KINCOPPAL-ROSE BAY
SCHOOL OF THE SACRED HEART

2001

EXTENSION 2 MATHEMATICS

Thursday 28th June, 2001

Time Allowed : Three hours
(plus five minutes reading time)

DIRECTIONS TO CANDIDATES:

- All questions may be attempted
- Start each question in a new booklet.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Only Board-approved calculators are to be used.
- Standard integrals are printed on a separate page.

Question 1

(a) Find $\int \tan^3\theta \sec^2\theta \, d\theta$ **2**

(b) By completing the square find $\int \frac{4}{x^2+16x+68} dx$ **2**

(c) (i) Find a, b and c such that $\frac{2x^2-17x+20}{x^2(4-x)} = \frac{ax+b}{x^2} + \frac{c}{4-x}$ **2**

(ii) Find $\int \frac{2x^2-17x+20}{x^2(4-x)} dx$ **2**

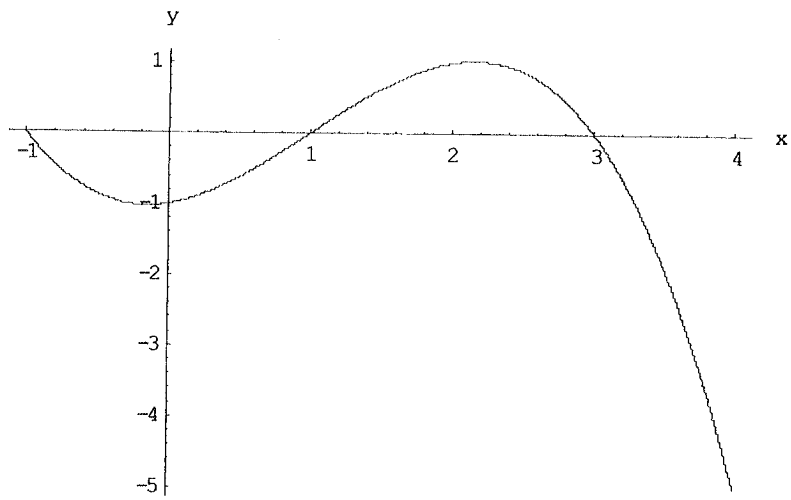
(d) Show that $\int_0^{\pi} e^{-x} \sin x \, dx = \frac{1+e^{-\pi}}{2}$ **3**

(e) Use the substitution $t = \tan\frac{x}{2}$ to find $\int \frac{1}{2\sin x + \cos x + 1} dx$ **4**

Questions continue over...

Question 2 : **Start a new booklet**

(a) A function $y = f(x)$ is drawn below. Sketch the following on separate neat number planes about one third of a booklet in size. Show all important features (including the vertices and endpoints) but **do not use calculus**.



(i) $y = |f(x)|$ **2**

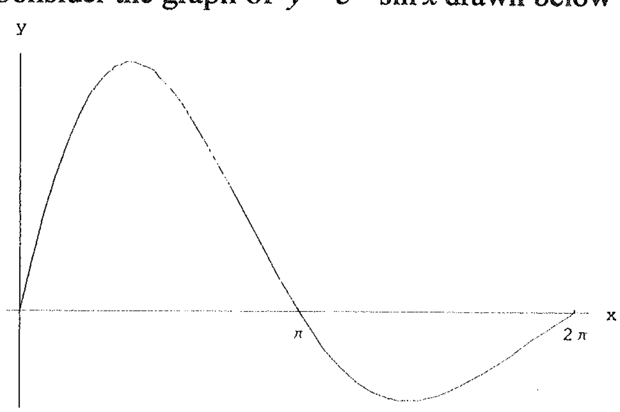
(ii) $y = f(|x|)$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(v) $y = f(-x)$ **2**

(b) Consider the graph of $y = e^{-x}\sin x$ drawn below over the domain $0 \le x \le 2\pi$.



(i) Copy the diagram into your answers and draw any horizontal tangents. **1**

(ii) Determine the range of this function over the given domain. **4**

Question 3 : **Start a new booklet**

(a) Two complex numbers are defined as $a = 2 + 5i$ and $b = \sqrt{3} - i$.

Evaluate the following, writing you answer in the form $x + iy$

(i) $\bar{a}b$ **2**

(ii) $\dfrac{b}{a}$ **1**

(b) If $b = \sqrt{3} - i$, find:

(i) $|b|$ **1**

(ii) the principal argument of b **1**

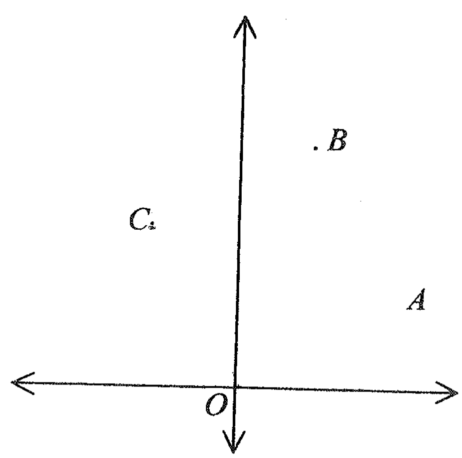
(iii) the value of b^9 **2**

(c) On an Argand Diagram draw a neat sketch of the locus of z defined by:

(i) the intersection of $-\dfrac{\pi}{3} \le \arg z \le \dfrac{\pi}{3}$ and $1 \le |z| \le 2$ **2**

(ii) $(z\bar{z})^{2} + (\text{Re}(z))^2 - 1 = 0$ **3**

(d) The points represented by $OABC$ form the vertices of a square.



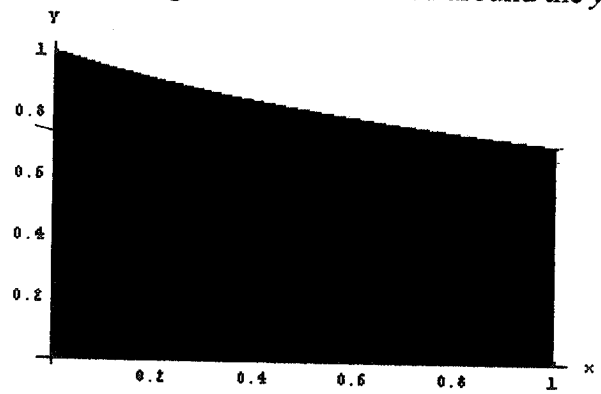
The vertex at A represents the complex number $\sqrt{3} + i$.

(i) Find the complex number represented by the vertex at C. **1**

(ii) Find the argument of the complex number represented by B. **2**

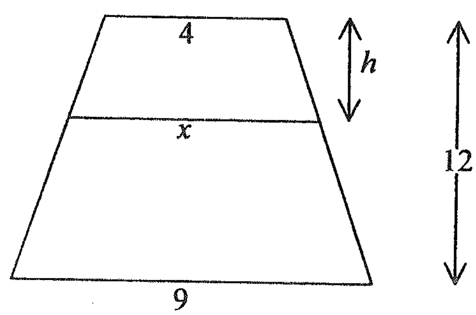
Question 4 : **Start a new booklet**

(a) The shaded area in the graph below represents the area between the curve $y = \frac{1}{\sqrt{x+1}}$, the coordinate axes and $x = 1$. Use the method of cylindrical shells to find the volume that is generated when the region below is rotated around the y-axis. 4

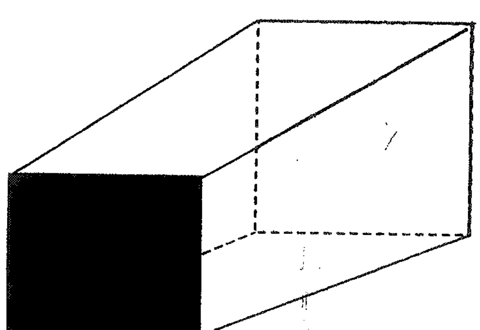


(b) Now suppose the area above is to be rotated around the line $x = 1$. Use the method of slicing to find the volume that is generated. 5

(c) (i) By using areas of trapezia or otherwise show that $x = \frac{5h-30}{12}$ 2



(ii) A model of Buffy's toy box is formed by joining 2 squares of area 16 cm^2 and 81 cm^2 which are 12 cm apart. Find the volume of the toy box. 4



Question 5 : **Start a new booklet**

(a) (i) If $(x-a)$ is a repeated factor of $P(x)$ of multiplicity n, prove that $(x-a)$ is a repeated factor of multiplicity $(n-1)$ of $P'(x)$ **3**

(ii) Given that the polynomial $P(x)=12x^3-32x^2+15x+9$ has a repeated factor, factorise $P(x)$ completely. **3**

(b) The roots of a certain cubic equation are α, β and γ. Given the following:

$$\alpha+\beta+\gamma=-3$$
$$\alpha^2+\beta^2+\gamma^2=29$$
$$\alpha\beta\gamma=-6$$

Form the cubic equation whose roots are α, β and γ. **3**

(c) Let ω be a non-real cube root of unity.

(i) Show that $1+\omega+\omega^2=0$ **1**

(ii) Hence simplify $(1+\omega)^2$ **1**

(iii) Show that $(1+\omega)^3=-1$

(iv) Use part (iii) and the fact that $\mathrm{Re}(\omega)=-\frac{1}{2}$ and $\mathrm{Re}(\omega^2)=-\frac{1}{2}$ to simplify $(1+\omega)^{3n}$ and hence show that **1**

$${}^{3n}C_0-\frac{1}{2}\left({}^{3n}C_1+{}^{3n}C_2\right)+{}^{3n}C_3-\frac{1}{2}\left({}^{3n}C_4+{}^{3n}C_5\right)+{}^{3n}C_6-\cdots+{}^{3n}C_{3n}=(-1)^n$$ **3**

Questions continue over...

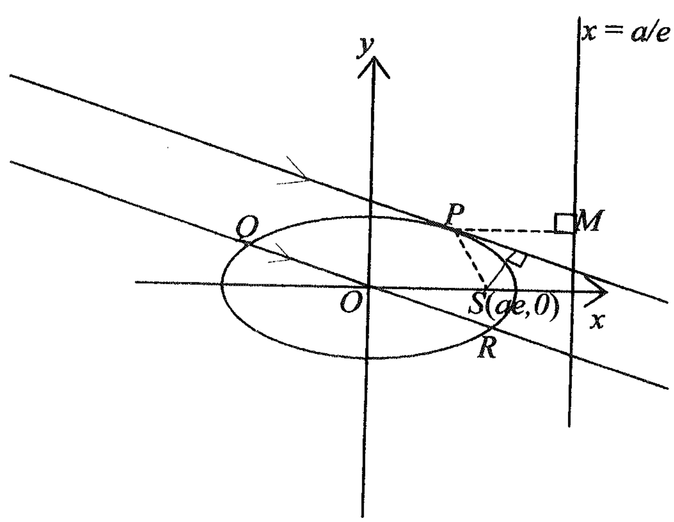
Question 6 : **Start a new booklet**

(a) P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) Show that the equation of the tangent at P is given by $bx\cos\theta + ay\sin\theta = ab$ **3**

(ii) Show that the equation of the diameter QR parallel to the tangent at P is given by $y=\frac{-bx\cos\theta}{a\sin\theta}$. **2**

(iii) Show that the coordinates of the ends of the diameter QR are $(a\sin\theta, -b\cos\theta)$ and $(-a\sin\theta, b\cos\theta)$. **2**



(You may use the diagram above in reference to your answers below.

(iv) If $PS = k$, show that $k = a(1 - e\cos\theta)$ **2**

(v) If q is the perpendicular distance from S to the tangent at P, show that
$$q = \frac{ab(1-e\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$ **2**

(vi) If $2h$ is the length of the diameter QR, show $h=\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}$ **2**

(vii) Hence prove that $hq = kb$ **2**

Questions continue over…

Question 7 : **Start a new booklet**

(a) Given a sequence such that $T_1 = 4$, $T_2 = 22$ and $T_{n+2} = T_{n+1} + 6 \times T_n$.
Prove by mathematical induction that $T_n = 2 \times 3^n + (-2)^n$, $n \geq 1$. **5**

(b) In the following question a, b, c, x, y and z are all positive. Prove that

(i) $a^2 + b^2 \geq 2ab$ **1**

(ii) $a^2 + b^2 + c^2 \geq ab + ac + bc$ **2**

(iii) Given that $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$,
prove that $a^3 + b^3 + c^3 \geq 3abc$ **2**

(iv) Hence show that $x + y + z \geq 3 \times \sqrt[3]{xyz}$ **1**

(c) (i) Show that $\tan^2 \theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$ **1**

(ii) Hence show that

$$\tan^2\left(\frac{\alpha + \beta}{2}\right) - \tan\alpha \tan\beta = \frac{\cos(\alpha + \beta)(1 - \cos(\alpha - \beta))}{\cos\alpha \cos\beta(1 + \cos(\alpha + \beta))}$$ **2**

(iii) Hence show that for $0 < \alpha < \dfrac{\pi}{4}$ and $0 < \beta < \dfrac{\pi}{4}$

$$\tan\left(\frac{\alpha + \beta}{2}\right) \geq \sqrt{\tan\alpha \tan\beta}$$ **1**

Questions continue over…

Question 8 : **Start a new booklet**

(a) An object of mass m kg is projected vertically upwards from ground level with an initial velocity U m/s. The air resistance is proportional to the square of the velocity, that is mkv^2. The only other force acting on the body is due to gravity.

(i) (α) Show that $\ddot{x} = -\left(g + kv^2\right)$. **1**

(β) Show that the time taken to reach the highest point of the flight is $T_u = \dfrac{1}{\sqrt{gk}} \tan^{-1}\left(U\sqrt{\dfrac{k}{g}}\right)$. **3**

(ii) Let T_d be the time taken to for the object to fall back to the ground, and for convenience let $w = U\sqrt{\dfrac{k}{g}}$. It can be shown that $\sqrt{gk}\,(T_d - T_u)$ simplifies to the function:

$$f(w) = \log_e\left(w + \sqrt{w^2+1}\right) - \tan^{-1} w$$

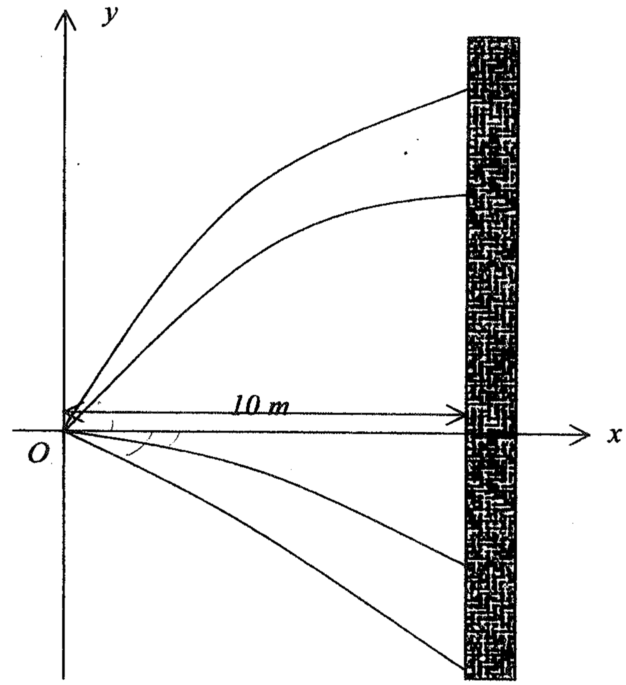
(a) Evaluate $f(0)$. **1**

(b) Determine $f'(w)$ and show that $f'(w) > 0$ for $w > 0$. **2**

(c) Hence show that it takes longer for the object to fall back to the ground than it does to reach the highest point. **1**

Question 8 continues over...

(b)



In the diagram above a bomb is detonated at O, firing simultaneously a large number of projectiles each with the same velocity V, but different angles of elevation. Several of these projectiles strike a wall, which is 10 metres away from O. You may consider that the projectiles are all fired in the same vertical plane that is perpendicular to the wall.

You may assume the equations of motion are given by:

$$x = Vt\cos\theta \text{ and } y = -\frac{1}{2}gt^2 + Vt\sin\theta.$$

(i) Use the identity $\cos^2\theta + \sin^2\theta = 1$ to eliminate θ from these two equations and hence prove that the relationship between height y and time t is:

$$4y^2 + 4gt^2y + k = 0, \text{ where } k = g^2t^4 + 4x^2 - 4V^2t^2.$$ **2**

(ii) Show that the first impact on the wall occurs at time $t = \dfrac{10}{V}$, and that this projectile was fired horizontally. **2**

(iii) Show that for $t > \dfrac{10}{V}$, there are two impacts at time t, and that the distance d between these impacts is given by:

$$d = 2\sqrt{V^2t^2 - 100}$$ **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$