KILLARA HIGH SCHOOL

2010

TRIAL HIGHER SCHOOL CERTIFICATE

EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) $\displaystyle\int_0^{\frac{\pi}{4}} 2\sec^3 x \tan x \, dx$ **2**

(b) Find $\displaystyle\int \frac{dx}{x^2+4x+6}$ **2**

(c) Use the substitution $x = 4\cos^2\theta$ to evaluate **4**

$$\int_0^2 \sqrt{\frac{x}{4-x}}\, dx$$

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{dx}{\sin x} = \ln 3$ **3**

(e) (i) Use the substitution $u = \pi - x$ to show that

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\pi - x}{\sin x}\, dx$$ **3**

(ii) Hence find the exact value of $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{x}{\sin x}\, dx$ **1**

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 3 - 5i$ and $w = 1 - i$. Find zw and $\frac{2}{iw}$ in the form of $x + iy$. **3**

(b) (i) Express $1 + i$ in modulus-argument form. **1**

(ii) Hence evaluate $(1+i)^{11}$ in the form of $x + iy$. **2**

(c) Sketch the region in the complex number plane where the following inequalities both hold.

$|z - i| \leq 2$ and $0 \leq \arg(z+1) \leq \frac{\pi}{4}$. **3**

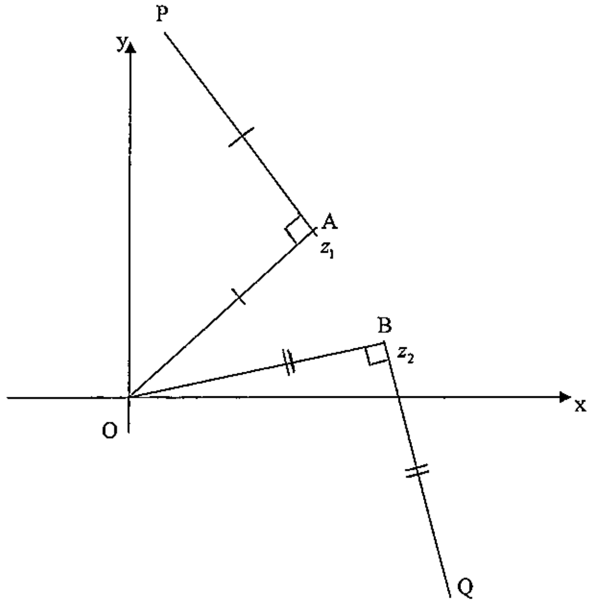
(d) Consider the equation $2z^3 - 3z^2 + 18z + 10 = 0$

(i) Given that $1 - 3i$ is a root of the equation, explain why $1 + 3i$ is also a root. **1**

(ii) Find all the roots of the equation. **1**

Question 2 continues over the page

(e)



The points A and B in the complex number plane correspond to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1+i)z_1$. **2**

(ii) Let M be the midpoint of PQ. What complex number corresponds to M? **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet.

Marks

(a) Consider the hyperbola $\mathcal{K}$ with equation $xy = 4$.

(i) Find the points of intersection of $\mathcal{K}$ with the major axis, the eccentricity and the foci of $\mathcal{K}$. **3**

(ii) Write down the equations of the directrices of $\mathcal{K}$. **1**

(iii) Sketch $\mathcal{K}$. **2**

(b) Consider the equation $z^3 + mz^2 + nz + 6 = 0$, where m and n are real. It is known that $1 - i$ is a root of the equation.

(i) Find the other two roots of the equation. **2**

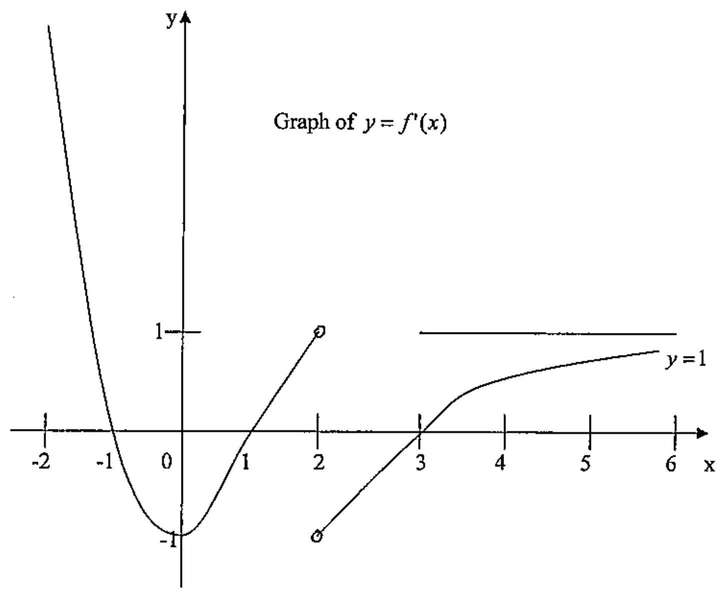
(ii) Find the values of m and n **2**

(c) Find the volume of the solid generated by rotating the area bounded by the curve $y = \log_e x$, the x axis and the line x = 4. Use the method of cylindrical shells. Rotate the area about the y-axis and give your answer correct to 1 decimal place. **5**

Question 4 (15 marks) Use a SEPARATE writing booklet.

Marks

(a)



The diagram shows a sketch of $y = f'(x)$, the derivative function of $y = f(x)$.

The curve $y = f'(x)$ has a horizontal asymptote $y = 1$.

(i) Identify and classify the turning points of the curve $y = f(x)$. **3**

(ii) Sketch the curve $y = f(x)$ given that $f(0) = 0 = f(2)$ and $y = f(x)$ is continuous. On your diagram, clearly identify and label any important features. **4**

Question 4 (continued) **Marks**

(b) The base of a solid is the segment of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Each cross-section, perpendicular to the major axis of the ellipse, is an equilateral triangle.

(i) Show that the area of the cross-section is $A=y^2\sqrt{3}$. **1**

(ii) Hence, or otherwise, find the volume of the solid. **3**

(c) The temperature T_1 of a beaker containing a chemical, and the temperature T_2 of a surrounding vat of cooler water satisfy in accordance with Newton's Law of cooling the equations:

$$\frac{dT_1}{dt}=-k(T_1-T_2) \text{ and } \frac{dT_2}{dt}=\frac{3}{4}k(T_1-T_2) \text{ where } k \text{ is a constant.}$$

(i) Show, by differentiation, that $\frac{3}{4}T_1+T_2=C$ where C is a constant. **2**

(ii) Find an expression for $\frac{dT_1}{dt}$ in terms of T_1, and show that $T_1=\frac{4}{7}C+Be^{\frac{-7}{4}kt}$ satisfies this differential equation for any constant B. **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta,\, b\sin\theta)$ has the equation $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. **3**

(ii) This ellipse meets the y-axis at C and D. Tangents drawn at C and D on the ellipse meet the tangent in (i) at the point E and F respectively. Prove that CE.DF = a^2. **3**

(b) A particle of mass M moves in a straight line with velocity v under the action of two propelling forces $\frac{Mu^2}{v}$ and Mk^2v where u and k are positive constants.

(i) Show that the acceleration equation $\frac{u^2+k^2v^2}{v}$ **1**

(ii) Show that the distance travelled by the particle in increasing its velocity from $\frac{u}{k}$ to $\frac{2u}{k}$ is $\frac{u}{k^3}\left(1-\tan^{-1}\frac{1}{3}\right)$. **4**

Question 5 is continued on the next page

Question 5 *continued*

(c) A code uses a string of the digit 0 and 1 to transmit messages. A message passes Through several relay machines each of which sometimes changes the value of an individual digit from 0 to 1 or from 1 to 0.The probability that a digit will be change by a machine is p.

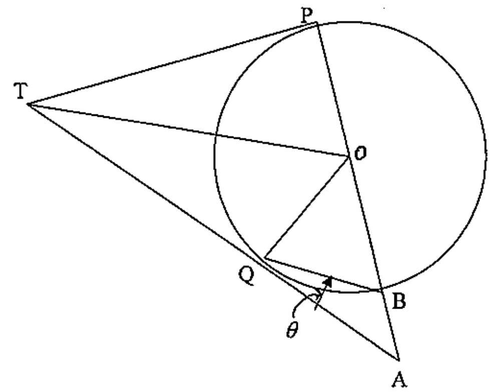
(i) Show that the probability that a single digit when received will be Different to what was sent after passing through two delay machines is $2p(1-p)$. **2**

(ii) If there is a 9.5% chance that a digit will be different to what was sent After passing through two relay machines, find the values of p given that p is less than 10 **2**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



From an external point T, tangents are drawn to a circle with centre O, touching the circle at P and Q. Angle PTQ is acute.

The diameter PB produced meets the tangent TQ at A. Let $\theta = \angle AQB$.

Copy the diagram above into your answer booklet.

(i) Prove that $\angle PTQ = 2\theta$. **2**

(ii) Prove that ΔPBQ and ΔTOQ are similar. **2**

(iii) Hence show that $BQ \times OT = 2(OP)^2$. **2**

Question 6 (continued) **Marks**

(b) If $(x-r)^2$ is a factor of the polynomial $P(x)$, prove that $x-r$ is a factor of the polynomial $P'(x)$. **2**

(c) The polynomial equation $x^4+x^3+1=0$ has roots x_1, x_2, x_3 and x_4.

Construct a polynomial equation whose four roots are x_1^2, x_2^2, x_3^2 and x_4^2. **3**

(d) The length of an arc joining $P(a,c)$ and $Q(b,d)$ on a smooth continuous curve $y=f(x)$ is given by

$$\text{arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Consider the curve defined by $y=\dfrac{x^2}{4}-\dfrac{\ln x}{2}$.

(i) Show that $1+\left(\dfrac{dy}{dx}\right)^2=\dfrac{1}{4}\left(x+\dfrac{1}{x}\right)^2$. **2**

(ii) Find the length of the arc between $x=1$ and $x=e$. **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The equation $ax^3+bx^2+d=0$, has a double root. **5**

Show that $27a^2d+4b^3=0$.

(b) Given that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are acute: **5**

(i) Show that: $\sin(\sin^{-1}x-\cos^{-1}x)=2x^2-1$

(ii) Solve the equation: $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(1-x)$.

(c) (i) If $\dfrac{z^2}{z-1}$ is always real, show that the locus of the point represented by z on the argand plane lies on a line and a circle. **4**

(i) State which line and which circle. **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let x, y, z and w be positive real numbers.

(i) Prove that $\frac{x}{y}+\frac{y}{x}\geq 2$. **2**

(ii) Deduce that $\frac{x+y+z}{w}+\frac{w+y+z}{x}+\frac{w+x+z}{y}+\frac{w+x+y}{z}\geq 12$. **2**

(iii) Hence prove that if $x+y+z+w=1$,

then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\geq 16$. **2**

(b) Let $J_n=\int_0^1 x^n e^{-x}\,dx$, where $n\geq 0$.

(i) Show that $J_0=1-\frac{1}{e}$. **1**

(ii) Show that $J_n=nJ_{n-1}-\frac{1}{e}$, for $n\geq 1$. **2**

(iii) Show that $J_n\to 0$ as $n\to\infty$. **1**

(iv) Deduce by the principle of mathematical induction that for all $n\geq 0$,

$$J_n=n!-\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}.$$ **4**

(v) Conclude that $e=\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right)$. **1**

THE END

EX2 _ Trial _ 2010

Q1 (a)

$$\int_0^{\pi/4} 2\sec^3 x\tan x\,dx = \frac{2}{3}\left[\sec^3 x\right]_0^{\pi/4} \checkmark$$

$$= \frac{2}{3}\left(2\sqrt{2}-1\right) \checkmark$$

(b)
$$\int \frac{dx}{x^2+4x+6} = \int \frac{dx}{(x+2)^2+2} \checkmark = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+2}{\sqrt{2}}\right)+c \checkmark$$

(c) let $x = 4\cos^2\theta$ $\quad x=0 \Rightarrow \theta = \pi/2$

$dx = -8\cos\theta\sin\theta$ $\quad x=2 \Rightarrow \theta = \pi/4$

$$\int_0^2 \sqrt{\frac{x}{4-x}}\,dx = -\int_{\pi/2}^{\pi/4} \sqrt{\frac{4\cos^2\theta}{4-4\cos^2\theta}} \cdot 8\cos\theta\sin\theta\,d\theta \checkmark$$

$$= 4\int_{\pi/4}^{\pi/2} \cos 2\theta + 1 \; d\theta \checkmark$$

$$= 4\left[\sin\frac{2\theta}{2} + \theta\right]_{\pi/4}^{\pi/2} \checkmark$$

$$= 4\left(\frac{\pi}{2} - \frac{1}{2} - \frac{\pi}{4}\right)$$

$$= \pi - 2 \checkmark$$

(d) let $t = \tan\frac{x}{2}$ $\quad x = \frac{2\pi}{3} \Rightarrow t = \sqrt{3}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$ $\quad x = \pi/3 \Rightarrow t = \frac{1}{\sqrt{3}}$

$$\frac{2\,dt}{1+t^2} = dx$$

$$\int_{\pi/3}^{2\pi/3} \frac{dx}{\sin x} = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} \checkmark$$

$$= \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dt}{t} = \left[\ln t\right]_{1/\sqrt{3}}^{\sqrt{3}} \checkmark$$

$$= \ln\sqrt{3} - \ln\left(\frac{1}{\sqrt{3}}\right) = \ln 3 \checkmark$$

(e) let $u = \pi - x$ $\quad x = \frac{2\pi}{3} \Rightarrow u = \frac{\pi}{3}$ $\quad$ and $du = -dx$

$x = \pi - u$ $\quad x = \pi/3 \Rightarrow u = \frac{2\pi}{3}$

$$\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \int_{2\pi/3}^{\pi/3} \frac{\pi - u}{\sin(\pi-u)} \cdot -du \checkmark$$

$$= -\int_{2\pi/3}^{\pi/3} \frac{\pi-u}{\sin u}\,du$$

$$= \int_{\pi/3}^{2\pi/3} \frac{\pi-u}{\sin u}\,du$$

$$= \int_{\pi/3}^{2\pi/3} \frac{\pi-x}{\sin x}\,dx \checkmark$$

$$= \int_{\pi/3}^{2\pi/3} \frac{\pi}{\sin x} - \frac{x}{\sin x}\,dx$$

$$\Rightarrow 2\int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \pi\int_{\pi/3}^{2\pi/3} \frac{dx}{\sin x} \checkmark$$

$$= \pi\ln 3 \quad \text{from (d)}$$

$$\therefore \int_{\pi/3}^{2\pi/3} \frac{x}{\sin x}\,dx = \frac{\pi}{2}\ln 3 \checkmark$$

Question 2

a) $z = 3-5i \qquad w = 1-i$

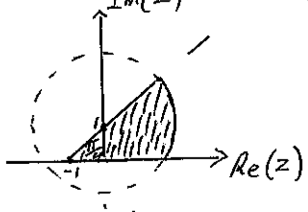
i $zw = (3-5i)(1-i)$
$= 3-5-3i-5i$
$= -2-8i$ ✓

ii $\frac{2}{iw} = \frac{2}{i(1-i)}$
$= \frac{2}{i+1} \times \frac{1-i}{1-i}$ ✓
$= \frac{2(1-i)}{2}$
$= 1-i$ ✓

b) i $1+i$
$r = \sqrt{2}, \quad \theta = \frac{\pi}{4}$
$\therefore 1+i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$ ✓

ii $(1+i)^{11} = \sqrt{2}^{11} \operatorname{cis} \frac{11\pi}{4}$
$= \sqrt{2}^{11} \operatorname{cis} \frac{3\pi}{4}$ ✓
$\cancel{= \sqrt{2}^{11} \operatorname{cis}\left(-\frac{\pi}{4}\right)}$
$= \sqrt{2}^{11}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$
$= -\sqrt{2}^{10} + i\sqrt{2}^{10}$
$= -32+32i$ ✓

c)



1 circle
1 arg (line)
1 shading

d) i The complex conjugate $1+3i$ of $1-3i$ is a root ✓ because the coefficients of $2z^3-3z^2+18z+10=0$ are real.

ii let α be the 3rd root
$\therefore \alpha + 1+3i + 1-3i = -\frac{b}{a} = \frac{3}{2}$
ie $\alpha + 2 = \frac{3}{2}$
$\alpha = -\frac{1}{2}$
✓ $\therefore$ the 3 roots are $1+3i, 1-3i, -\frac{1}{2}$

Question 2

e i $\vec{OA} = z_1 \qquad \vec{AP} = iz_1$
$\vec{OP} = \vec{OA} + \vec{AP}$ ✓
$= z_1 + iz_1$
$= (1+i)z_1$ ✓

ii $M = \frac{\vec{OP} + \vec{OQ}}{2}$ where $\vec{OQ} = (1-i)z_2$ ✓
$= \frac{z_1(1+i) + z_2(1-i)}{2}$ ✓

or $\frac{z_1 + z_2 + (z_1 - z_2)i}{2}$

Q3

(a) $x \cdot y = 4$
$= \frac{1}{2}a^2$

$\Rightarrow a = \sqrt{8} = 2\sqrt{2}$

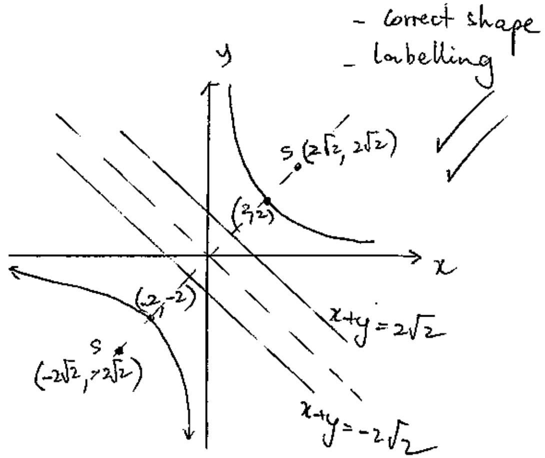
$c^2 = 4 \Rightarrow c = 2$

(i) $(c, c) = (2, 2)$ ✓

eccentricity $e = \sqrt{2}$ ✓

foci $(\pm 2\sqrt{2}, \pm 2\sqrt{2})$ ✓

iii) Directrices $x + y = \pm 2\sqrt{2}$ ✓



(b) (i) The coefs' of $P(x)$ are real $\therefore$ complex roots are conjugate

$\therefore$ $1+i$ is also a root of $P(x)$ ✓

product of root: $(1+i)(1-i)\cdot\alpha = -6$

$2\alpha = -6$

$\alpha = -3$

$\therefore$ $1+i$ & 3 are the other 2 roots ✓

(ii) Sum of roots : $1+i + 1-i + 3 = m$

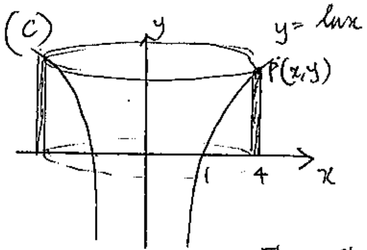
$\Rightarrow m = 1$ ✓

Σ product of roots 2 at a time : $(1+i)(1-i) + (1+i)3 + (1-i)3 = n$

$2 + 3 - 3 = n$

$\therefore n = -4$ ✓

(c)



each cylindrical shell has

radius $= x$

height $= y$

thickness $= \partial x$

The shell is cut opened to become a rectangula sheet

$\partial V = 2\pi x y \, dx$

$= 2\pi x \ln x \, dx$ ✓

Volume of the solid

$V = \lim_{\delta x \to 0} \sum_{x=1}^{4} 2\pi x \ln x \, dx$

$= 2\pi \int_1^4 x \ln x \, dx$ ✓

$= 2\pi \left[\frac{x^2}{2} \ln x \right]_1^4 - \frac{2\pi}{2} \int_1^4 x^2 \cdot \frac{1}{x} \, dx$ ✓

$= 2\pi \cdot 8 \ln 4 - \pi \left[\frac{x^2}{2} \right]_1^4$

$= 16\pi \ln 4 - \pi \left(\frac{16}{2} - \frac{1}{2} \right)$ ✓

$= 16\pi \cdot \ln 4 - \frac{15\pi}{2}$

$= 46.1$ cube units ✓

Question 4

a i the turning points are at $x = -1, 1$ and 3 ✓

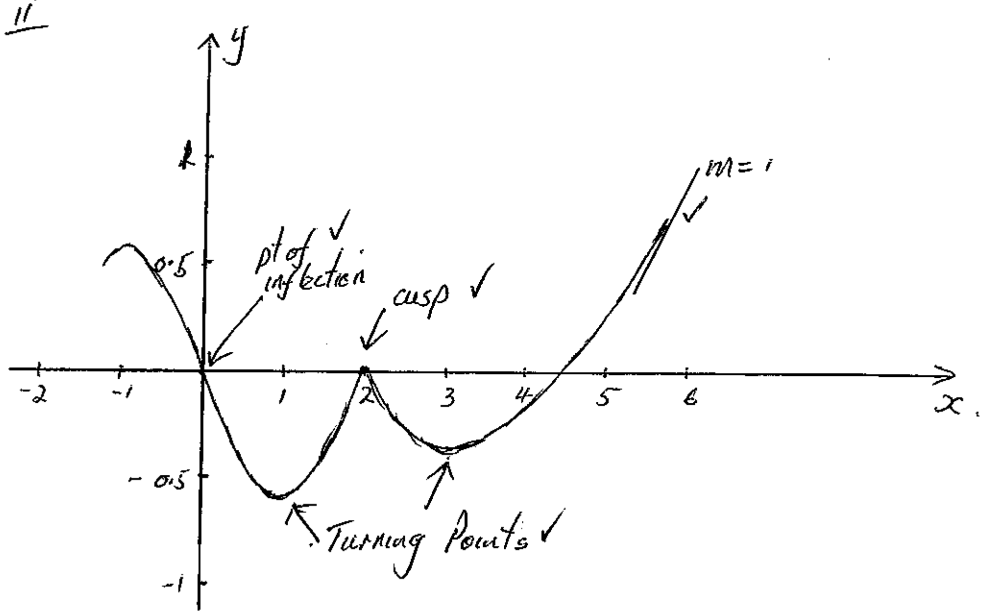
at $x=-1$, $f'(x)$ is decreasing, $\therefore$ Max T. Pt at $x=-1$

at $x=1$, $f'(x)$ is increasing, $\therefore$ Min T. Pt at $x=1$

at $x=3$, $f'(x)$ is increasing $\therefore$ Min T. Pt at $x=3$

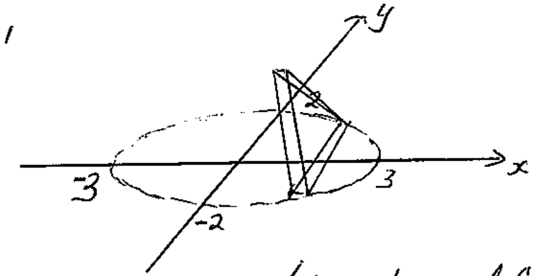
3 Marks (1 off for each error).

ii

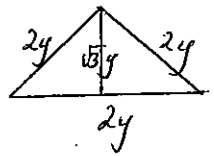


Question 4

b $\frac{x^2}{9} + \frac{y^2}{4} = 1$



i let the cross section by of "dx" thick with the base '$2y$' in length.



$$\text{Area} = \frac{1}{2} \times b \times h = \frac{1}{2} \times 2y \times \sqrt{3}y \quad ✓$$
$$= \sqrt{3}\,y^2$$

ii Volume $= \lim_{dx\to 0} \sum_{-3}^{3} \sqrt{3}(y^2)\,dx$.

$$\text{Volume} = 2\int_0^3 \sqrt{3}\,y^2\,dx \quad ✓$$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\frac{y^2}{4} = 1 - \frac{x^2}{9}$

$y^2 = 4\left(1 - \frac{x^2}{9}\right)$

$$= 2\sqrt{3}\int_0^3 4\left(1 - \frac{x^2}{9}\right)dx$$
$$= 8\sqrt{3}\int_0^3 \left(1 - \frac{x^2}{9}\right)dx \quad ✓$$
$$= 8\sqrt{3}\left[x - \frac{x^3}{27}\right]_0^3$$
$$= 8\sqrt{3}\left[3 - 1\right]$$
$$= \cancel{24}\ 16\sqrt{3} \text{ units}^3 \quad ✓$$

Question 4

c i $\frac{dT_1}{dt} = -k(T_1 - T_2)$ $\frac{dT_2}{dt} = \frac{3}{4}k(T_1 - T_2)$

Consider $\frac{3}{4}T_1 + T_2$

$$\frac{d}{dt}\left(\frac{3}{4}T_1 + T_2\right) = \frac{3}{4}\frac{d}{dt}T_1 + \frac{d}{dt}T_2 \quad ✓$$
$$= -\frac{3}{4}k(T_1 - T_2) + \frac{3}{4}k(T_1 - T_2)$$
$$= 0 \quad ✓$$

$\therefore \frac{3}{4}T_1 + T_2 = c$ (a constant) — ②

ii From ② $T_2 = c - \frac{3}{4}T_1$

Thus $\frac{dT_1}{dt}$ becomes.

$$\left.\begin{aligned}\frac{dT_1}{dt} &= -k\left(T_1 - c + \frac{3}{4}T_1\right)\\ &= kc - \frac{7}{4}kT_1\end{aligned}\right\} \quad ✓$$

Now Consider $T_1 = \frac{4}{7}c + Be^{-\frac{7}{4}kt}$ — ①

$$\frac{dT_1}{dt} = 0 - \frac{7kB}{4}e^{-\frac{7}{4}kt} \quad — ②$$

✓ but $\frac{dT_1}{dt} = kc - \frac{7}{4}kT_1$

$= kc - \frac{7}{4}k\left(\frac{4}{7}c + Be^{-\frac{7}{4}kt}\right)$ from ①

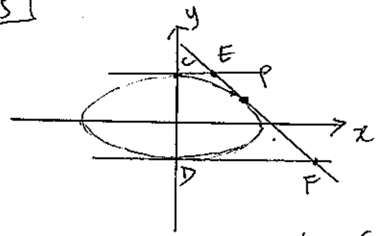
$= kc - kc - \frac{7kB}{4}e^{-\frac{7}{4}kt}$

$= 0 - \frac{7kB}{4}e^{-\frac{7}{4}kt}$ which is equal to the above result (②)

$\therefore T_1 = \frac{4}{7}c + Be^{-\frac{7}{4}kt}$ satisfies $\frac{dT_1}{dt}$.

Q5

(i)



gradient of the tangent

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy = 0$$

$$\Rightarrow m = -\frac{\cancel{2}b^2x}{\cancel{2}a^2y}$$

at $P(a\cos\theta, b\sin\theta) \Rightarrow m = -\frac{b^{\cancel{2}}\cdot\cancel{a}\cos\theta}{a^{\cancel{2}}\cancel{b}\sin\theta}$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

equation of the tangent at P

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\frac{y\sin\theta}{b} - \frac{\cancel{b}\sin^2\theta}{\cancel{b}} = -\frac{x\cos\theta}{a} + \frac{\cancel{a}\cos^2\theta}{\cancel{a}}$$

$$\Rightarrow \frac{y\sin\theta}{b} + \frac{x\cos\theta}{a} = \sin^2\theta + \cos^2\theta \quad (1)$$

$\therefore \frac{y\sin\theta}{b} + \frac{x\cos\theta}{a} = 1$ is the equation of the tangent at P.

ii) equation of the tangent at C is $y = b$

equation of the tangent at D is $y = -b$

Sub. $y = \pm b$ into (1)

$$\frac{+b\sin\theta}{\cancel{b}} + \frac{x\cos\theta}{a} = 1 \quad \text{and} \quad \frac{-b\sin\theta}{b} + \frac{x\cos\theta}{a} = 1$$

$$x = \frac{a(1-\sin\theta)}{\cos\theta} \qquad x = \frac{a(1+\sin\theta)}{\cos\theta}$$

coordinates of $E\left(b, \frac{a(1-\sin\theta)}{\cos\theta}\right)$ and $F\left(\frac{a(1+\sin\theta)}{\cos\theta}, -b\right)$

$$CE \cdot DF = \frac{a(1-\sin\theta)}{\cos\theta} \cdot \frac{a(1+\sin\theta)}{\cos\theta}$$

$$= \frac{a^2(1-\sin^2\theta)}{\cos^2\theta} = a^2$$

(b) The force acts on the particle is:

$$ma = \left(\frac{Mu^2}{v} + Mk^2v\right)$$

$$= M\left(\frac{u^2}{v} + vk^2\right)$$

$$\therefore a = \frac{u^2 + v^2k^2}{v}$$

$\therefore$ The equation of motion is

$$\ddot{x} = \frac{u^2 + v^2k^2}{v}$$

$$v\frac{dv}{dx} = \frac{u^2 + v^2k^2}{v}$$

$$\frac{dv}{dx} = \frac{u^2 + v^2k^2}{v^2} \Rightarrow dx = \frac{v^2}{u^2 + v^2k^2}dv$$

$$\therefore k^2dx = \left(1 - \frac{u^2}{u^2 + v^2k^2}\right)dv$$

integrate both sides

$$\Rightarrow k^2x = v - \frac{u}{k}\tan^{-1}\frac{kv}{u} + c$$

when $x = 0$ $v = \frac{u}{k}$ $\Rightarrow c = \frac{u}{k}\tan^{-1}1 - \frac{u}{k}$

$$\therefore x = \frac{v}{k^2} - \frac{u}{k^3}\tan^{-1}\frac{kv}{u} + \frac{u}{k^3}\tan^{-1}1 - \frac{u}{k^3}$$

when $v = \frac{2u}{k}$

$$x = \frac{2u}{k^3} - \frac{u}{k^3} - \frac{u}{k^3}\left(\tan^{-1}2 - \tan^{-1}1\right)$$

$$= \frac{u}{k^3}\left(1 + \tan^{-1}\frac{1}{3}\right)$$

Note: let $A = \tan^{-1}2$, $B = \tan^{-1}1$

$$\tan(A - B) = \frac{2-1}{1+2\times1} = \frac{1}{3}$$

$$\therefore A - B = \tan^{-1}\frac{1}{3}$$

hence $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{3}$

(c) Q5 Cont

$P(\text{digit will be changed}) = p$

$P(\text{digit won't be changed}) = 1-p$

(i) $P = p(1-p) + (1-p)\cdot p$

$= 2p(1-P)$ ✓

ii) $2p(1-p) = 0.095$

$2p^2 - 2p + 0.095 = 0$ ✓

$$p = \frac{2 \pm \sqrt{4 - 4\times2\times0.095}}{4}$$

$$= \frac{2 \pm 1.8}{4}$$

$\Rightarrow p = 95\%$ or 5% ✓

Since p is less than 10%, $P = 5\%$. ✓

Question 6

a i $\angle BQA = \theta$

$\therefore \angle BPQ = \theta$ (alternate segment theorem) ✓

$\therefore \angle PQO = \theta$ ($\triangle OQP$ is isosceles)

$\therefore \angle POQ = 180 - 2\theta$ (angles sum of $\triangle POQ$)

Now $\angle TPO = \angle TQO = 90°$ (radius and tangent) ✓

$\therefore PTQ = 2\theta$ (angle sum of quadrilateral $TPOQ$)

ii Consider $\triangle PBQ$ and $\triangle TOQ$

* $\angle PQB = 90$ (angle in a semi circle)
$= \angle TQO$ ✓

* $\angle OTQ = \theta$
$= \angle QPO$ ($\angle QPB$) from i ✓

Hence $\triangle PBQ \parallel\!\parallel \triangle TOQ$ (AA).

iii $\frac{PB}{TO} = \frac{BQ}{OQ}$ (corresponding sides of similar triangles)

However, $PB = 2OP$ and $OQ = OP$ ✓

$\therefore \frac{2OP}{TO} = \frac{BQ}{OP}$ ✓

Thus $BQ \times OT = 2(OP)^2$

b let $P(x) = (x-r)^2 \cdot Q(x)$ where $Q(x)$ ✓ is a polynomial

$\therefore P'(x) = 2(x-r)\cdot Q(x) + (x-r)^2 \cdot Q'(x)$

$= (x-r)\left[2Q(x) + (x-r)Q'(x)\right]$ ✓

$\therefore x-r$ is a factor of $Q(x)$.

Question 6

c Replace x with $\sqrt{x}$

$\therefore (\sqrt{x})^4 + (\sqrt{x})^3 + 1 = 0$ ✓

$x^2 + x^{3/2} + 1 = 0$

$x^{3/2} = -(x^2+1)$

$x^3 = (x^2+1)^2$ ✓

$x^3 = x^4 + 2x^2 + 1$

$x^4 - x^3 + 2x^2 + 1 = 0$ ✓

d i

$y = \frac{x^2}{4} - \frac{\ln x}{2}$

$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$

$= \frac{1}{2}\left(x - \frac{1}{x}\right)$ ✓

Now, $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(x - \frac{1}{x}\right)^2$

$= 1 + \frac{1}{4}\left(x^2 + \frac{1}{x^2} - 2\right)$

$= \frac{1}{4}\left(x^2 + \frac{1}{x^2} + 2\right)$

$= \frac{1}{4}\left(x + \frac{1}{x}\right)^2$ ✓

ii Arc length $= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_1^e \sqrt{\frac{1}{4}\left(x + \frac{1}{x}\right)^2}\, dx$ ✓

$= \frac{1}{2}\int_1^e \left(x + \frac{1}{x}\right) dx$

$= \frac{1}{2}\left[\frac{x^2}{2} + \ln x\right]_1^e$

$= \frac{1}{2}\left(\frac{e^2}{2} + 1 - \frac{1}{2}\right)$

$= \frac{1}{2}\left(\frac{e^2}{2} + \frac{1}{2}\right)$

$= \frac{1}{4}\left(e^2 + 1\right)$ units ✓

Q7

(a) $ax^3 + bx^2 + d = 0$

$P'(x) = 3ax^2 + 2bx$

$P''(x) = 6a + 2b$

for double root $P'(x) = 0$

$\therefore \ x(3ax + 2b) = 0$

$x = 0 \quad \text{or} \quad x = -\frac{2b}{3a}$

* if $x = 0$ is a double root then $P(0) = d = 0$
and if $27a^2d + 4b^3 = 0$ then $4b^3 = 0 \Rightarrow b = 0$
$\therefore P(x) = ax^3$ hence 0 is a tripple root of $P(x)$ and
$27a^2d + 4b^3 \neq 0$

* if $x = -\frac{2b}{3a}$ is a double root then

$$P\left(-\frac{2b}{3a}\right) = -a\left(\frac{2b}{3a}\right)^3 + b\left(\frac{2b}{3a}\right)^2 + d = 0$$

$$\frac{-8b^3}{27a^2} + \frac{4b^3}{9a^2} + d = 0$$

$$\Rightarrow \quad \frac{4b^3 + 27a^2d}{27a^2} = 0$$

$$\therefore \quad 27a^2d + 4b^3 = 0$$

(b) let $A = \sin^{-1}x$, $B = \cos^{-1}x$

(i) $\sin(\sin^{-1}x - \cos^{-1}x) = \sin(A - B)$

$= \sin A \cos B - \cos A \sin B$

$= x \cdot x - \sqrt{1-x^2} \cdot \sqrt{1-x^2}$

$= x^2 - 1 + x^2$

$= 2x^2 - 1$

ii)

$\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$

$\sin(\sin^{-1}x - \cos^{-1}x) = \sin(\sin^{-1}(1-x))$

$2x^2 - 1 = 1 - x$ from (i)

$2x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1 + 4 \times 2 \times 2}}{4}$$

$$= \frac{-1 \pm \sqrt{17}}{4}$$

as $-1 \le x \le 1$, $x = \frac{-1+\sqrt{17}}{4}$

(c) if $\frac{z^2}{z-1}$ is always real then $Im(z) = 0$

let $z = a + ib$

$$\frac{z^2}{z-1} = \frac{(a+ib)^2}{(a-1)+ib}$$

$$= \frac{(a^2 - b^2 + 2abi)}{(a-1)+ib} \frac{(a-1-ib)}{(a-1-ib)}$$

$$= \frac{(a^2-b^2)(a-1) + 2ab^2 + (a-1)2abi - (a^2-b^2)ib}{(a-1)^2 + b^2}$$

$$Im\, z = \frac{(a-1)2ab - (a^2-b^2)b}{(a-1)^2 + b^2} = 0$$

$\Rightarrow b(2a^2 - 2a - a^2 + b^2) = 0$

$b(a^2 - 2a + 1 + b^2 - 1) = 0$

$b((a-1)^2 + b^2 - 1) = 0$

$\therefore \ b = 0 \quad \text{or} \quad (a-1)^2 + b^2 = 1$

let $x = a$ and $y = b$

Then $y = 0$ and $(x-1)^2 + y^2 = 1$

ii) The line is the x-axis except $x = 1$ and the circle has radius $= 1$ unit & the centre $(1, 0)$

Question 8

a i $\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 = \frac{x}{y} + \frac{y}{x} - 2$ ✓

But $\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)^2 \geq 0$ ✓

$\therefore \frac{x}{y} + \frac{y}{x} - 2 \geq 0$

$\frac{x}{y} + \frac{y}{x} \geq 2.$

ii $\frac{x+y+z}{w} + \frac{w+y+z}{x} + \frac{w+x+z}{y} + \frac{w+x+y}{z}$ — ①

$= \frac{x}{w} + \frac{y}{w} + \frac{z}{w} + \frac{w}{x} + \frac{y}{x} + \frac{z}{x} + \frac{w}{y} + \frac{x}{y} + \frac{z}{y} + \frac{w}{z} + \frac{x}{z} + \frac{y}{z}$

$= \left(\frac{x}{w} + \frac{w}{x}\right) + \left(\frac{y}{w} + \frac{w}{y}\right) + \left(\frac{z}{w} + \frac{w}{z}\right) + \left(\frac{y}{x} + \frac{x}{y}\right) + \left(\frac{x}{z} + \frac{z}{x}\right) + \left(\frac{z}{y} + \frac{y}{z}\right)$ ✓

$\geq 2 + 2 + 2 + 2 + 2 + 2 = 12$ (from i) ✓

iii $x + y + z + w = 1$

$\therefore x + y + w = 1 - z,\ x + y + z = 1 - w,\ x + w + z = 1 - y,\ w + y + z = 1 - x$

substitute into ① in ii

$\therefore \frac{1-w}{w} + \frac{1-x}{x} + \frac{1-y}{y} + \frac{1-z}{z} \geq 12$ ✓

$\frac{1}{w} + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 4 \geq 12$

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} \geq 16$ ✓

Question 8

b i $J_0 = \int_0^1 e^{-x}\,dx$

$= -\left[e^{-x}\right]_0^1$

$= -(e^{-1} - e^0)$ ✓

$= 1 - \frac{1}{e}$

ii let $u = x^n$ and $v = -e^{-x}$

$u' = nx^{n-1}$ and $v' = e^{-x}$

$\therefore J_n = \int_0^1 x^n e^{-x}\,dx$

$= \int_0^1 u v'\,dx$

Now $\int_0^1 u v'\,dx = uv - \int v u'\,dx$

$\int_0^1 x^n e^{-x}\,dx = -\left[x^n e^{-x}\right]_0^1 - \int_0^1 n x^{n-1}(-e^{-x})\,dx$ ✓

$J_n = -e^{-1} + n\int_0^1 x^{n-1} e^{-x}\,dx$

$= -e^{-1} + nJ_{n-1}$ ✓

iii Since $0 \leq e^{-x} \leq 1$

for $0 \leq x \leq 1$

$\therefore 0 \leq \int_0^1 x^n e^{-x}\,dx \leq \int_0^1 x^n\,dx$

However, $\int_0^1 x^n\,dx = \frac{1}{n+1} \to 0$ as $n \to \infty$

$\therefore \int_0^1 x^n e^{-x}\,dx \to 0$ as $n \to \infty$ ✓

ie $J_n \to 0$ as $n \to \infty$

v ✓ From iv (PTO)

$J_n = n! - \frac{n!}{e}\sum_{r=0}^{n} \frac{1}{r!}$

$\therefore \sum_{r=0}^{n} \frac{1}{r!} = -\frac{eJ_n}{n!} + e$

as $n \to \infty$, RHS $\to 0 + e$ from (iii)

Thus $\lim_{n\to\infty}\left(\sum_{r=0}^{n} \frac{1}{r!}\right) = e$ ✓

Question 8

b iv * When $n = 0$, $J_0 = 0! - \frac{0!}{e} \sum_{r=0}^{0} \frac{1}{r!}$

$= 1 - \frac{1}{e}$

$\therefore$ true for $n = 0$ ✓

* Assume true for $n = k$, $k > 0$

ie $J_k = k! - \frac{k!}{e} \sum_{r=0}^{k} \frac{1}{r!}$

* We need to prove true for $n = k+1$

ie $J_{k+1} = (k+1)! - \frac{(k+1)!}{e} \sum_{r=0}^{k+1} \frac{1}{r!}$

* LHS $= J_{k+1}$

$= (k+1) J_k - \frac{1}{e}$ ✓ from (ii)

$= (k+1)\left[k! - \frac{k!}{e} \sum_{r=0}^{k} \frac{1}{r!}\right] - \frac{1}{e}$

$= (k+1)! - \frac{(k+1)!}{e} \sum_{r=0}^{k} \frac{1}{r!} - \frac{1}{e}$

$= (k+1)! - \frac{(k+1)!}{e} \sum_{r=0}^{k} \frac{1}{r!} - \frac{(k+1)!}{e}\left(\frac{1}{(k+1)!}\right)$

$= (k+1)! - \frac{(k+1)!}{e}\left[\sum_{r=0}^{k} \frac{1}{r!} - \frac{1}{(k+1)!}\right]$ ✓

$= (k+1)! - \frac{(k+1)!}{e} \sum_{r=0}^{k+1} \frac{1}{r!}$ (required form)

* Thus, if true for $n = k$ then true for $n = k+1$
It is true for $n = 0$, $\therefore$ true for $n = 1, 2, 3$ and so on. Thus true for all $n \geq 0$. ✓