C) $p^4 - 2q$ (D) p^4

5. The equation of the conic with eccentricity $\sqrt{2}$ and asymptotes $y=\pm x$ is

   (A) xy=2 (B) $x^2 - y^2 = 4$ (C) xy=1 (D) $\frac{x^2}{4} - y^2 = 1$

6. With a suitable substitution $\int_1^2 x^2\sqrt{2-x}\,dx$ can be expressed as

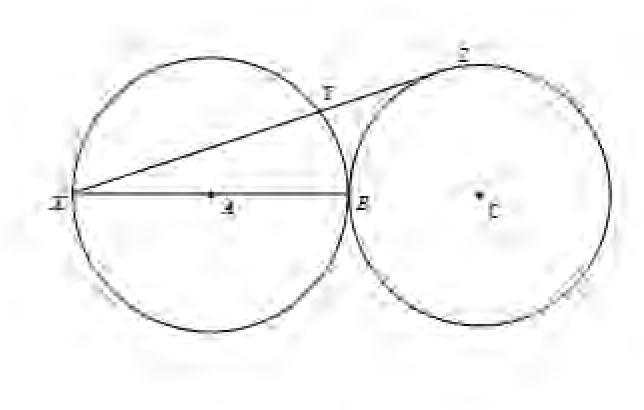
(A) $-\int_1^2 (4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}})\mathrm{du}$ (B) $\int_1^2 (4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}})du$

(C) $\int_0^1 (-4u^{\frac{1}{2}} + 4u^{\frac{3}{2}} - u^{\frac{5}{2}})du$ (D) $-\int_1^0 (4u^{\frac{1}{2}} - 4u^{\frac{3}{2}} + u^{\frac{5}{2}})du$

7. In how many distinct ways can 5 letters be chosen from the letters of the word CAREERS?

(A) 9 (B) 12 (C) 21 (D) 30

8. Two equal circles touch externally at B. XB is a diameter of one circle. XZ is a tangent from X to the other circle and cuts the first circle at Y



Which is the correct expression that relates XZ to XY?

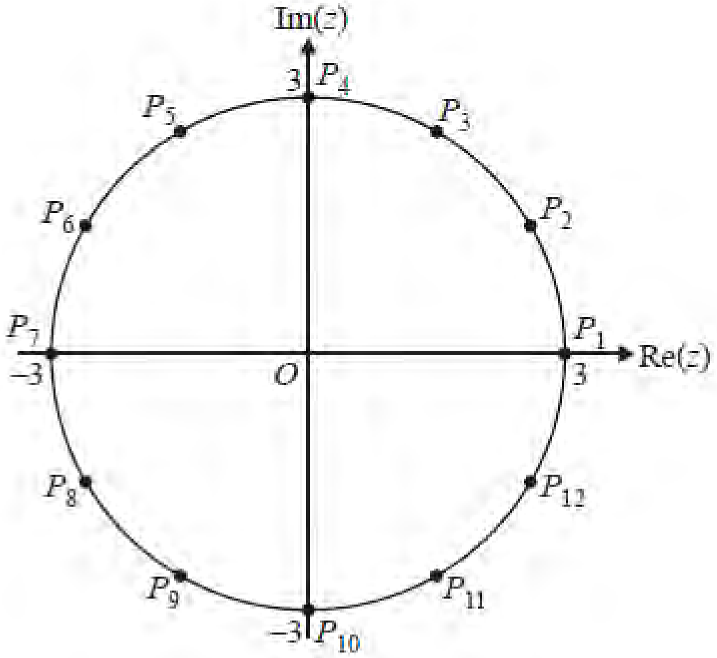
(A) 3XZ =4XY

(B) XZ =2XY

(C) 2XZ=3 XY

(D) 2XZ= 5XY

9. On the argand diagram below, the twelve points $P_1, P_2, P_3, \dots\dots, P_{12}$ are evenly spaced around a circle of radius 3



The points which represent complex numbers such that $z^3 = -27i$ are

(A) P_4 only (B) P_4, P_6, P_{10} (C) P_3, P_7, P_{11} (D) P_4, P_8, P_{12}

10. Let g(x) be a function with first derivative given by $g'(x) = \int_0^x e^{-t^2} dt$. Which of the following must be true on the interval 0< x < 2?

(A) g(x) is increasing and the graph of g(x) is concave up

(B) g(x) is increasing and the graph of g(x) is concave down

(C) g(x) is decreasing and the graph of g(x) is concave up

(D) g(x) is decreasing and the graph of g(x) is concave down

Section II

90 Marks

Attempt Question 11-16

Allow about 2 hours 45 minutes for this section

QUESTION 11 (15 Marks)

(a) Let z = $1 - i\sqrt{3}$ and w = $5 + i\sqrt{3}$

(i) Find z + $\overline{w}$ 1

(ii) Express z in modulus-argument form 2

(iii) Write z^{21} in its simplest form 2

(b) Sketch the region on the Argand diagram defined $|z + 3i| \leq 2|z|$ 3

(c) Find 2

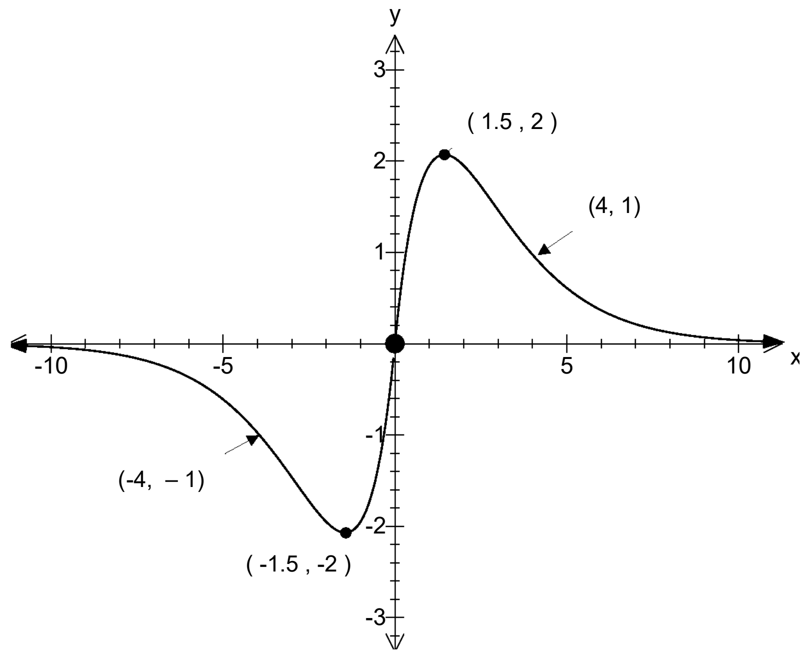
$$\int \frac{dx}{\sqrt{12 + 4x - x^2}}$$

(d) Use the trigonometric substitution $x = 5\sin\theta$ to evaluate 5

$$\int \frac{x^2 dx}{\sqrt{25 - x^2}}$$

QUESTION 12 (15 Marks) Start a new page

(a) The diagram shows the graph of $y = f(x)$



Without any use of Calculus, draw careful sketches of the following curves showing all intercepts, asymptotes and turning points

(i) $y = (f(x))^2$ 1

(ii) $y = f\,'(x)$ 2

(iii) $y = \int f(x)dx$, given that y = 0 when x = 0 2

(iv) y = x+ f(x) 2

(b) (i) On the same axes sketch the graphs of $y = \sqrt{1 - x^2}$ and $y = \frac{1}{\sqrt{1-x^2}}$. 2

(ii) The region bounded by the curve $y = \frac{1}{\sqrt{1-x^2}}$ the coordinate axes and the line $x = \frac{1}{2}$ is rotated through one complete revolution about the line $x = 6$. Use the method of cylindrical shells to show that the volume, V units^3, of the solid of revolution is given by 3

$$V = 2\pi \int_0^{\frac{1}{2}} \frac{6 - x}{\sqrt{1 - x^2}} dx$$

(iii) Hence find the value of V in exact simplest form. 3

QUESTION 13 (15 Marks) Start a new page

(a) The roots of $x^3 + 3px + q = 0$ are α, β and γ (none of which are equal to 0).

(i) Find the monic equation with roots $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}$ and $\frac{\alpha\gamma}{\beta}$, giving the coefficients in terms of p and q. 4

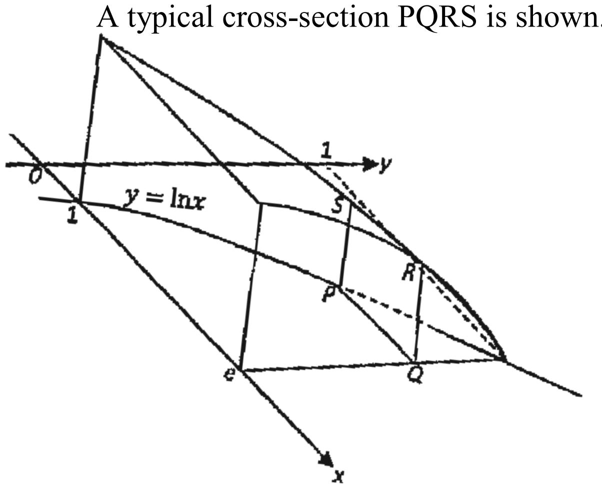
(ii) Deduce that if $\gamma = \alpha\beta$ then $(3p - q)^2 + q = 0$ 2

(b) (i) Draw a sketch of the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ showing foci,vertices and directrices. 2

(ii) Prove that the equation of the asymptotes to the above curve are $y = \pm \frac{bx}{a}$ 2

(iii) Prove that the directrices meet the asymptotes on the auxiliary circle. 2

(c) The base of a solid is the region bounded by the curve y=ln(x),the x-axis,and the lines $x = 1$ and $x = e$, as shown in the diagram. 3
Vertical cross-sections taken through this solid in a direction parallel to the x-axis are squares.
A typical cross-section PQRS is shown. Find the volume of the solid.



QUESTION 14 (15 Marks) Start a new page

(a) For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution? 3

(b) A wire of length *h* metres, attached to the top of a pole of height $2h$ metres, suspends at its end a ball of mass *m* that is rotating about the pole. The situation assumes no air resistance and *g*, in ms^{-2}, is the gravitational acceleration at the surface of the Earth.

(i) Draw a diagram of the situation, showing the forces acting on the mass and write down expressions for the vertical and radial forces. 2

The wire can support a weight no more than twice that of the ball. If the speed of rotation is steadily increased to v ms^{-1}, whereupon the wire breaks:

(ii) Show the height of the ball above the ground at this time is $\frac{3h}{2}$ metres. 3

(iii) Show that the speed of the ball at this time is given by $v = \sqrt{\frac{3gh}{2}}$. 3

(iv) Show that the horizontal distance the ball is from the breakpoint, where the ball is connected to the wire, when it hits the ground is given by $\frac{3\sqrt{2}h}{2}$ metres. 4

QUESTION 15 (15 Marks) Start a new page

(a) Let $I_n = \int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n} dx$, where *n* is a positive integer.

(i) Find the value of I_1. 1

(ii) Using integration by parts show that $I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$ 4

(iii) Hence evaluate $I_3 = \int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^3} dx$ 2

(b) A hotel has four vacant rooms. Each room can accommodate a maximum of four people. In how many ways can six people be accommodated in the four rooms? 2

(c) A sequence of numbers T_n, n=1,2,3…. is defined by $T_1 = 2$ and $T_2 = 0$ and 6

$$T_n = 2T_{n-1} - 2T_{n-2} \text{ for } n=3,4,5,\ldots$$

Use mathematical induction to show that $T_n = (\sqrt{2})^{n+2} \cos\frac{n\pi}{4},\ where\ n = 1, 2, 3, \ldots..$

QUESTION 16 (15 Marks) Start a new page

(a) On the Argand diagram P represents the complex number z and R represents the number 1/z. A square PQRS is drawn with PR as a diagonal. If P lies on the circle $|z| = 2$, prove that Q, represented by X+iY, will lie on an ellipse whose equation has the form: 6

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

and specify numerical values for a and b.
(The X-Y axes do not need to be parallel to the x-y axes.)

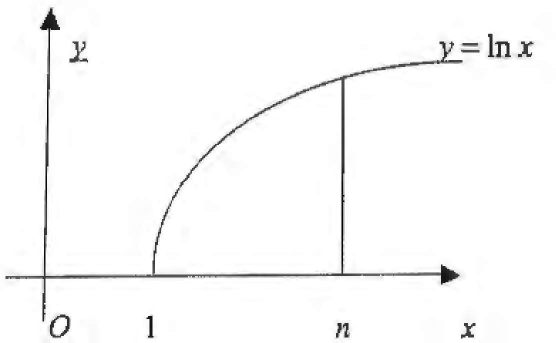
(b) (i) If a and b are positive numbers show that 1

$$\frac{a+b}{2} \geq \sqrt{ab}$$

(ii) Hence, or otherwise, if $a + b = 1$ prove that 2

$$a^2 + b^2 \geq \frac{1}{2}$$

(c)



(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x \, dx$ 2

(ii) Show that $\frac{d}{dx}(x \ln x - x) = \ln x$ and hence find the exact value of $\int_1^n \ln x \, dx$ 2

(iii) Deduce that $\ln(n!) < \left(n + \frac{1}{2}\right)\ln(n) - n + 1$ 2

Q11 Ext 2 TRIAL 2019

Q11 a b /8

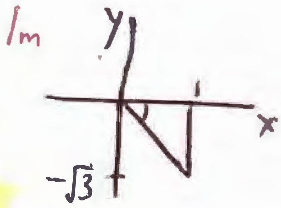
c d /7

a) $z = 1 - i\sqrt{3}$ $w = 5 + i\sqrt{3}$ $\bar{w} = 5 - i\sqrt{3}$

1m i) $z + \bar{w} = 6 - 2i\sqrt{3}$

2m ii) $z = 2, \text{cis} -\frac{\pi}{3}$ 1

$z = 2(\cos\frac{\pi}{3} + i\sin -\frac{\pi}{3})$



2m iii) $z^{21} = 2^{21}\text{cis}(-\frac{\pi}{3} \times 21)$ (De Moivre's Th.) 1

$= 2^{21}\text{cis} -7\pi$

$= 2^{21}(\text{cis}\,\pi)$

$= -2^{21} = -2097152$ 1

3m b) $|x + (y+3)i| \le 2|x + iy|$

$\sqrt{x^2 + (y+3)^2} \le 2\sqrt{x^2 + y^2}$ 1

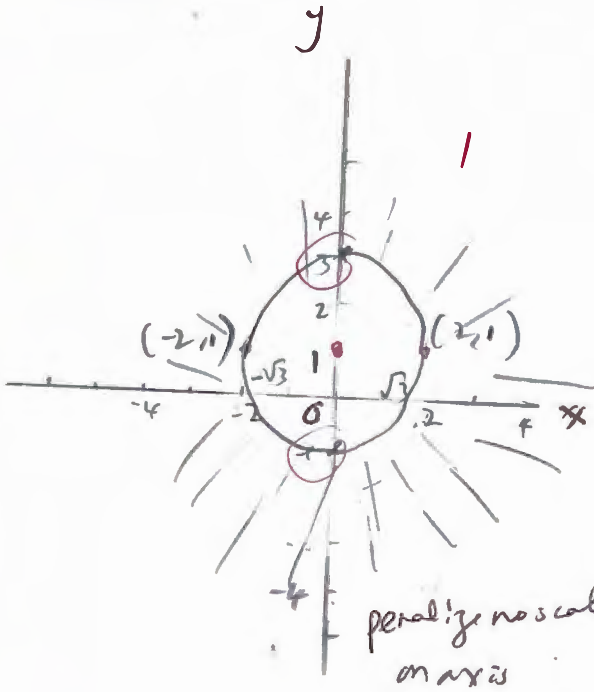
$x^2 + (y+3)^2 \le 4x^2 + 4y^2$

$x^2 + y^2 + 6y + 9 \le 4x^2 + 4y^2$

$0 \le 3x^2 + 3y^2 - 6y - 9$

$0 \le x^2 + y^2 - 2y - 3$

$4 \le x^2 + (y-1)^2$ 1



penalize no scale on axis

ignore x-intercepts

2m c) $\int \frac{dx}{\sqrt{-(x^2 - 4x - 12}}$

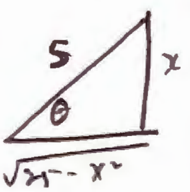
$= \int \frac{dx}{\sqrt{16 - (x^2 - 4x + 4)}} = \int \frac{dx}{\sqrt{4^2 - (x-2)^2}} = \sin^{-1}\left(\frac{x-2}{4}\right) + c$

5m d) $x = 5\sin\theta$ $dx = 5\cos\theta\, d\theta$ $25 - x^2 = 25 - 25\sin^2\theta = 25\cos^2\theta$

$\therefore \sqrt{25 - x^2} = 5\cos\theta$

$\int \frac{x^2\,dx}{\sqrt{25 - x^2}} = \int \frac{25\sin^2\theta\, \cancel{5}\cancel{\cos\theta}\, d\theta}{\cancel{5}\cancel{\cos\theta}} = 25\int \sin^2\theta\, d\theta = \frac{25}{2}\int 1 - \cos 2\theta\, d\theta$

$= \frac{25}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) = \frac{25}{2}\theta - \frac{25\sin\theta\cos\theta}{2}$



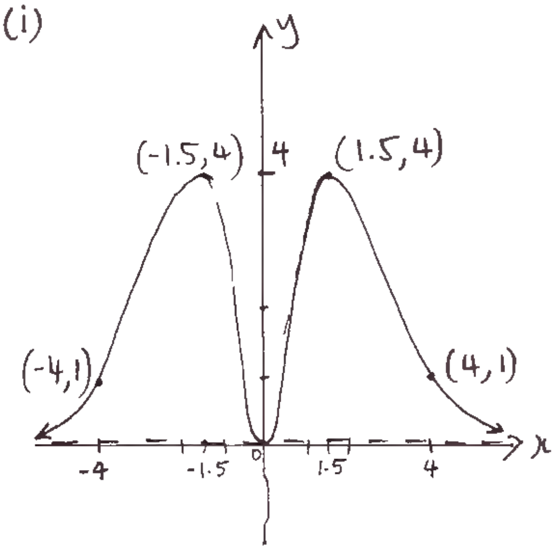
$= \frac{25}{2}\left[\sin^{-1}\frac{x}{5} - \frac{x(\sqrt{25 - x^2})}{25}\right] + c$

or $= \frac{25}{2}\sin^{-1}\frac{x}{5} - \frac{x}{2}\sqrt{25 - x^2} + c$

$\frac{25}{2}\left(\sin^{-1}\frac{x}{5}\right.$ $- \frac{\sin(2\sin^{-1}?}{2}$

MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments
a) (i)	1	Students should note that since points are given on original curve – all further graphs should also show coordinates of points which can be found



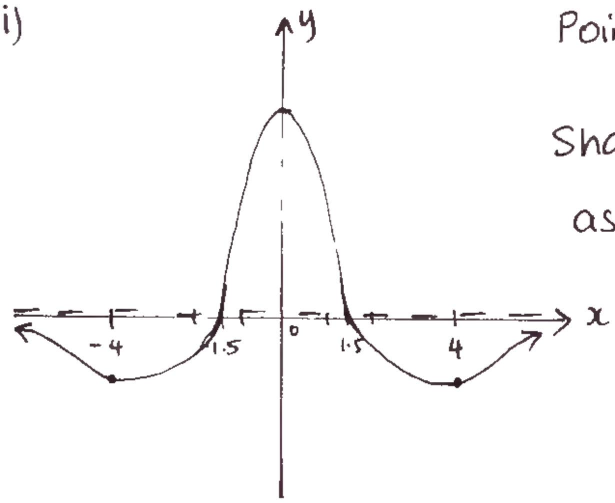
Points $(\pm 1.5, 4)$

$(\pm 4, 1)$

Shape: Rounded at $(0,0)$

Asymptote $y=0$.

(ii) — Marks: 2



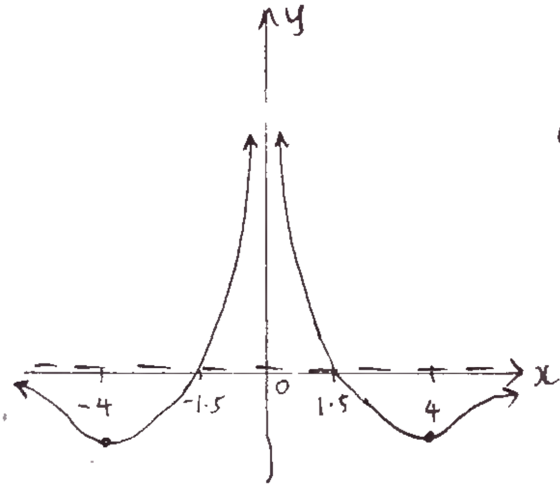
Points $(\pm 1.5, 0)$

$(\pm 4, f'(4))$

Shape $f'(0) > f'(4)$

asymptote $y=0$

OR

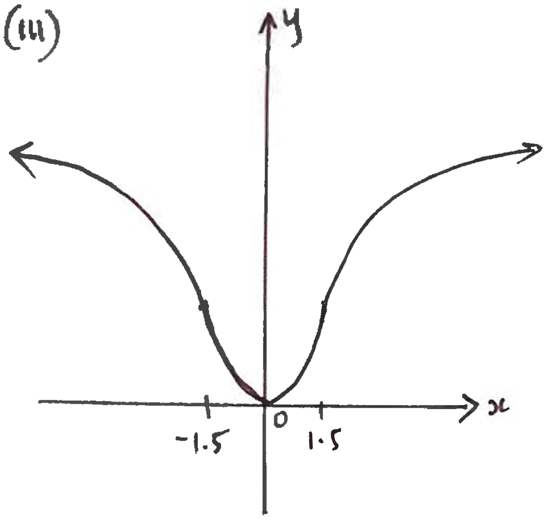


OR $f'(0) \to \infty$

MATHEMATICS Extension 2: Question ___

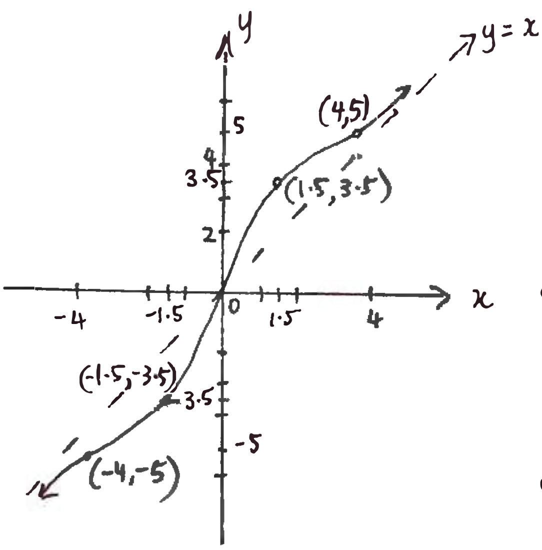
Suggested Solutions	Marks	Marker's Comments

(iii)



Points of inflection $x = \pm 1.5$
shape: Rounded at $(0, 0)$

Marks: 2

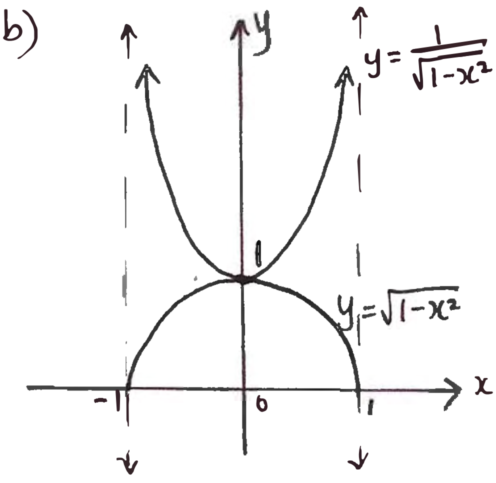


Symmetrical
Graph is an increasing function
Points:

- $(\pm 1.5, \pm 3.5)$ Both are turning points with respect to $y = x$.
- $(\pm 4, \pm 5)$ Both are POIs.

Marks: 2

b)

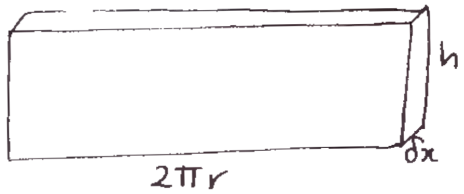


semi-circle
$\therefore$ x & y axes should have same scale

Marks: 2

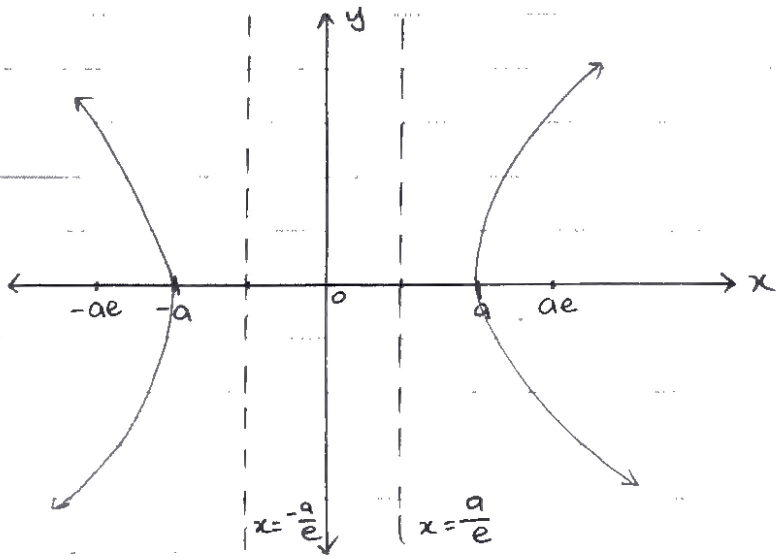
Marker's Comments: Students should make sure point of intersection at $(0, 1)$ is clear.

MATHEMATICS Extension 2: Question 12

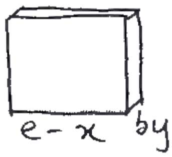
Suggested Solutions	Marks	Marker's Comments
(ii)  $r = 6-x$	1	diagram
$h = y = \frac{1}{\sqrt{1-x^2}}$	1	r and h
$\delta V = 2\pi r h \delta x$ $= 2\pi(6-x)y\delta x$ $= 2\pi(6-x)\frac{1}{\sqrt{1-x^2}}\delta x$ $V = \lim_{\delta x \to 0}\sum_{x=0}^{1/2}\delta V$		
$= \lim_{\delta x \to 0}\sum_{x=0}^{1/2}\frac{2\pi(6-x)}{\sqrt{1-x^2}}$	1	limiting som
$= \int_0^{1/2}\frac{2\pi(6-x)}{\sqrt{1-x^2}}dx$		
(iii) $V = 2\pi\int_0^{1/2}\frac{6}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\,dx$		
$= 2\pi\left[6\sin^{-1}x + \sqrt{1-x^2}\right]_0^{1/2}$	1	
$= 2\pi\left(6\sin^{-1}\frac{1}{2} + \sqrt{\frac{3}{4}} - 6\sin^{-1}0 - \sqrt{1}\right)$	1	
$= 2\pi\left(6 \times \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1\right)$		
$= 2\pi^2 + \sqrt{3}\pi - 2\pi$ cubic units	1	

MATHEMATICS Extension 2: Question 13		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $x^3 + 3px + q = 0$		
$\alpha + \beta + \gamma = 0$		
$\alpha\beta + \alpha\gamma + \beta\gamma = 3p$		
$\alpha\beta\gamma = -q$		
i) For polynomial with roots $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}$:		
let $y = \frac{\alpha\beta}{\gamma} = \frac{\alpha\beta\gamma}{\gamma^2} = \frac{-q}{\gamma^2}$		
$\gamma^2 = -\frac{q}{y}$		
$\therefore \gamma = \pm\sqrt{\frac{-q}{y}}$	1	
sub. γ into $x^3 + 3px + q = 0$		
$\left(\pm\sqrt{\frac{-q}{y}}\right)^3 + 3p\left(\pm\sqrt{\frac{-q}{y}}\right) + q = 0$		
$\pm\sqrt{\frac{-q}{y}}\left(\left(\pm\sqrt{\frac{-q}{y}}\right)^2 + 3p\right) = -q$		
$-\frac{q}{y}\left(\frac{-q}{y} + 3p\right)^2 = q^2$		
$-q\left(\frac{q^2}{y^2} - \frac{6pq}{y} + 9p^2\right) = q^2 y$	1	
$\frac{q^2}{y^2} - \frac{6pq}{y} + 9p^2 = -qy$		
$q^2 - 6pqy + 9p^2y^2 = -qy^3$		
$qy^3 + 9p^2y^2 - 6pqy + q^2 = 0$	1	
$\therefore$ monic polynomial is: $y^3 + \frac{9p^2}{q}y^2 - 6py + q = 0$		
$x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0$ (as y is a dummy variable)	1	

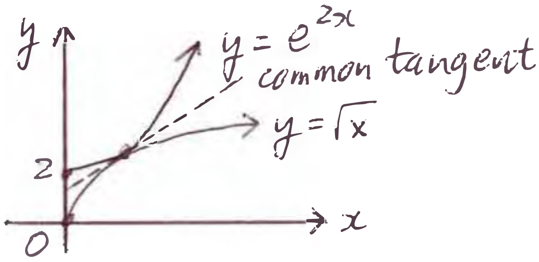
MATHEMATICS Extension 2: Question 13

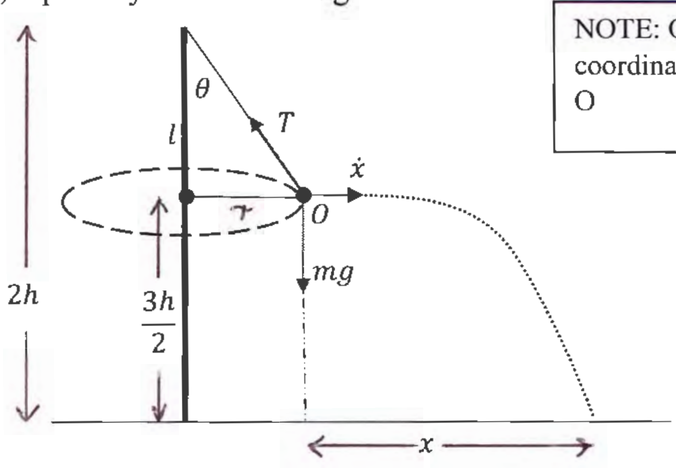
Suggested Solutions	Marks	Marker's Comments
ii) if $\gamma = \alpha\beta$		
roots are: $\frac{\alpha\beta}{\alpha\beta}$, $\frac{\alpha\beta^2}{\alpha}$, $\frac{\alpha^2\beta}{\beta}$		
$= 1, \beta^2, \alpha^2$	1	
sub. 1 into polynomial.		
$(1)^3 + \frac{9p^2}{q}(1)^2 - 6p(1) + q = 0$		
$q + 9p^2 - 6pq + q^2 = 0$		
$q + (3p)^2 - 2(3p)q + q^2 = 0$	1	
$q + (3p-q)^2 = 0$		
$\therefore (3p-q)^2 + q = 0$		
b) i) 		1 foci & directrices 1 vertices & shape
ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$		
$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$		
$y^2 = \frac{b^2x^2}{a^2} - b^2$	1	
as $x \to \infty$, $\frac{b^2x^2}{a^2} - b^2 \to \frac{b^2x^2}{a^2}$	1	
$\therefore y^2 = \frac{b^2x^2}{a^2}$		
$\therefore y = \pm\frac{bx}{a}$ are asymptotes.		

MATHEMATICS Extension 2: Question 13

Suggested Solutions	Marks	Marker's Comments
iii) when $x = \pm \frac{a}{e}$, $y = \pm \frac{b}{e}$	1	
For auxiliary circle: $x^2 + y^2 = a^2$		
LHS $= \left(\pm \frac{a}{e}\right)^2 + \left(\pm \frac{b}{e}\right)^2$		
$= \frac{a^2}{e^2} + \frac{b^2}{e^2}$		
for hyperbola: $b^2 = a^2(e^2-1)$		
$= \frac{a^2 + a^2(e^2-1)}{e^2}$		
$= \frac{a^2 + a^2e^2 - a^2}{e^2}$	1	
$= \frac{a^2e^2}{e^2}$		
$= a^2$		
$=$ RHS		
$\therefore$ points satisfy the auxiliary circle		
$\therefore$ directrices meet the asymptotes on the auxiliary circle.		
c) 		
$y = \ln x$		
$\therefore x = e^y$		
$V = \lim_{\delta y \to 0} \sum_{y=0}^{1} (e - e^y)^2 \delta y$		
$= \int_0^1 (e^2 - 2e \times e^y + e^{2y})\,dy$	1	
$= \int_0^1 (e^2 - 2e^{y+1} + e^{2y})\,dy$		
$= \left[e^2 y - 2e^{y+1} + \frac{1}{2}e^{2y}\right]_0^1$	1	
$= e^2 - 2e^2 + \frac{1}{2}e^2 - (0 - 2e + \frac{1}{2})$		
$= -\frac{1}{2}e^2 + 2e - \frac{1}{2}\ u^3$	1	

Extension 2 **Question 14**	**TRIAL**	**Term 3 2019**	**JRAHS**

(a) 		
$e^{2x} = k\sqrt{x}$		
$\Rightarrow \frac{d}{dx}(e^{2x}) = \frac{d}{dx}\left(k\sqrt{x}\right)$ . . . (i)	1	For equating the derivatives
$\Rightarrow 2e^{2x} = \frac{k}{2\sqrt{x}}$ . . . (ii)		
Solving: (i) / (ii): $\frac{1}{2} = \frac{\sqrt{x}}{\frac{1}{2\sqrt{x}}}$		
Yields: $x = \frac{1}{4}$	1	Correct value of x
Solve for k: $k = 2\sqrt{e}$	1	Correct value of k
Less obvious and less common methods: (α) Showing when $y = e^{2x} - k\sqrt{x}$ is increasing and decreasing, and when stationary ($x = \frac{1}{4}$) (β) Considering $y = 2x - \ln\sqrt{x} = \ln k$ could also get the desired results		**NOTE**: Almost 30% of students had no idea what to do Many recognised equal roots, or tangents and attempted to use the discriminant (Δ), which was fruitless as no quadratic or cubic could be obtained from the given informtaion

(b) Here, students enjoyed great success !!!! Important to have a decent diagram, showing the various quantities involved, especially when resolving forces: NOTE: Origin of coordinate system, O 		
(i) <u>Resolving forces</u>: Vertically: $mg = T\cos\theta$	1	For vertical force
Radially : $\frac{mv^2}{r} = T\sin\theta$ OR: $mw^2r = T\sin\theta$ when $v = wr$	1	For radial force
(ii) At breakpoint, $T = 2mg$ Hence $mg = 2mg\cos\theta$	1	For correct use of tensile strength condition ($T \leq 2mg$
$\Rightarrow \cos\theta = \frac{1}{2}$ Clearly, from the diagram $\cos\theta = l/h$, from which $l = \frac{h}{2}$	1	For correctly obtaining $\cos\theta = \frac{1}{2}$
Hence the height above ground is $2h - \frac{h}{2}$ which gives $\frac{3h}{2}$	1	For subtracting to get $\frac{3h}{2}$
(iii) Dividing the two equations in (i) gives $$v^2 = rg\tan\theta$$	1	Obtaining an expression for v^2
But $\tan\theta = \frac{\frac{3}{2}h}{r}$ With $\theta = \frac{\pi}{3}$, $r = \frac{\sqrt{3}}{2}h$	1	For correct r
Substituting: $v^2 = \frac{\sqrt{3}}{2}hg\left(\frac{\frac{3}{2}h}{r}\right)$ $\Rightarrow v = \sqrt{\frac{3gh}{2}}$	1	For algebraic manipulation

<table>
<tr><td>(iv) Horizontal projectile motion: $(\ddot{x} = 0 \ \ \& \ \ \ddot{y} = -g)$

$\ddot{x} = 0$
$\Rightarrow \dot{x} = c$

But at $t = 0, \dot{x} = \sqrt{\frac{3gh}{2}} = c$
$\Rightarrow x = \sqrt{\frac{3gh}{2}}t + k$

But at $t = 0, x = 0 \Rightarrow k = 0$

$\therefore x = \sqrt{\frac{3gh}{2}}t \qquad . . . (\alpha)$</td><td>1

1</td><td>For recognising horizontal motion together with the 2 relevant equations of motion.

For getting an expression for the horizontal distance in terms of time (with all the bells and whistles)</td></tr>
<tr><td>$\ddot{y} = -g$

$\Rightarrow \dot{y} = -gt + C$

But $\qquad t = 0, \dot{y} = 0 = C$

$\Rightarrow y = -\frac{1}{2}gt^2 + K$

But $\qquad t = 0, y = 0 = K$

$\therefore y = -\frac{1}{2}gt^2$

But when the ball lands, $y = -\frac{3h}{2} \ \ \Rightarrow t = \sqrt{\frac{3h}{g}}$

Substitute this into (α), gives the horizontal distance from the breakpoint

i.e $x = \sqrt{\frac{3gh}{2}} \times \sqrt{\frac{3h}{g}}$
$= \frac{3\sqrt{gh}}{\sqrt{2}}$
$= \frac{3\sqrt{2}h}{2}$</td><td>1

1</td><td>For time of flight (with all bells and whistles)

For the actual bells and whistles i.e. <u>demonstrating integration with boundary conditions</u></td></tr>
</table>

Q15 TRIAL JR 2019 a/7 bc/8 P1/3

a i) $I_1 = \int_0^{\frac{1}{2}} \frac{dx}{4(\frac{1}{4}+x^2} = \frac{2}{4} \tan^{-1}\left(\frac{x}{\frac{1}{2}}\right)\Big]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}0\right) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}.$ 1m

ii) $I_n = \int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n} = \frac{x}{(1+4x^2)^n}\Big]_0^{\frac{1}{2}} - -\int_0^{\frac{1}{2}} \frac{n\,8x^2dx}{(1+4x^2)^{n+1}}$ 1m integrate by Parts

$dx = dv$, $x = v$

$u = \frac{1}{(1+4x^2)^n}$

$du = \frac{-n\,8x}{(1+4x^2)^{n+1}}$

$I_n = \frac{\frac{1}{2}}{(1+1)^n} + 2n\int_0^{\frac{1}{2}} \frac{4x^2+1-1}{(1+4x^2)^{n+1}}dx$ 1m for $\frac{1}{2^{n+1}}$

$I_n = \frac{1}{2^{n+1}} + 2n\int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^n} - 2n\int_0^{\frac{1}{2}} \frac{dx}{(1+4x^2)^{n+1}}$ separating &

$I_n = \frac{1}{2^{n+1}} + 2nI_n - 2n\,I_{n+1}$ 1m for identifying I_n, I_{n+1}

$\frac{2n\,I_{n+1}}{2n-1} - \frac{1}{(2n-1)2^{n+1}} = I_n$

$\therefore I_n = \frac{2n\,I_{n+1}}{2n-1} + \frac{1}{(1-2n)2^{n+1}}$ 1m

iii) $I_1 = \frac{\pi}{8}$ $\qquad I_{n+1} = \left(\frac{2n-1}{2n}\right)I_n + \frac{1}{2^{n+1}(2\cdot n)}$

$n=1$ $\quad I_2 = \frac{1}{2}I_1 + \frac{1}{2^2\cdot 2} = \frac{1}{2}\cdot\frac{\pi}{8} + \frac{1}{8} = \frac{\pi}{16} + \frac{1}{8}$ 1m

$n=2$ $\quad I_3 = \frac{3}{4}I_2 + \frac{1}{2^3\cdot 4} = \frac{3}{4}\left(\frac{\pi}{16}+\frac{1}{8}\right) + \frac{1}{32}$

$I_3 = \frac{3\pi}{64} + \frac{3+1}{32} = \frac{3\pi}{64} + \frac{1}{8}$ 1m

Q 15

b) $\square\,\square\,\square\,\square$

Each person has 4 rooms to choose from

$\therefore 4^6 = 4096$ ways.

1 m for 4^6 or ${}^6C_5 \cdot {}^4C_1 \cdot {}^3C_1$

However max 4 in a room only

6 people in 1 room $= {}^4C_1 = 4$

5 people in one room, 1 person in different room

$= {}^6C_5 \times {}^4C_1 \times {}^3C_1 = 72$

5 people room last person

$\therefore 4^6 - 4 - 72 = 4020$ ways

1 m correct answer

Alternatively

		Room	People	Swap room	
2 rooms	(4, 2)	${}^4C_2$	${}^6C_4\ {}^2C_2$	$2!$	$= 180$
	(3, 3)	${}^4C_2$	${}^6C_3\ {}^3C_3$		$= 120$
3 rooms	(2, 2, 2)	${}^4C_3$	${}^6C_2\ {}^4C_2\ {}^2C_2$		$= 360$
	(1, 2, 3)	${}^4C_3$	${}^6C_3\ {}^3C_2\ {}^1C_1$	$\times 3!$	$= 1440$
	(1, 1, 4)	${}^4C_3$	${}^6C_4\ {}^2C_1$	$\times \frac{3!}{2!}$	$= 360$
4 rooms	(1, 1, 1, 3)		${}^6C_3 \times {}^3C_1 \times {}^2C_1$	$\times \frac{4!}{3!}$	$= 480$
	(1, 1, 2, 2)		${}^6C_2\ {}^4C_2\ {}^2C_1$	$\times \frac{4!}{2!\,2!}$	$= 1080$

Total 4020 ways

1 m for 1 of the 7 cases correct

c) $n = 1$ $\quad T_1 = (\sqrt{2})^3 \cos\frac{\pi}{4} = 2\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 2$ $\quad$ T

$n = 2$ $\quad T_2 = (\sqrt{2})^4 \cos\left(\frac{\pi}{4}\cdot 2\right) = (\sqrt{2})^4 \cdot 0 = 0$ $\quad$ T $\quad$ 1m

Assume the statement is true up to some integer k,

$k = 1, 2, 3 \ldots$ $\quad$ ie $T_k = (\sqrt{2})^{k+2}\cos\frac{k\pi}{4}$ $\quad$ 1m

RTP for $n = k+1$, $\quad T_{k+1} = (\sqrt{2})^{k+3}\cos\frac{(k+1)\pi}{4}$

$$T_{k+1} = 2(T_k) - 2(T_{k-1})$$

$$= 2\sqrt{2}^{\,k+2}\cos\frac{k\pi}{4} - 2\cdot\sqrt{2}^{\,k+1}\cos\frac{(k-1)\pi}{4} \qquad \text{1m}$$

$$= (\sqrt{2})^{k+3}\left(\sqrt{2}\cos\frac{k\pi}{4} - \cos\frac{(k-1)\pi}{4}\right)$$

$$= (\sqrt{2})^{k+3}\left(\sqrt{2}\cos\frac{k\pi}{4} - \left(\cos\frac{k\pi}{4}\cancelto{\frac{1}{\sqrt{2}}}{\cos\frac{\pi}{4}} + \sin\frac{k\pi}{4}\cancelto{\frac{1}{\sqrt{2}}}{\sin\frac{\pi}{4}}\right)\right) \qquad \text{1m for expansion}$$

$$= (\sqrt{2})^{k+3}\left[\cos\frac{k\pi}{4}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\sin\frac{k\pi}{4}\right)$$

$$= (\sqrt{2})^{k+3}\left(\cos\frac{k\pi}{4}\left(\frac{2-1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\sin\frac{k\pi}{4}\right)$$

$$= (\sqrt{2})^{k+3}\left(\cos\frac{k\pi}{4}\cdot\frac{1}{\sqrt{2}} - \sin\frac{k\pi}{4}\cdot\frac{1}{\sqrt{2}}\right) \qquad \text{1m for simplifying}$$

$$= (\sqrt{2})^{k+3}\left(\cos\frac{k\pi}{4}\cos\frac{\pi}{4} - \sin\frac{k\pi}{4}\cos\frac{\pi}{4}\right)$$

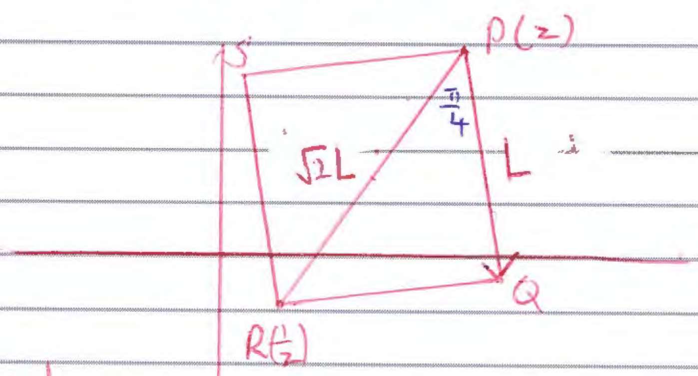
$$= (\sqrt{2})^{k+3}\left(\cos\frac{(k+1)\pi}{4}\right) \qquad \text{1m}$$

$\therefore$ By the Principle of Math Induction,

the statement is true for $k = 1, 2, 3, \ldots$

MATHEMATICS Extension 2: Question 16		
Suggested Solutions	**Marks**	**Marker's Comments**

a)



Method 1

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$$

$$= \overrightarrow{OP} + \overrightarrow{PR} \times \frac{1}{\sqrt{2}} \text{cis}\, \frac{\pi}{4} \quad \text{where } \text{cis}\,\theta = \cos\theta + i\sin\theta$$

$$\overrightarrow{OP} = x+iy, \quad |z| = 2 \Rightarrow x^2+y^2 = 4$$

$$\overrightarrow{OR} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x-iy}{4}$$

$$\therefore \overrightarrow{PR} = \left(\frac{x-iy}{4}\right) - (x+iy)$$

$$= \frac{-3x}{4} - \frac{i5y}{4}$$

$$\therefore \overrightarrow{OQ} = (x+iy) + \left(\frac{-3x}{4} - \frac{5iy}{4}\right)\left(\frac{1}{2} + \frac{1}{2}i\right)$$

$$= (x+iy) + \left(\frac{-3x}{8} + \frac{5y}{8}\right) - i\left(\frac{5y}{8} + \frac{3x}{8}\right)$$

$$= \left(\frac{5x+5y}{8}\right) + i\left(\frac{-3x+3y}{8}\right)$$

$$\therefore = \frac{5}{8}(x+y) + \frac{3i}{8}(-x+y)$$

$$= X + iY$$

MATHEMATICS Extension 2: Question……..

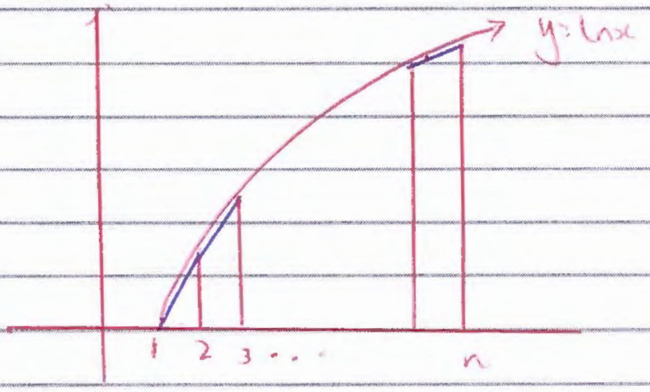
Suggested Solutions	Marks	Marker's Comments
$X = \frac{5}{8}(x+y)$ $\quad Y = \frac{3}{8}(y-x)$		
$\frac{8X}{5} = x+y$ — ① $\quad \frac{8Y}{3} = y-x$ — ②		
$①^2 + ②^2 \rightarrow \frac{64X^2}{25} + \frac{64Y^2}{9} = (x+y)^2 + (y-x)^2$		
$\frac{64X^2}{25} + \frac{64Y^2}{9} = 2(x^2+y^2)$		
$\frac{64X^2}{25} + \frac{64Y^2}{9} = 8$		
$\frac{8X^2}{25} + \frac{8Y^2}{9} = 1$		
$a^2 = \frac{25}{8}$ $\quad b^2 = \frac{9}{8}$		
$a = \frac{5}{2\sqrt{2}}$ $(a>0)$ $\quad b = \frac{3}{2\sqrt{2}}$ $(b>0)$		
① Working out R in terms of x and y		
① Writing an expression for $\overrightarrow{PQ}$ in terms of z		
① Writing an expression for $\overrightarrow{OQ}$ in terms of x and y.		
① Simplify $\overrightarrow{OQ}$		
① Getting the locus ① values for a and b.		

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**
b) i $(\sqrt{a}-\sqrt{b})^2 \geq 0$ $a+b-2\sqrt{ab} \geq 0$ $a+b \geq 2\sqrt{ab}$ $\frac{a+b}{2} \geq \sqrt{ab}$ Note: Must finish with final statement! ii $(a+b)=1 \Rightarrow (a+b)^2=1$ $a^2+b^2+2ab=1$ $a^2+b^2=1-2ab$ However, from i) $\frac{a+b}{2} \geq \sqrt{ab}$ $\frac{1}{2} \geq \sqrt{ab}$ if $a+b=1$ $\frac{1}{4} \geq ab$ $\frac{1}{2} \geq 2ab$ $\therefore a^2+b^2 \geq 1-\frac{1}{2}$ $a^2+b^2 \geq \frac{1}{2}$ What NOT to do: Sub $a+b=1$ in part i) $\rightarrow \frac{1}{2} \geq \sqrt{ab}$ Since $a^2+b^2 \geq 2ab$ (by substituting $a=a^2$ and $b=b^2$) then $a^2+b^2 \geq 2(\sqrt{ab})^2$ $a^2+b^2 \geq 2(\frac{1}{2})^2$ $a^2+b^2 \geq \frac{1}{2}$ ✗		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
c) i $\int_1^n \ln x \, dx$		
$\approx \frac{1}{2}\left[\ln(1) + \ln(n) + 2(\ln(2) + \ln(3) + \ldots \ln(n-1))\right]$	① for correct h or correct expression but wrong h.	
$= \frac{1}{2}\ln(n) + \ln[(n-1)!]$		
$= \ln(n!) - \frac{1}{2}\ln(n)$	① correct simplification.	
ii $\frac{d}{dx}(x\ln x - x) = \left(x \cdot \frac{1}{x} + \ln x - 1\right)$		
$= \ln x$		①
$\therefore \int_1^n \ln x \, dx = \left[x\ln x - x\right]_1^n$		
$= n\ln(n) - n + 1$		①

iii)



It can be seen that the area of the trapezoids sum to a value less than the area of the curve since the curve is concave down.

$\therefore \ln(n!) - \frac{1}{2}\ln(n) < n\ln(n) - n + 1$

$\ln(n!) < (n + \frac{1}{2})\ln(n) - n + 1$

* Full marks <u>ONLY</u> awarded if mentions concave down of $y = \ln x$