Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2020 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Question 11 – 16, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks: 100

Section I – 10 marks

- Attempt Questions 1–10
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section

Section II – 90 marks

- Attempt Questions 11–16
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 2 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate stapled bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section I

10 marks
Attempt Questions 1 to 10
Allow approximately 15 minutes for this section

Answer on the Multiple Choice sheet provided.

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1. Let z be a complex number such that $z^2 = i\bar{z}$. Which of the following is a possible value for z?

(A) $e^{-\frac{i\pi}{3}}$ (B) $e^{\frac{i\pi}{3}}$ (C) $e^{\frac{i\pi}{6}}$ (D) $e^{-\frac{i\pi}{6}}$

2. Consider the following statement, "If you have no treasure, I have no kingdom."

Which of the following is logically equivalent to this statement?

(A) If I have no kingdom then you have no treasure.

(B) If you have treasure then I have a kingdom.

(C) If you have no kingdom then I have no treasure.

(D) If I have a kingdom then you have treasure.

3. Which of the following integrals has the largest value?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx$

(B) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} (1 - \tan x) \, dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \left(1 - \tan^2 x\right) \, dx$

4. The unit vector in the same direction as $\underset{\sim}{u} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is which of the following?

(A) $\dfrac{-1}{36}\begin{bmatrix}-2\\-2\\1\end{bmatrix}$

(B) $\dfrac{1}{3}\begin{bmatrix}2\\2\\-1\end{bmatrix}$

(C) $\dfrac{1}{5}\begin{bmatrix}2\\2\\-1\end{bmatrix}$

(D) $\dfrac{1}{6}\begin{bmatrix}2\\2\\-1\end{bmatrix}$

5. Which of the following is an expression for $\displaystyle\int \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\,dx$?

(A) $x + \dfrac{1}{2}\cos 2x + c$

(B) $x - \dfrac{1}{2}\cos 2x + c$

(C) $x + \dfrac{1}{2}\sin^2 x + c$

(D) $x - \dfrac{1}{2}\sin^2 x + c$

6. ω is a complex cube root of unity and $\omega \neq 1$. What is the value of $\left(1+\dfrac{1}{\omega}\right)^{2020}\left(1+\dfrac{1}{\omega^2}\right)^{2021}$?

(A) $-\dfrac{1}{\omega}$ (B) ω (C) $-\omega$ (D) 1

7. The projection of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$ for $A(4,2,-3)$ and $B(-1,1,1)$ is which of the following?

(A) $\dfrac{5}{3}\begin{bmatrix}1\\-1\\-1\end{bmatrix}$

(B) $\dfrac{5}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$

(C) $\dfrac{5}{3}\begin{bmatrix}4\\2\\-3\end{bmatrix}$

(D) $-\dfrac{5}{3}\begin{bmatrix}4\\2\\-3\end{bmatrix}$

8. If the complex number z satisfies $|z| - z - 4(1-2i) = 0$, which of the following is $|z|^2$?

(A) 80 (B) 180 (C) 100 (D) 400

9. The value of $\dfrac{d}{dx}\left(\displaystyle\int_x^{2x} \frac{1}{1+t^2}\,dt\right)$ is which of the following?

(A) $\dfrac{1}{1+x^2}$ (B) $\dfrac{2}{1+4x^2} - \dfrac{1}{1+x^2}$ (C) $\dfrac{1}{1+4x^2} - \dfrac{1}{1+x^2}$ (D) $\dfrac{1}{1+4x^2} + \dfrac{1}{1+x^2}$

10. Which of the following is the negation of the statement

"$\forall p \in P\,(p$ is of the form $4m+1 \implies p$ can be written as a sum of two squares$)$"?

(A) $\forall p \in P\,(p$ is of the form $4m+1$ and p cannot be written as a sum of two squares$)$

(B) $\exists p \in P\,(p$ is not of the form $4m+1$ and p can be written as a sum of two squares$)$

(C) $\forall p \in P\,(p$ is not of the form $4m+1$ or p cannot be written as a sum of two squares$)$

(D) $\exists p \in P\,(p$ is of the form $4m+1$ and p can not be written as a sum of two squares$)$

End of Section I

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers on the paper supplied.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)

(a) Find $\int \cos^3 x \, dx$. **2**

(b) Use integration by parts to find $\int x \sec^2 x \, dx$. **2**

(c) i. Express $\sqrt{-24-10i}$ in the form $a+ib$. **2**

ii. Hence, solve $z^2-(1-i)z+6+2i=0$. **2**

(d) i. Prove that **2**

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \{f(x)+f(a+b-x)\}\,dx$$

ii. Hence, evaluate **2**

$$\int_3^7 \frac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$$

(e) i. Sketch on the Argand diagram the set of all points satisfying $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$. **2**
Show all important features.

ii. On your diagram in part (i) shade the region where $\frac{\pi}{4} \le \arg\left(\frac{z-1}{z+1}\right) \le \frac{\pi}{2}$. **1**
Show all important features.

End of Question 11

Question 12 (15 Marks)

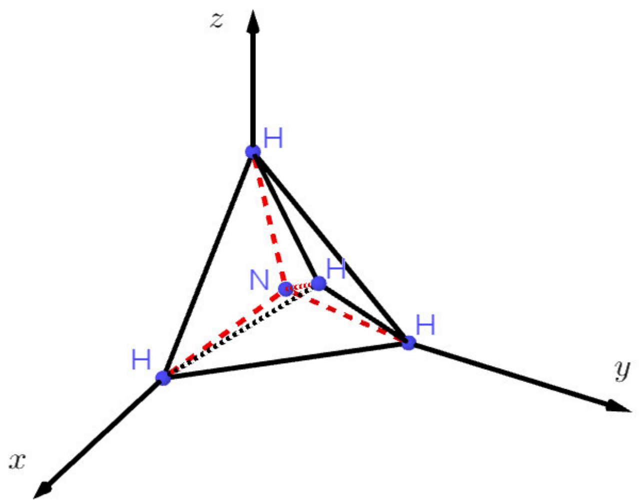
(a) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x-axis is given by $v^2 = 8 - 2x - x^2$ where x is in metres.

i. Find the acceleration of the particle in term of x. **1**

ii. State the centre and period of the motion. **2**

iii. What is the maximum speed of the particle? **1**

(b) A molecule of ammonia ion $[NH_4]^+$ is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the nitrogen atom at the centroid. The bond angle is the angle formed by the $H - N - H$ combination. It is the angle between the lines that join the nitrogen atom to two of the hydrogen atoms.



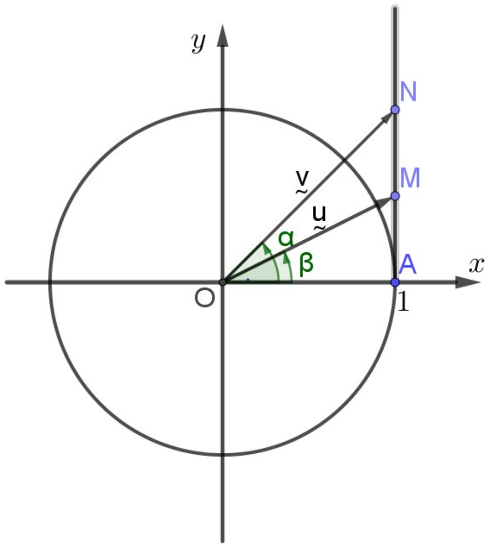
Take the vertices of the tetrahedron to be the points $A(1,0,0)$, $B(0,1,0)$, $D(0,0,1)$ and $E(1,1,1)$ where, A,B,D and E representing the hydrogen atoms as shown in the figure. The nitrogen atom is represented by point N.

i. Given that $\overrightarrow{NA} + \overrightarrow{NB} + \overrightarrow{ND} + \overrightarrow{NE} = 0$, show that the centroid is given by **2**

$$N\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

ii. Show that the bond angle is approximately $109.5°$. **2**

(c) Consider the points $M = (1, \tan\beta)$ and $N = (1, \tan\alpha)$, where $\alpha = \angle AON$ and $\beta = \angle AOM$ as shown on the diagram below.



i. Find a point P such that $OMPN$ is a parallelogram. **1**

ii. Show that the area of a parallelogram with adjacent sides $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is **3**

$$\frac{\left|\underset{\sim}{v}|\underset{\sim}{u}|^2 - (\underset{\sim}{v}.\underset{\sim}{u})\underset{\sim}{u}\right|}{|\underset{\sim}{u}|}$$

iii. Hence find the area of $OMPN$. **3**

End of Question 12

Question 13 (15 Marks)

(a) Assume that the tides rise and fall in simple harmonic motion. At low tide, the channel in a harbour is 8m deep and at high tide, it is 12m deep. Low tide is at 9 AM and high tide is at 4 PM.

i. What is the depth of the channel at 12:30 PM? **1**

ii. What is the amplitude of the motion? **1**

iii. What is the period of the motion? **1**

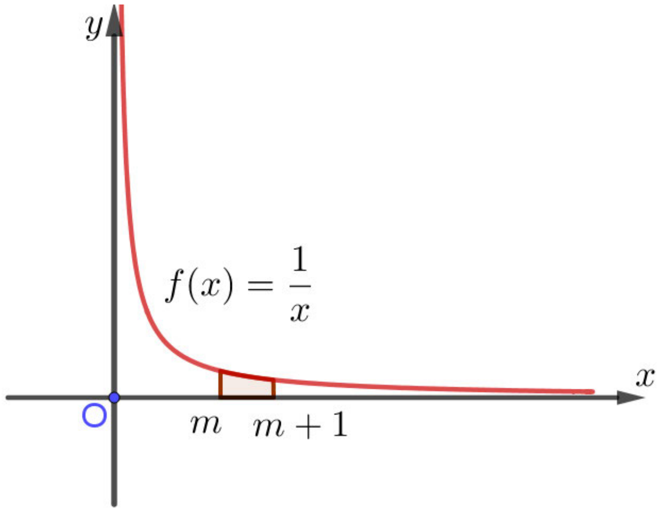
iv. Show that the depth y m of the water in the channel is given by **2**

$y = 10 - 2\cos\left(\frac{\pi}{7}t\right)$, where t is the number of hours after low tide.

v. A ship needs at least 9 m of water to pass through the channel safely. **2**
Find the times the ship can navigate safely through the channel.

(b) Assuming $n > 0$, prove using contrapositive that if $4^n - 1$ is prime, then n is odd. **2**

(c) Consider the function $f(x) = \dfrac{1}{x}$ for $x > 0$.



i. Let m be a positive real number, $m \geq 1$. By integrating $f(x)$ between m and $m+1$, **2**

show that

$$\frac{1}{m+1} < \ln(m+1) - \ln m < \frac{1}{m}$$

ii. Let m take on successively the values $1, 2, \ldots, n-1$. Hence show that **4**

$$\ln(k+1) < S_k < 1 + \ln k$$

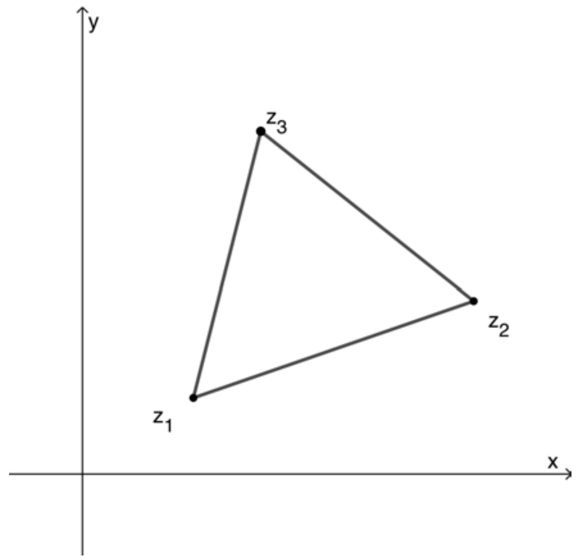
where

$$S_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

End of Question 13

Question 14 (15 Marks)

(a) Complex numbers z_1, z_2 and z_3 are given on the Argand diagram below. Consider the complex number z_3 such that $(z_3 - z_1)^2 - (z_3 - z_1)(z_2 - z_1) + (z_2 - z_1)^2 = 0$.



i. Find $\dfrac{z_3 - z_1}{z_2 - z_1}$ and express your answer in the form $r(\cos\theta + i\sin\theta)$. **2**

ii. Hence show that z_1, z_2, z_3 are the vertices of an equilateral triangle. **2**

iii. Show that $\dfrac{1}{z_2 - z_1} - \dfrac{1}{z_1 - z_3} - \dfrac{1}{z_2 - z_3} = 0$. **3**

(b) Consider the integral

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx \quad \text{for} \quad n \geq 1$$

i. Use integration by parts to show that **2**

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

ii. Find the values of A and B for which **2**

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$$

iii. Show that **2**

$$I_{n+1} = \frac{1}{n2^{n+1}} + \frac{2n-1}{2n}I_n$$

iv. Hence obtain the exact value of **2**

$$\int_0^1 \frac{1}{(1+x^2)^3}\,dx$$

End of Question 14

Question 15 (15 Marks)

(a) i. By using the identity

$$x^5 + y^5 = (x+y)^5 - 5xy(x+y)^3 + 5xy(x+y)$$

and $z^n + z^{-n} = 2\cos n\theta$, show that **2**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

ii. Hence, show that the equation $16x^5 - 20x^3 + 5x - 1 = 0$ has roots **2**

$$1,\ \cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5} \text{ and } \cos\frac{8\pi}{5}$$

iii. By using division, or otherwise, deduce from (ii) that **2**

$16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ has roots

$$x = \cos\frac{2\pi}{5},\ \cos\frac{4\pi}{5},\ \cos\frac{6\pi}{5} \text{ and } \cos\frac{8\pi}{5}$$

iv. By equating coefficients, or otherwise, find the values of b and c for which **2**

$$16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + bx + c)^2$$

v. Hence find the exact values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$. **2**

(b) Given that a, b and c are positive real numbers;

i. Prove that $a^3 + b^3 \geq a^2b + ab^2$. **2**

ii. Hence prove **3**

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3}$$

You may also assume $a + b + c \geq 3\sqrt[3]{abc}$ and

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc.$$

End of Question 15

Question 16 (15 Marks)

A sequence of polynomials $T_k(x)$ is defined by the recurrence formula

$$T_0(x) = 1, \quad T_1(x) = 2x, \quad T_2(x) = 8x^2 - 1$$

and

$$T_k(x) = 4xT_{k-1}(x) - T_{k-2}(x) \quad \text{for} \quad k \geq 2 \quad \text{and} \quad x > 1.$$

i. Show that $T_3(x)$ and $T_4(x)$ are **2**

$$T_3(x) = 32x^3 - 6x$$
$$T_4(x) = 128x^4 - 32x^2 + 1$$

To find a formula for $T_k(x)$, let $F(Z)$ be the power series in Z with the coefficient of Z^k being $T_k(x)$; that is, let

$$F(Z) = 1 + 2xZ + (8x^2 - 1)Z^2 + \cdots + T_k(x)Z^k + \ldots$$

ii. Find $(1 - 4xZ + Z^2)F(Z)$ and hence show that **3**

$$F(Z) = \frac{1 - 2xZ}{1 - 4xZ + Z^2}$$

iii. Given that α and β are the roots of $1 - 4xZ + Z^2 = 0$, find α and β in terms of x and explain why α and β are reciprocals of each other. **1**

iv. Hence show that **1**

$$1 - 4xZ + Z^2 = \left(1 - \frac{Z}{\alpha}\right)\left(1 - \frac{Z}{\beta}\right)$$

v. Using partial fractions to express $F(Z)$ in the form **2**

$$F(Z) = \frac{1 - 2xZ}{1 - 4xZ + Z^2} = \frac{A}{1 - \dfrac{Z}{\alpha}} + \frac{B}{1 - \dfrac{Z}{\beta}},$$

find A and B where A and B are real constants.

vi. For Z sufficiently small, explain why $\dfrac{1}{1 - \dfrac{Z}{\alpha}}$ is equal to **1**

$$1 + \frac{Z}{\alpha} + \left(\frac{Z}{\alpha}\right)^2 + \cdots + \left(\frac{Z}{\alpha}\right)^k + \ldots.$$

vii. Hence show that the coefficient $T_k(x)$ is $A\left(\dfrac{1}{\alpha}\right)^k + B\left(\dfrac{1}{\beta}\right)^k$. **2**

viii. Deduce that the formula for $T_k(x)$ is **1**

$$T_k(x) = \frac{1}{2}\left(\frac{1}{2x + \sqrt{4x^2 - 1}}\right)^k + \frac{1}{2}\left(\frac{1}{2x - \sqrt{4x^2 - 1}}\right)^k$$

ix. Find $\displaystyle\lim_{n \to \infty} \frac{T_{n+1}(x)}{T_n(x)}$. **2**

End of paper.

MATHEMATICS Extension 2: Multiple Choice __		
Suggested Solutions	**Marks**	**Marker's Comments**
1. C 2. D 3. D 4. B 5. D 6. A 7. A 8. C 9. B 10. D		

MATHEMATICS Extension 2: Question ...11...

Suggested Solutions	Marks	Marker's Comments
a) $\int \cos^3 x \, dx$		Most students receive full marks. A few mistakes in the negative signs.
$= \int \cos x \, (1-\sin^2 x) \, dx \qquad (\cos^2 x = 1-\sin^2 x)$		
$= \int \cos x - \sin^2 x \cos x \, dx \qquad (\cos x = (\sin x)')$		
$= \sin x - \frac{\sin^3 x}{3} + k.$	2	1 mark for $\sin x$ 1 mark for $-\frac{\sin^3 x}{3}$.
b) $\int x \sec^2 x \, dx \qquad$ (integration by parts)		(same as the question above)
$= x \tan x - \int \tan x \, dx.$		
$= x \tan x - \int \frac{\sin x}{\cos x} \, dx$		
$= x \tan x + \int \frac{-\sin x}{\cos x} \, dx \qquad (-\sin x = (\cos x)')$		
$= x \tan x + \ln\|\cos x\| + c$	2	1 mark for $x \tan x$ 1 mark for $\ln\|\cos x\|$
c) let $a + ib = \sqrt{-24-10i}$ (a and b are real numbers)		
square both sides. $a^2 - b^2 + 2abi = -24 - 10i$		
$\begin{cases} a^2 - b^2 = -24 & ① \\ 2ab = -10 & ② \end{cases}$	1	
from ② $b = \frac{-5}{a}$ ③		
Sub ③ into ① $a^2 - \frac{25}{a^2} = -24$		
$a^4 + 24a^2 - 25 = 0$		
$(a^2-1)(a^2+25) = 0$		Many students make calculation mistakes for a.
$a^2 = 1$ or $a^2 = -25$		
$\because$ a is a real number $\therefore a^2 = 1 \quad \therefore a = \pm 1$		Or have four answers
$\therefore$ when $a = 1$, $b = -5$ when $a = -1$, $b = 5$		Or have one answer $1-5i$, eliminating $-1+5i$ b/c it has a negative sign for the real part.
$\therefore \sqrt{-24-10i} = \pm(1-5i)$	1	

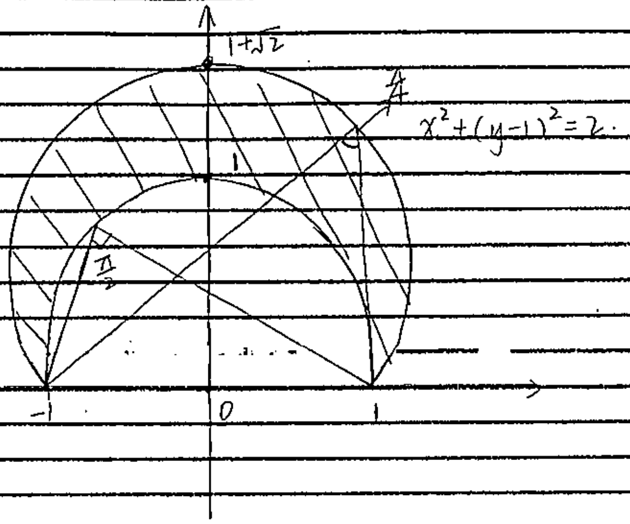
MATHEMATICS Extension 2: Question...11...

Suggested Solutions	Marks	Marker's Comments
c) ii) $z = \dfrac{(1-i) \pm \sqrt{(1-i)^2 - 4(6+2i)}}{2}$		Some students have $-(1-i)$ in the expression.
$= \dfrac{1-i \pm \sqrt{-24-10i}}{2}$		Some calculation mistakes in the last step.
$= \dfrac{1-i+(1-5i)}{2}$ or $\dfrac{1-i-(1-5i)}{2}$		
$= 1-3i$ or $2i$	2	1 mark for $1-3i$ 1 mark for $2i$
d) i)		
let $I = \int_a^b f(x)\,dx$		Some students conclude $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ without necessary proof steps.
$u = a+b-x$ $x=b,\ u=a$		
$x = a+b-u$ $x=a,\ u=b$		
$dx = -du$		
$I = \int_b^a f(a+b-u)(-du) = \int_a^b f(a+b-u)\,du$		
$= \int_a^b f(a+b-x)\,dx$ (dummy variable)	1	
$\int_a^b \{f(x) + f(a+b-x)\}dx$		
$= \int_a^b f(x)\,dx + \int_a^b f(a+b-x)\,dx$		
$= I + I = 2I$		
$I = \int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \{f(x) + f(a+b-x)\}dx$	1	

MATHEMATICS Extension 2: Question…….

Suggested Solutions	Marks	Marker's Comments
d) i) Method 2.		
$\frac{1}{2}\int_a^b \{f(x)+f(a+b-x)\}\,dx$		
$=\frac{1}{2}\left[F(x)+F(a+b-x)\right]_a^b$		
$=\frac{1}{2}\left[F(b)-F(a)+F(a)-F(b)\right]$		
$= F(b)-F(a)$	1	
$\int_a^b f(x)\,dx$		
$=\left[F(x)\right]_a^b = F(b)-F(a)$		
$\therefore \int_a^b f(x)\,dx = \frac{1}{2}\int_a^b \{f(x)+f(a+b-x)\}\,dx$	1	
d ii) $b=7$, $a=3$,		Some students do not recognise the connection between (i) and (ii).
$f(x)=\dfrac{\ln(x+2)}{\ln(24+10x-x^2)}$		
$f(a+b-x)=f(7+3-x)=f(10-x)$		Many students do not find the correct expression for $f(a+b-x)$.
$=\dfrac{\ln(10-x+2)}{\ln(24+10(10-x)-(10-x)^2}$		
$=\dfrac{\ln(12-x)}{\ln\{(x+2)(12-x)\}}$	1	
$\int_3^7 \dfrac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$		Some students don't use log law very well.
$=\frac{1}{2}\int_3^7 \dfrac{\ln(x+2)}{\ln\{(12-x)(x+2)\}}+\dfrac{\ln(12-x)}{\ln\{(x+2)(12-x)\}}\,dx$		
$=\frac{1}{2}\int_3^7 \dfrac{\ln(x+2)(12-x)}{\ln(12-x)(x+2)}\,dx$		
$=\frac{1}{2}\int_3^7 dx = \frac{1}{2}\left[x\right]_3^7 = 2$	1	

MATHEMATICS Extension 2: Question........

Suggested Solutions	Marks	Marker's Comments
e) i)+ii) 	i) 2	Need to satisfy any two of three listed items below to receive 1 mark. * major segment * x-intercepts w/ open circles * $\frac{\pi}{4}$ labeled or y-intercep
	ii) 1	

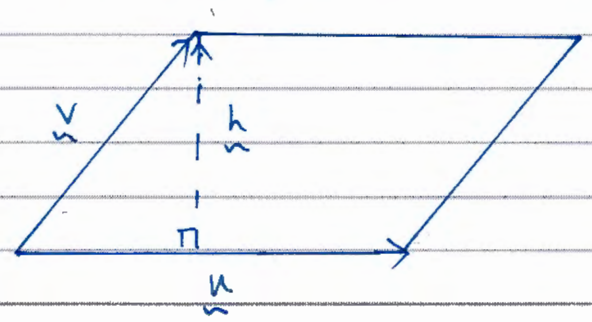
MATHEMATICS Extension 2: Question 12

Suggested Solutions	Marks	Marker's Comments
a) i $v^2 = 8 - 2x - x^2$		
$\frac{1}{2}v^2 = 4 - x - \frac{x^2}{2}$		
$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -1 - x$	1	
$a = -(1+x)$		
ii centre at $x = -1$	1	
$n = 1, \therefore T = \frac{2\pi}{1} = 2\pi$ seconds	1	
iii Max v at $x = -1$,		
$\therefore v^2 = 8 - 2(-1) - (-1)^2$		
$= 8 + 2 - 1$		
$= 9$		
$\therefore \lvert v \rvert = 3$		
$\therefore$ Max Speed at $v = 3\,ms^{-1}$.		
b) i) $\vec{NA} + \vec{NB} + \vec{ND} + \vec{NE} = 0$		
$(\vec{OA} - \vec{ON}) + (\vec{OB} - \vec{ON}) + (\vec{OD} - \vec{ON}) + (\vec{OE} - \vec{ON}) = 0$		1
$\therefore (\vec{OA} + \vec{OB} + \vec{OD} + \vec{OE}) = 4\vec{ON}$		
$\therefore 4\vec{ON} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$		
$= \begin{pmatrix}2\\2\\2\end{pmatrix}$		
$\therefore \vec{ON} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix} \quad \therefore N = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	1	

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
ii) $\vec{NA} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$		
$\vec{NB} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$		1
$\vec{NA} \cdot \vec{NB} = \lvert\vec{NA}\rvert \lvert\vec{NB}\rvert \cos\theta$		
$\cos\theta = \dfrac{\vec{NA} \cdot \vec{NB}}{\lvert\vec{NA}\rvert \lvert\vec{NB}\rvert}$		
$= \dfrac{\left(\frac{1}{2} \times \frac{-1}{2}\right) + \left(\frac{-1}{2} \times \frac{1}{2}\right) + \left(\frac{-1}{2} \times \frac{-1}{2}\right)}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} \times \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2}}$		
$= \dfrac{\frac{-1}{4}}{3\left(\frac{1}{2}\right)^2}$		
$= \dfrac{\frac{-1}{4}}{\frac{3}{4}}$		
$\cos\theta = \dfrac{-1}{3}$		
$\therefore \theta = \cos^{-1}\left(\frac{-1}{3}\right)$		1
$= 109.5^\circ$		
Note: Those who uses the dot product wrong and got 70.5° and said that the angle is obtuse based on diagram DOES NOT get full marks.		
Dot formula is incorrect when it is used head to tail!		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
c) i $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$		
$= \begin{pmatrix} 1 \\ \tan\beta \end{pmatrix} + \begin{pmatrix} 1 \\ \tan\alpha \end{pmatrix}$		
$= \begin{pmatrix} 2 \\ \tan\alpha + \tan\beta \end{pmatrix}$.		
$\therefore P = (2, \tan\alpha + \tan\beta)$	1	
ii 		
$\text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} + \underset{\sim}{h} = \underset{\sim}{v}$		
$\therefore \underset{\sim}{h} = \underset{\sim}{v} - \text{proj}_{\underset{\sim}{u}} \underset{\sim}{v}$		1 Recognising height is given by a subtraction with projection
$= \underset{\sim}{v} - \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert^2} \underset{\sim}{u}$		
$\therefore \text{Area} = \lvert\underset{\sim}{u}\rvert \, \lvert\underset{\sim}{h}\rvert$		
$= \lvert\underset{\sim}{u}\rvert \left\lvert \underset{\sim}{v} - \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert^2} \underset{\sim}{u} \right\rvert$		1 Correct substitution into area of parallelogram formula.
$= \left\lvert \lvert\underset{\sim}{u}\rvert \underset{\sim}{v} - \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert} \underset{\sim}{u} \right\rvert$		
$= \left\lvert \dfrac{\underset{\sim}{v} \lvert\underset{\sim}{u}\rvert^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v}) \underset{\sim}{u}}{\lvert\underset{\sim}{u}\rvert} \right\rvert$		
$= \dfrac{\left\lvert \underset{\sim}{v} \lvert\underset{\sim}{u}\rvert^2 - (\underset{\sim}{u} \cdot \underset{\sim}{v}) \underset{\sim}{u} \right\rvert}{\lvert\underset{\sim}{u}\rvert}$		1

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**

iii) $\underset{\sim}{u} = \begin{pmatrix} 1 \\ \tan\beta \end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix} 1 \\ \tan\alpha \end{pmatrix}$ $|\underset{\sim}{u}|^2 = 1 + \tan^2\beta$

$= \sec^2\beta$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 1 + \tan\alpha\tan\beta$

$$\therefore \text{Area} = \frac{\left| \sec^2\beta \begin{pmatrix} 1 \\ \tan\alpha \end{pmatrix} - (1 + \tan\alpha\tan\beta)\begin{pmatrix} 1 \\ \tan\beta \end{pmatrix} \right|}{|\sec\beta|}$$

$$= |\cos\beta| \left| \begin{pmatrix} \sec^2\beta \\ \sec^2\beta\tan\alpha \end{pmatrix} - \begin{pmatrix} 1 + \tan\alpha\tan\beta \\ \tan\beta + \tan\alpha\tan^2\beta \end{pmatrix} \right|$$

$$= |\cos\beta| \left| \begin{pmatrix} \sec^2\beta - 1 - \tan\alpha\tan\beta \\ \sec^2\beta\tan\alpha - \tan\beta - \tan\alpha\tan^2\beta \end{pmatrix} \right|$$

$$= |\cos\beta| \left| \begin{matrix} \tan^2\beta - \tan\alpha\tan\beta \\ \tan\alpha(\sec^2\beta - \tan^2\beta) - \tan\beta \end{matrix} \right|$$

$$= |\cos\beta| \left| \begin{matrix} \tan\beta(\tan\beta - \tan\alpha) \\ \tan\alpha - \tan\beta \end{matrix} \right|$$

$$= |\cos\beta| \, |\tan\alpha - \tan\beta| \left| \begin{pmatrix} \tan\beta \\ 1 \end{pmatrix} \right|$$

$$= |\cos\beta| \sqrt{\tan^2\beta + 1} \, |\tan\alpha - \tan\beta|$$

$$= |\cos\beta| \, |\sec\beta| \, |\tan\alpha - \tan\beta|$$

$$= \tan\alpha - \tan\beta \quad (\alpha > \beta)$$

Marker's Comments:

(1) Expressions of $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{u} \cdot \underset{\sim}{v}$ and $|\underset{\sim}{u}|$.

(1) Eliminating the magnitude sign by using the magnitude formula (i.e $\left|\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right| = \sqrt{a_1^2 + a_2^2}$)

(1) Final answer.

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<tr><th>Suggested Solutions</th><th>Marks</th><th>Marker's Comments</th></tr>
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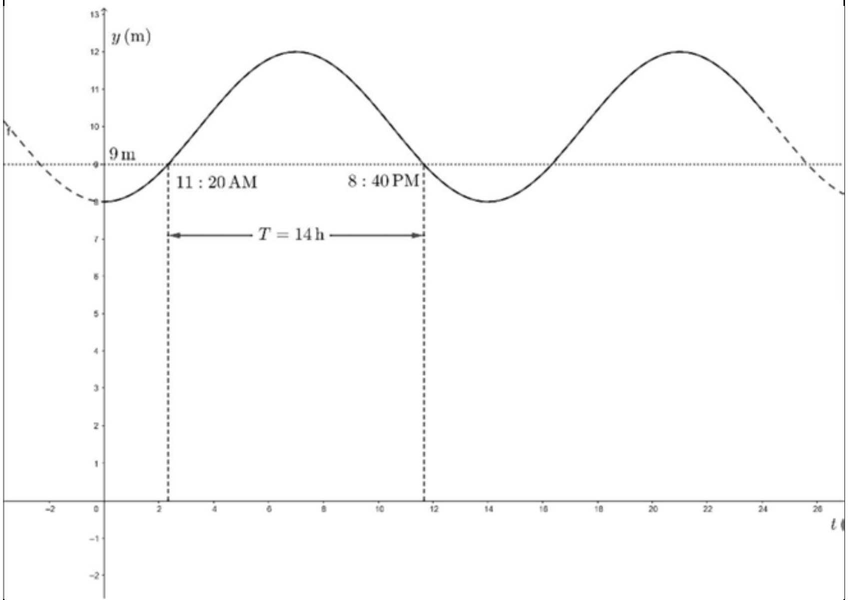

(a)

(i) We're told low tide is at 9 AM, high tide is at 4 PM. Given 12:30 PM is the midpoint in time between low tide and high tide, we expect the depth to be the centre line of the motion:

$$\frac{(8+12)}{2} = 10 \text{ m}$$

(ii) Amplitude is half the distance between peak and trough, hence amplitude is

$$\frac{12-8}{2} = 2 \text{ m}$$

(iii) Period of the motion is the time taken to complete one cycle. If it takes 7 hours to move from low tide to high tide, it takes another 7 hours to complete the cycle. Hence the period is 14 hours.

(iv) The general equation for displacement y for an entity moving in simple harmonic motion is

$$y = a\cos(nt+\alpha) + c$$

Now, from parts (i) and (ii), we have

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Single-mark questions, one mark only available for each of (i) to (iii), so correct answer necessary irrespective of any working.

This is a 'show that' question, meaning the candidate must demonstrate every step entirely. Here, you needed to start with a general

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$$y = 2\cos(nt + \alpha) + 10$$

From (iii), $T = 14 \rightarrow n = \frac{2\pi}{T} = \frac{\pi}{7}$, so

$$y = 2\cos\left(\frac{\pi}{7}t + \alpha\right) + 10$$

Finally, at 9 AM, $t = 0$ and we are told that this occurs when we are at low tide, $y = 8$ m:

$$8 = 2\cos(\alpha) + 10$$

$$-1 = \cos(\alpha)$$

$$\rightarrow \alpha = \pi$$

so

$$y = 2\cos\left(\frac{\pi}{7}t + \pi\right) + 10$$

$$= 2\left(\cos\left(\frac{\pi}{7}t\right)\cancelto{-1}{\cos\pi} - \sin\left(\frac{\pi}{7}t\right)\cancelto{0}{\sin\pi}\right) + 10$$

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trigonometric function, and then **justify** each of the components: a, n, α, c.

All but α had been dealt with in (i) to (iii).

It was necessary to explain away the phase shift. This could have been done with verbal reasoning, but it really couldn't be ignored.

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$$= 10 - 2\cos\left(\frac{\pi}{7}t\right)$$

(v) Require t such that $y \geq 9$ m; so, solve

$$10 - 2\cos\left(\frac{\pi}{7}t\right) = 9$$

$$\cos\left(\frac{\pi}{7}t\right) = \frac{1}{2}$$

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Then

$$\frac{\pi}{7}t = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \frac{\pi}{7}t = \frac{5\pi}{3} + 2k\pi$$

for $k \in \mathbb{Z}$.

So

$$t = \frac{7}{3} + 14k \quad \text{or} \quad t = \frac{35}{3} + 14k$$

As the tide is increasing just after $t = 0$, we reach 9 m when $t = \frac{7}{3}$ hours after 9 AM; i.e., at

11:20 AM

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First mark for identifying the correct arguments of cosine.

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The tide maximises, then retreats, hitting 9 m again at $t = \frac{35}{3}$ hours after 9 AM; i.e. at
8:40 PM
This block of time where the ship may pass safely repeats every 14 hours ($k = 1, 2, ...$), but explicit calculation of these times is not necessary for the question (such a list would comprise of 12 pairs).
□
(b) Original statement (taking as a premise that $n > 0$ (i.e. it's not part of the intended contraposition)):
"If $4^n - 1$ is prime, then n is odd."
Contrapositive:
"If not (n is odd), then not ($4^n - 1$ is prime)."
$\equiv$
"If n is even, then $4^n - 1$ is composite."
So, suppose $n > 0$ is even. Then $n = 2k$ for $k \in \mathbb{Z}^+$. Hence
$$\begin{aligned} 4^{2k} - 1 &= \left(4^k\right)^2 - 1 \\ &= (4^k - 1)(4^k + 1) \end{aligned}$$
Since $k > 0$, neither factor is trivial (i.e. neither is equal to 1).
Hence $4^{2k} - 1$ is composite.
Hence the contraposition is true, hence the original statement is true.
□
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Second mark for identifying the correct times in the first period.
There are many additional times if we're to be accurate, but this is a 2-mark question, and the next round of times would be beyond the first 24 hours.
First mark awarded for correctly identifying contrapositive, either explicitly (sentence) or implicitly through the mathematics (question did not ask for a statement of the contrapositive).
Many students used proofs that were incomplete, such as justifying by example that 4^{2k} will always give a number with last digit 6, hence $4^{2k} - 1$ will always end in a 5, hence divisible by 5 (or similar). There is nothing wrong with this line of reasoning, but it doesn't reach the threshold of a proof. An induction of some sort would be needed.
Many also made this far more difficult than necessary; if you're reaching for an induction in a 2-mark question, something's been missed.
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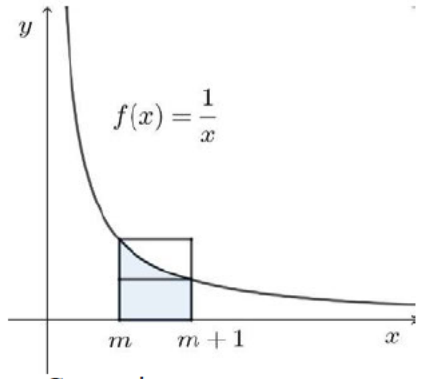
(c)  (i) Comparing areas: $\frac{1}{m+1} < \int_{m}^{m+1} \frac{dx}{x} < \frac{1}{m}$	**1**	First mark for setting up the area argument.
So $\frac{1}{m+1} < \ln(m+1) - \ln m < \frac{1}{m}$	**1**	Second mark for deriving the required expression.
(ii) Let m successively take on the values $1, 2, \dots, k-1$ and form the sum from $m = 1$ to $m = k-1$: $\sum_{m=1}^{k-1} \frac{1}{m+1} < \sum_{m=1}^{k-1} (\ln(m+1) - \ln m) < \sum_{m=1}^{k-1} \frac{1}{m}$ $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} < \ln k - \overset{0}{\cancel{\ln 1}} < 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}$	**1**	First mark for appropriate pattern search.
That is, for $S_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$, $S_k - 1 < \ln k < S_{k-1}$	**1**	Second mark for reaching this conclusion (or similar) after identifying the pattern.
From the left inequality: $S_k - 1 < \ln k$ so $S_k < 1 + \ln k \quad \dots (1)$	**1**	Third mark for one part of the inequality proved.
From the right inequality: $\ln k < S_{k-1}$ Replace k with $k+1$ (since statement is true for any $k \in \mathbb{Z}$ where $k \geq 2$), then $\ln(k+1) < S_k \quad \dots (2)$ Hence by (1) and (2): $\ln(k+1) < S_k < 1 + \ln k$	**1**	Fourth mark for second part of inequality proved.

Many students again made this question more difficult than necessary.

Working was haphazard and cogent arguments were left to be made for the student by the marker. It’s not enough to leave ‘bits and pieces’ of relevant parts of your proof on your paper and fail to tie things together adequately.

If you prove results and wish to use them later, you need to identify them and refer to them explicitly. It’s your job to prove your results.

General remarks:

- students writing facts and hoping for relevancy = doesn’t work.
- setting out was, overall, awful. Yes, you’re in exam conditions, but you’re trying to communicate your thoughts to someone else. Would you attempt to speak to someone in a broken and warped version of English and expect success? Why do it with your mathematics?
- Pause before answering. Sketch out a solution (on spare paper), then write something cogent. You think you’re wasting time, but a waste of time is writing a bunch of unconnected passages in the hope that someone will see something of value and award you marks. That just implies you don’t really know what you’re doing, so why would an examiner just hand over points in the HSC when you’re in competition with thousands of others?

Q14 a) JRAHS TRIAL 2020 4U

P

☆ Given that $(z_3-z_1)^2-(z_3-z_1)(z_2-z_1)+(z_2-z_1)^2=0$

(i) Find $\frac{z_3-z_1}{z_2-z_1}$ and express answer in the form of $r(\cos\theta+i\sin\theta)$.

In ☆, Divide by $(z_2-z_1)^2$ $\left(\frac{z_3-z_1}{z_2-z_1}\right)^2-\left(\frac{z_3-z_1}{z_2-z_1}\right)+1=0$

$\therefore \frac{z_3-z_1}{z_2-z_1}=\frac{1\pm\sqrt{1^2-4}}{2}$ (Quadratic Formula)

$=\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$ — 1m

$=1.(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ or $1.(\cos\frac{-\pi}{3}+i\sin(-\frac{\pi}{3}))$

From the diagram $\operatorname{Arg}\left(\frac{z_3-z_1}{z_2-z_1}\right)>0$ } 1m

$\therefore \frac{z_3-z_1}{z_2-z_1}=\underline{1.\cos(\frac{\pi}{3}+i\sin\frac{\pi}{3})}$

Don't accept 1st quadrant nor acute nor z_3-z_1 is left of z_2-z_1

(ii) Since $r=1$, $\left|\frac{z_3-z_1}{z_2-z_1}\right|=1$ $\therefore |z_3-z_1|=|z_2-z_1|$

1m $\operatorname{Arg}(z_3-z_1)=\operatorname{Arg}(z_2-z_1)+\frac{\pi}{3}$

ie angle between the 2 equal sides is $\frac{\pi}{3}$

1m angle sum of a triangle is 2π so this is an isosceles triangle with apex angle $\frac{\pi}{3}$. Thus other angles are also $\frac{\pi}{3}$. (Also accept rotation of $\frac{\pi}{3}$)

Hence z_1, z_2, z_3 are the vertices of an equilateral triangle.

☆ Alternate method to part (i) $a^3+b^3=(a+b)(a^2-ab+b^2)$

$(z_3-z_1)^2-(z_3-z_1)(z_2-z_1)+(z_2-z_1)^2=0$

$(z_3-z_1)^3+(z_2-z_1)^3=0$ $\Rightarrow$ $\left(\frac{z_3-z_1}{z_2-z_1}\right)^3=-1=\operatorname{cis}(\pm\pi)$

$\therefore \frac{z_3-z_1}{z_2-z_1}=\operatorname{cis}\pm\frac{\pi}{3}$

14b) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$

$u=(1+x^2)^n \qquad dv = dx$

$du = \frac{-2xn}{(1+x^2)^n}\,dx \qquad v = x$ 1m

i) $I_n = \frac{x}{(1+x^2)^n}\Big]_0^1 - \int_0^1 \frac{-2nx^2}{(1+x^2)^{n+1}}\,dx$ 1m

$= \left(\frac{1}{2^n} - 0\right) + 2n\int_0^1 \frac{x^2\,dx}{(1+x^2)^{n+1}}$ #

If these are not shown must see $-\frac{-2n\,x\,x}{(1+x^2)^{n+1}}$ for 1st mark

Note: $u' \neq \frac{-2xn\,dx}{(1+x^2)^n}$

ii) $\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} \equiv \frac{x^2}{(1+x^2)^{n+1}}$

$A(1+x^2) + B \equiv x^2$ 1m

Equating coeff $A=1, \quad B=-1$ 1m

iii) From (i) & (ii) $I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{dx}{(1+x^2)^n} - \int_0^1 \frac{dx}{(1+x^2)^{n+1}}$

$I_n = \frac{1}{2^n} + 2n\left(I_n - I_{n+1}\right)$ 1m identify I_n, I_{n+1}

$2n\,I_{n+1} = \frac{1}{2^n} + 2nI_n - I_n$

$I_{n+1} = \frac{1}{2n\,2^n} + \frac{(2n-1)\,I_n}{2n}$ 1m collect terms

$I_{n+1} = \frac{1}{n\,2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n$

Some students evaluate I_0 max 1m

iv) $I_3 = \frac{1}{2\times 2^3} + \frac{3}{4}I_2 = \frac{1}{16} + \frac{3}{4}I_2$

$I_2 = \frac{1}{2^2} + \frac{1}{2}I_1 = \frac{1}{4} + \frac{1}{2}I_1$

$I_1 = \int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1}x\right)\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ 1m

$I_2 = \frac{1}{4} + \frac{\pi}{8}$

$I_3 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right) = \frac{1}{16} + \frac{3}{16} + \frac{3\pi}{32} = \frac{1}{4} + \frac{3\pi}{32}$ 1m

Trials 2020

MATHEMATICS Extension 2: Question 15 — 1/5

Suggested Solutions	Marks	Marker's Comments
a) i) using identity $x^5+y^5=(x+y)^5-5xy(x+y)^3+5xy(x+y)$ and $z^n+z^{-n}=2\cos n\theta$		
show $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	1	mark to show LHS using given identity.
Let $x=z$ $y=z^{-1}$		
$x^5+y^5 = z^5+z^{-5} = 2\cos 5\theta$; $xy = z\cdot z^{-1} = 1$; $5xy=5$	1	mark to show RHS using given identity.
$(x+y) = (z+z^{-1}) = 2\cos\theta$		
$(x+y)^5 = (2\cos\theta)^5 = 32\cos^5\theta$; $(x+y)^3 = (2\cos\theta)^3 = 8\cos^3\theta$		* no marks awarded if us DMT or any other method.
from i: $2\cos 5\theta = 32\cos^5\theta - 5(8\cos^3\theta) + 5(2\cos\theta)$		
$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ (shown)		
ii) Let $x=\cos\theta$ (must show this) $\cos 5\theta = 16x^5-20x^3+5x$ $\therefore$ consider $16x^5-20x^3+5x-1=0$		
$\therefore \cos 5\theta - 1 = 0$		
$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$		
$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$	1	
Roots are $x = 1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$	1	

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MATHEMATICS Extension 2: Question 15		2/5
Suggested Solutions	**Marks**	**Marker's Comments**
a) iii) can do any method, inspection, synthetic ie my preferred method long division from ii, we know one of the root is 1 $$\begin{array}{r} 16x^4 + 16x^3 - 4x^2 - 4x + 1 \\ x-1 \overline{\big) 16x^5 + 0x^4 - 20x^3 + 0x^2 + 5x - 1} \\ -(16x^5 - 16x^4) \\ \hline 16x^4 - 20x^3 \\ -(16x^4 - 16x^3) \\ \hline -4x^3 + 0 \\ -(-4x^3 + 4x^2) \\ \hline -4x^2 + 5x \\ -(-4x^2 + 4x) \\ \hline x - 1 \\ x - 1 \\ \hline - \end{array}$$ $16x^5 - 20x^3 + 5x - 1 = (x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1)$ The other zeroes of the polynomials must be zeroes of $16x^4 + 16x^3 - 4x^2 - 4x + 1$ so the roots of the quartic are $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\cos\frac{6\pi}{5}$, $\cos\frac{8\pi}{4}$		1 to show that $x-1$ is a root 1 state the quartic has roots $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\cos\frac{6\pi}{5}$ and $\cos\frac{8\pi}{5}$.
iv) expand and equate coefficients. RHS $(4x^2+bx+c)(4x^2+bx+c)$ $= 16x^4 + 4bx^3 + 4cx^2 + 4bx^3 + b^2x^2 + bcx + 4cx^2 + bcx + c^2$ $= 16x^4 + 8bx^3 + (8c + b^2)x^2 + 2bcx + c^2$ $x^3 : 16 = 8b \quad \therefore \quad b = 2$ $x^2 : -4 = 8c + b^2 \quad c = -1 \quad$ ie $(4x^2 + 2x - 1)^2$		1 mark for b 1 mark for c

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MATHEMATICS Extension 2: Question 15 3/5

Suggested Solutions	Marks	Marker's Comments

a) v) $p(x)$ has 2 repeated roots

$$\cos\frac{8\pi}{5} = \cos\frac{2\pi}{5}$$

and $\cos\frac{6\pi}{5} = \cos\frac{4\pi}{5}$

from pt iii) $(4x^2 + 2x - 1)^2 = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

Let $4x^2 + 2x - 1 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-2 \pm \sqrt{20}}{8}$$

$$= \frac{-1 \pm \sqrt{5}}{4}$$

1 — to show correct roots.

$\cos\frac{2\pi}{5}$ is in quad 1 $\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

$\cos\frac{4\pi}{5}$ is in quad 2 $\therefore \cos\frac{4\pi}{5} = \frac{-1-\sqrt{5}}{4}$

1 to show corresponding roots to $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$

b) i) $(a-b)^2 \geq 0$

$a^2 + b^2 - 2ab \geq 0$

$a^2 + b^2 \geq 2ab$ ——— *

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a, b \geq 0$

$a^3 \geq 0$, $b^3 \geq 0$

$\therefore a^3 + b^3 \geq 0$

if no explaination of any sort but just 2 lines. (1/2 marks awarded) only a handful of students got 1 mark.

MATHEMATICS Extension 2: Question 15		4/5
Suggested Solutions	**Marks**	**Marker's Comments**
$(a+b)(a^2-ab+b^2) \geq (a+b)(2ab-ab$	from *	most students got full marks.
$(a+b)(a^2-ab+b^2) \geq (a+b)ab$		
$a^3+b^3 \geq a^2b+ab^2$ — ①		
b ii) given $a+b+c \geq 3\sqrt[3]{abc}$ and		
$(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+6abc$		
consider left, middle and right.		
METHOD1		
from $a+b+c \geq 3\sqrt[3]{abc}$		
$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$		
*1 $\left(\frac{a+b+c}{3}\right)^3 \geq abc$	1	
from ① and using $(a+b+c)^3 = a^3+b^3 \ldots$		
$(a+b+c)^3 \leq (a^3+b^3+c^3)+3(a^3\ldots)+6abc$		
$\leq 7(a^3+b^3+c^3)+6abc$	1	
but using $a+b+c \geq 3\sqrt[3]{abc}$		
$a^3+b^3+c^3 \geq 3abc$		
$6abc \leq 2(a^3+b^3+c^3)$		
Hence $(a+b+c)^3 \leq 9(a^3+b^3+c^3)$		
*2 $\left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3}$	1	
combining *1 and *2		

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3}$$

MATHEMATICS Extension 2: Question ___ 5/5

Suggested Solutions	Marks	Marker's Comments

METHOD 2

bii) $a+b+c \geq 3\sqrt[3]{abc}$ (given)

$$\left(\frac{a+b+c}{3}\right)^3 \geq abc \qquad ①$$

RHS $\dfrac{9a^3+9b^3+9c^3}{3^3}$

$$\geq \frac{(3a^3+3b^3)+(3a^3+3c^3)+(3b^3+3c^3)+3(a^3+b^3+c^3)}{3^3}$$

$$\geq \frac{3(a^2b+b^2a)+3(a^2c+c^2a)+3(b^2c+c^2b)+(a^3+b^3+c^3)+2(a^3+b^3+c^3)}{3^3} \qquad ①$$

but $a^3+b^3+c^3 \geq 3\sqrt[3]{a^3b^3c^3}$ from (i)

$a^3+b^3+c^3 \geq 3abc$

$$\text{RHS} \geq \frac{a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+2(a^3+b^3+c^3)}{3^3}$$

$$\geq \frac{a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+6abc}{3^3} \qquad ①$$

$\geq \dfrac{(a+b+c)^3}{3^3}$ from given

$\geq \left(\dfrac{a+b+c}{3}\right)^3$

$$\therefore abc \leq \left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3+b^3+c^3}{3}$$

MATHEMATICS Extension 2: Question 16

Suggested Solutions	Marks	Marker's Comments
(i) $T_3 = 4xT_2 - T_1$ $= 4x(8x^2-1) - 2x$ $= 32x^3 - 6x$	1	Well done by nearly all students. Make sure to include the first line for T_3 and T_4 to show the recursive result.
$T_4 = 4xT_3 - T_2$ $= 4x(32x^3-6x) - (8x^2-1)$ $= 128x^4 - 24x^2 - 8x^2 + 1$ $= 128x^4 - 32x^2 + 1$	1	
(ii) $F(Z) = 1 + 2xZ + (8x^2-1)Z^2 + T_3Z^3 + \cdots + T_kZ^k + \cdots$ $-4xZF(Z) = -4xZ - 8x^2Z^2 - 4xT_2Z^3 + \cdots - 4xT_{k-1}Z^k + \cdots$ $Z^2F(Z) = Z^2 + T_1Z^3 + \cdots + T_{k-2}Z^k + \cdots$	$\leftarrow$ 1	For systematically finding an expression for $(1-4xZ+Z^2)F(Z)$
$(1-4xZ+Z^2)F(Z) = 1 + 2xZ - 4xZ + (8x^2-1-8x^2+1)Z^2$ $+ (T_3 - 4xT_2 + T_1)Z^3 + \cdots + (T_k - 4xT_{k-1} + T_{k-2})Z^k$ $+ \cdots$		
$= 1 - 2xZ + 0Z^2 + (T_3 - T_3)Z^3 + \cdots$ $+ (T_k - T_k)Z^k + \cdots$ (Since $T_k = 4xT_{k-1} - T_{k-2}$)	$\leftarrow$ 1	For showing at least the term in Z^2 has coefficient of 0.
$= 1 - 2xZ$ $\therefore F(Z) = \dfrac{1-2xZ}{1-4xZ+Z^2}$	1	using $T_k(x)$ relationship to show all 0 terms. Most students struggled to explain why the terms after the Z term cancel out.

MATHEMATICS Extension 2: Question 16		
Suggested Solutions	**Marks**	**Marker's Comments**
(iii) $1-4xz+z^2=0$ $z=\dfrac{4x\pm\sqrt{16x^2-4}}{2}=\dfrac{4x\pm2\sqrt{4x^2-1}}{2}$ $\therefore \alpha=2x+\sqrt{4x^2-1},\ \beta=2x-\sqrt{4x^2-1}$ Product of roots: $\alpha\beta=\dfrac{c}{a}=1$ $\therefore$ roots are reciprocals of each other.	1	Many students showed $\alpha\beta=1$ but did not find α,β in terms of x as required.
(iv) If α,β are the roots of $1-4xz+z^2$ then $1-4xz+z^2=(z-\alpha)(z-\beta)$ (dividing by $\alpha\beta=1$) $=\left(\dfrac{z}{\alpha}-1\right)\left(\dfrac{z}{\beta}-1\right)$ $=\left(1-\dfrac{z}{\alpha}\right)\left(1-\dfrac{z}{\beta}\right)$	1	
(v) From partial fractions, numerator is $A\left(1-\dfrac{z}{\beta}\right)+B\left(1-\dfrac{z}{\alpha}\right)$ Equating numerators: $A\left(1-\dfrac{z}{\beta}\right)+B\left(1-\dfrac{z}{\alpha}\right)=1-2xz$ $A+B-\left(\dfrac{A}{\beta}+\dfrac{B}{\alpha}\right)z=1-2xz$ Equating coefficients:		

MATHEMATICS Extension 2: Question ___

Suggested Solutions	Marks	Marker's Comments
$A+B=1$, $\frac{A}{\beta}+\frac{B}{\alpha}=2x$ $A\alpha+B\beta=2x$ $A(2x+\sqrt{4x^2-1})+B(2x-\sqrt{4x^2-1})=2x$ $(A+B)2x+(A-B)\sqrt{4x^2-1}=2x$ $\therefore\ A+B=1,\ A-B=0$ $\therefore\ A=B=\frac{1}{2}$	1 1	Most students could find $A+B=1$. Most struggled to find another relationship for A and B.
vi) Now for the infinite geometric series: $1+\frac{z}{\alpha}+\left(\frac{z}{\alpha}\right)^2+\dots+\left(\frac{z}{\alpha}\right)^k+\dots$ $a=1$, $r=\frac{z}{\alpha}$, and if $\lvert z\rvert<\alpha$ then $\left\lvert\frac{z}{\alpha}\right\rvert<1$ and so the series has a limiting sum $\frac{a}{1-r}$. $\therefore\ 1+\frac{z}{\alpha}+\left(\frac{z}{\alpha}\right)^2+\dots=\frac{1}{1-\frac{z}{\alpha}}$	1	Many students used $\left\lvert\frac{z}{\alpha}\right\rvert\le 1$ or $\frac{z}{\alpha}<1$ or left out the condition for a limiting sum altogether
vii) $F(z)=\frac{A}{1-\frac{z}{\alpha}}+\frac{B}{1-\frac{z}{\beta}}$ from v)		

MATHEMATICS Extension 2: Question 16___

Suggested Solutions	Marks	Marker's Comments
vii) "coefficient of z^k being $T_k(x)$". $F(z) = A \times \left(1 + \frac{z}{\alpha} + \left(\frac{z}{\alpha}\right)^2 + \cdots + \left(\frac{z}{\alpha}\right)^k + \cdots\right)$ $+ B \times \left(1 + \frac{z}{\beta} + \left(\frac{z}{\beta}\right)^2 + \cdots + \left(\frac{z}{\beta}\right)^k + \cdots\right)$	1	Many students did not acknowledge where the $T_k(x)$ came from.
Coefficient of z^k is $A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k$ Equating coefficients of z^k: $T_k(x) = A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k$	1	
viii) $T_k(x) = A\left(\frac{1}{\alpha}\right)^k + B\left(\frac{1}{\beta}\right)^k$ $A = B = \frac{1}{2}$ from (v) $\alpha = 2x + \sqrt{4x^2-1}$, $\beta = 2x - \sqrt{4x^2-1}$ from (iii) $\therefore T_k(x) = \frac{1}{2}\left(\frac{1}{2x+\sqrt{4x^2-1}}\right)^k + \frac{1}{2}\left(\frac{1}{2x-\sqrt{4x^2-1}}\right)^k$	1	Students needed to state where $\frac{1}{2}$ and $2x \pm \sqrt{4x^2-1}$ came from This was a gift – the opportunity to go back and fix mistakes in (iii) and (v)

MATHEMATICS Extension 2: Question 16___

Suggested Solutions	Marks	Marker's Comments
(ix) $\displaystyle\lim_{n\to\infty} \frac{T_{n+1}(x)}{T_n(x)}$		
$\displaystyle = \lim_{n\to\infty} \frac{\frac{1}{2}\left(\frac{1}{2x+\sqrt{4x^2-1}}\right)^{n+1} + \frac{1}{2}\left(\frac{1}{2x-\sqrt{4x^2-1}}\right)^{n+1}}{\frac{1}{2}\left(\frac{1}{2x+\sqrt{4x^2-1}}\right)^{n} + \frac{1}{2}\left(\frac{1}{2x-\sqrt{4x^2-1}}\right)^{n}}$	1	Correct expression with $\lim_{n\to\infty}$
$\displaystyle = \lim_{n\to\infty} \frac{(2x-\sqrt{4x^2-1})^{n+1} + (2x+\sqrt{4x^2-1})^{n+1}}{(2x-\sqrt{4x^2-1})^{n} + (2x+\sqrt{4x^2-1})^{n}}$		
[since $\alpha = 2x+\sqrt{4x^2-1}$ and $\beta = 2x-\sqrt{4x^2-1}$ are reciprocals of each other.]		Some students successfully substituted $\alpha = 2x+\sqrt{4x^2-1}$ and $\beta = 2x-\sqrt{4x^2-1}$
$\displaystyle = (2x+\sqrt{4x^2-1}) \lim_{n\to\infty} \left\{ \frac{\left(\frac{2x-\sqrt{4x^2-1}}{2x+\sqrt{4x^2-1}}\right)^{n+1} + 1}{\left(\frac{2x-\sqrt{4x^2-1}}{2x+\sqrt{4x^2-1}}\right)^{n} + 1} \right\}$		
$= (2x+\sqrt{4x^2-1}) \times 1$ since $\left\lvert \frac{2x-\sqrt{4x^2-1}}{2x+\sqrt{4x^2-1}} \right\rvert < 1$	1	Correct limit expression from correct working.