2008

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

BLANK PAGE

Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) $\displaystyle\int \frac{2x}{\sqrt{1-x^4}}\,dx$ **2**

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ **3**

(c) $\displaystyle\int_1^{e^2} 3x^2 \ln x \;\; dx$ **3**

(d) $\displaystyle\int \frac{dx}{\sqrt{x^2-x\;\;+1}}$ **2**

(e) (i) Show that $\displaystyle\int_0^a f(x)\;dx = \int_0^a f(a-x)\;dx$ **2**

(ii) Use this property to show that $\displaystyle\int_0^1 x^3(1-x)^6\;dx \;=\; \frac{1}{840}$ **3**

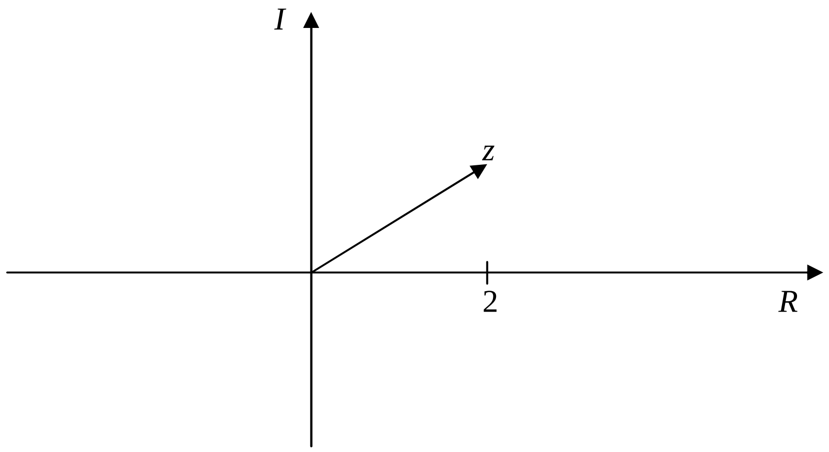
Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A complex number z is given by $z = \sqrt{3} + i$

(i) Evaluate $\bar{z}$. Verify that $z\bar{z}$ is real. **2**

(ii) Find $\frac{1}{z}$ in the form $a + ib$, where a and b are real. **1**

(b) A point z on the Argand Diagram is given below:



Copy this diagram into your examination booklet and use it to plot the following points. Clearly label each point.

(i) $\bar{z}$ **1**

(ii) $2iz$ **1**

(iii) $\frac{1}{z}$ **1**

(c) Express $i - 1$ in modulus argument form, and hence simplify $(i-1)^5$ **2**

Question 2 continues on page 4

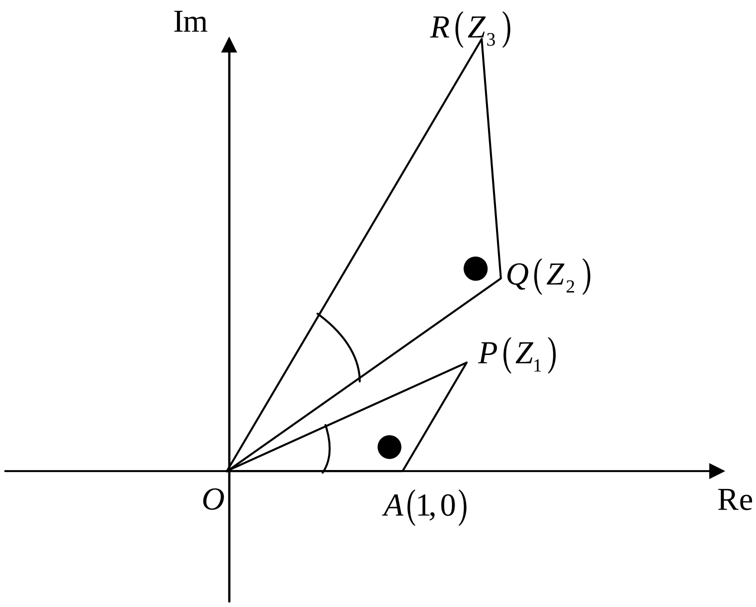
Question 2 (continued)

(d) Sketch the locus and state its equation:

(i) $|z-2| = |z-2i|$ **2**

(ii) $z\bar{z} - 3(z+\bar{z}) \le 0$ **2**

(e)



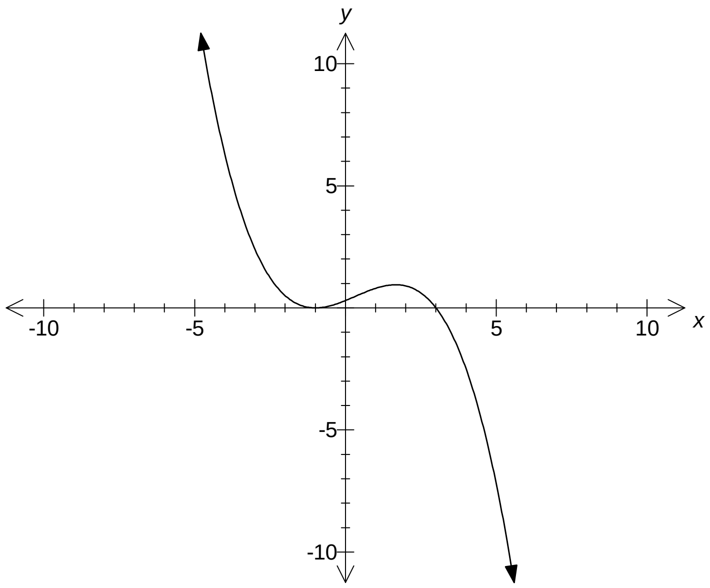
In the figure above, the points *P*, *Q* and *A* represent the complex numbers Z_1, Z_2 and $(1,0)$ respectively. Given $\angle OAP = \angle OQR$ and $\angle AOP = \angle QOR$.

Explain why $R(Z_3)$ represents the complex number $Z_1 Z_2$. **3**
You must support your answer with clear and complete mathematical reasons.

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph of $f(x) = \frac{1}{10}(x+1)^2(3-x)$ is drawn above.

On separate diagrams, draw a neat sketch showing the main features of each of the following:

(i) $y = f(x-1)$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = \{f(x)\}^2$ **2**

(iv) $y = xf(x)$ **2**

(v) $y^2 = f(x)$ **2**

(vi) $y = e^{f(x)}$ **2**

(b) Given that $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \; dx$ show that :

(i) $I_n = \frac{1}{n-1}\left((\sqrt{2})^{n-2} + (n-2)I_{n-2}\right)$ **4**

(ii) Hence or otherwise evaluate I_4 **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) If $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, show that $P(x) = 0$ has a multiple root, find this root and its multiplicity. **3**

(ii) Hence factorise $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ into its linear factors. **1**

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ. Find the monic equations with roots

(i) $\alpha^2, \beta^2, \gamma^2$. **2**

(ii) $\alpha\beta,\ \beta\gamma,\ \alpha\gamma$ **3**

(iii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

(c) A point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola is given by $x + t^2 y = 2ct$. **2**

(ii) Prove that the area bounded by the tangent and the asymptotes of the rectangular hyperbola is a constant. **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) *ABC* is an equilateral triangle, inscribed in a circle. *X* is a point on the minor arc *BC*.

(i) Prove that $\Delta BDX \parallel \Delta ACX$ **3**

(iii) Prove that $XB + XC = XA$ **3**

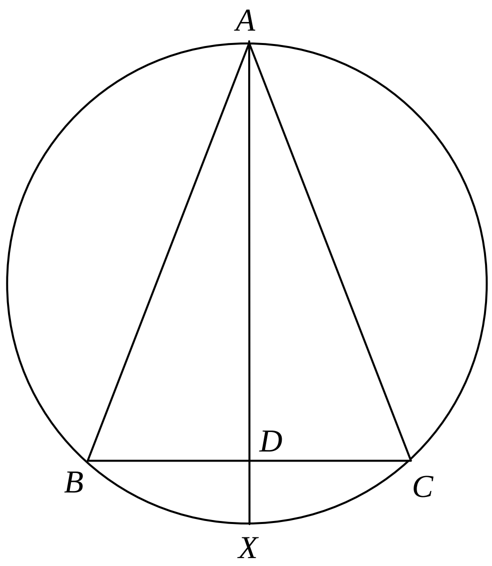


DIAGRAM NOT TO SCALE

(b) State whether each of the following are true or false giving brief reasons for your answers:

(i) $\int_0^{\pi} \sin 9x \ dx = 0$ **1**

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \ dx = 0$ **2**

(c) Find the equation of the tangent to the curve $\cos 2x + \sin y = 1$ at the point $x = \frac{\pi}{6}$. **3**

(d) Use the substitution $x = a \sin\theta$ to show that **3**

$$\int \sqrt{(a^2 - x^2)} \ dx = \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{(a^2 - x^2} + C$$

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Given $a\alpha^2 + b\alpha + c = 0$ where $a, b, c \in \square$ and $\alpha \in \square$, prove that $a\left(\overline{\alpha}\right)^2 + b\overline{\alpha} + c = 0$ **2**

(ii) A polynomial $P(x)$ with real coefficients , has two of its zeros $3i$ and $1 + 2i$. Find in expanded form, a possible polynomial $P(x)$. **3**

(b) Use De Moivres Theorem and binomial expansion to find an expression for $cos\, 4\theta$ in terms of $cos\,\theta$. **3**

(c) (i) Given $z = cos\,\theta + i\,sin\,\theta$, prove $z^n + \dfrac{1}{z^n} = 2\,cos\,n\theta$ **2**

(i) Hence by considering the expansion $\left(z + \dfrac{1}{z}\right)^4$ show that **3**

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(iii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} cos^4\theta\, d\theta$ **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of the polynomial $p(x)=x^3+ax^2+bx+c=0$ are three consecutive terms of an arithmetic series. Prove that the relationship between the coefficients is given by $2a^3-9ab+27c=0$ **4**
Hint: make an appropriate choice for the roots in arithmetic progression.

(b) A point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>0$ and $b>0$.
The equation of the normal at the point $P(a\cos\theta, b\sin\theta)$ is given by
$xa\sin\theta - yb\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

(i) Show that the ellipse intersects the rectangular hyperbola $xy=c^2$ in four points if $ab>2c^2$ **3**

(ii) Show that for $0<\theta<\frac{\pi}{2}$, the normal at *P* on the ellipse intersects the hyperbola in two distinct points, say *A* and *B*. **3**

(iii) If *M* is the mid-point of *AB*, show that the coordinates of *M* are given by **2**

$$\left(\frac{(a^2-b^2)\cos\theta}{2a}, -\frac{(a^2-b^2)\sin\theta}{2b}\right)$$

(iv) Hence find the locus of *M* as θ varies. **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the function $y = \cos^{-1}(e^x)$,

(i) Find the domain and the range. **2**

(ii) Draw a neat sketch the graph of $y = \cos^{-1}(e^x)$. **2**

(iii) Hence draw a neat sketch of the curve $y = \dfrac{1}{(\cos^{-1}(e^x))}$ **2**

(b) (i) Using induction, show that for each positive integer *n*, there are unique positive integers p_n and q_n such that: $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$ **4**

(ii) Show also that $p_n{}^2 - 2q_n{}^2 = (-1)^n$. **1**

(c) If $f(xy) = f(x) + f(y)$, for all $x, y \neq 0$, prove that

(i) $f(1) = f(-1) = 0$ **2**

(ii) $f(x)$ is an even function. **2**

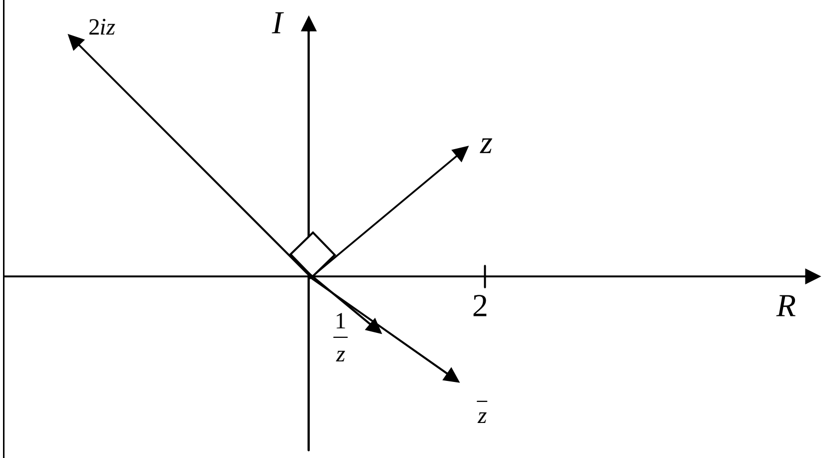
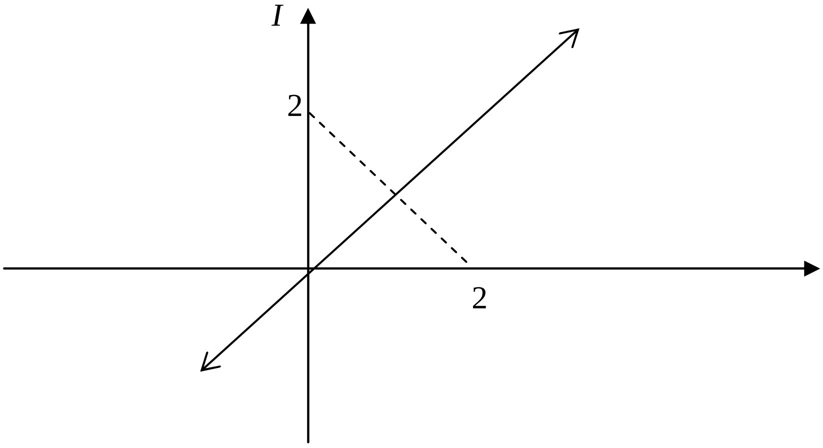
End of paper

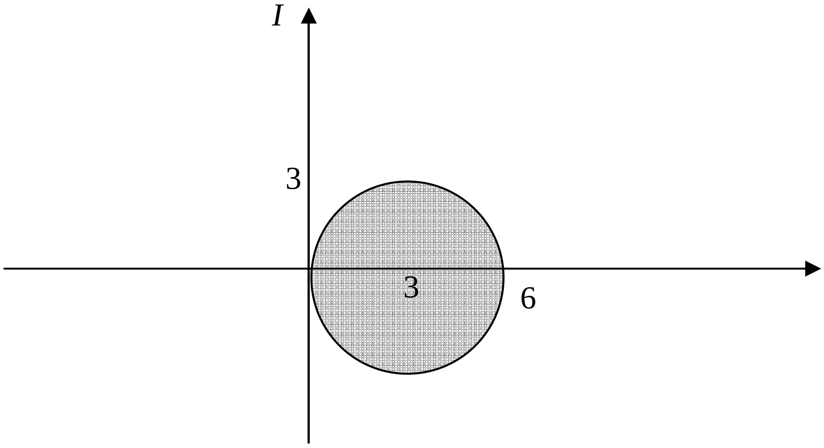
BLANK PAGE

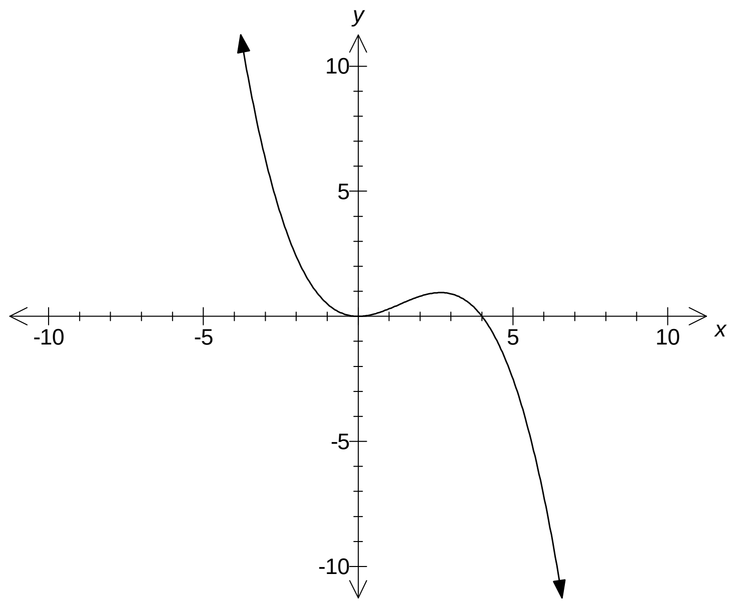
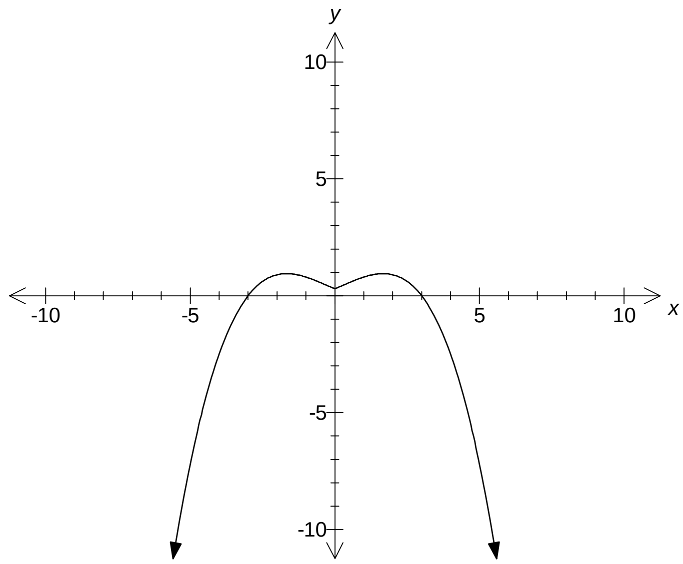
Mathematics Extension 2 HSC Trial Examination 2008 Solutions

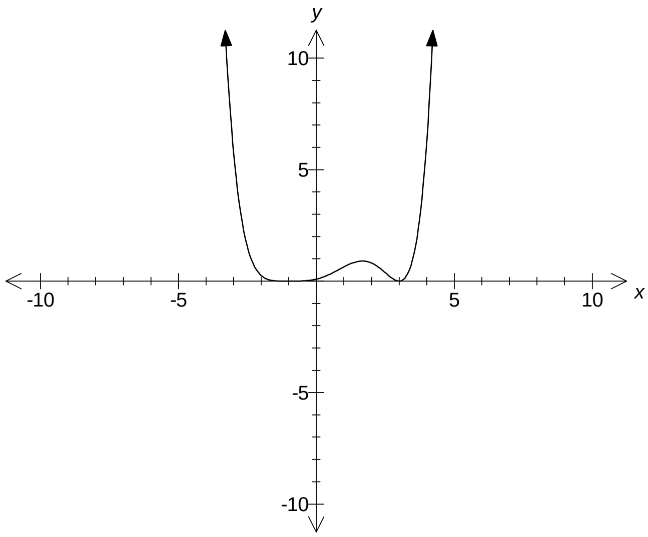
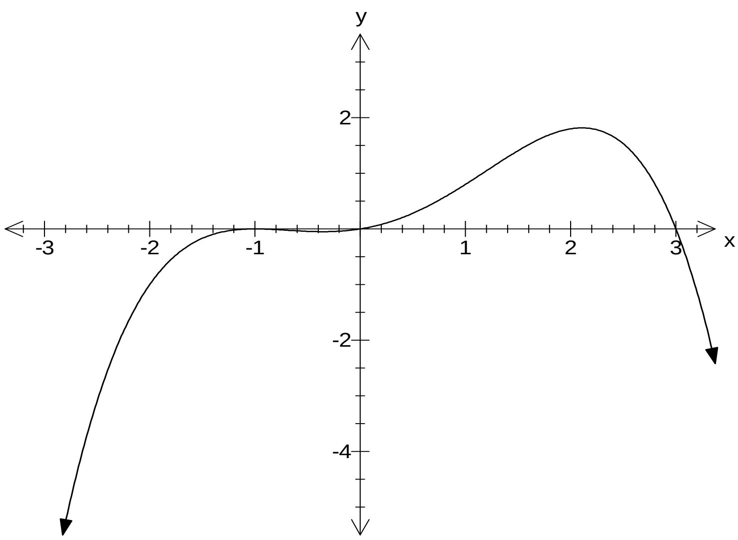
Question	Criteria	Marks	Bands
1(a)	$\int \frac{2x}{\sqrt{1-x^4}}\,dx$ *Let* $u=x^2$ $\therefore \frac{du}{dx}=2x$ *or* $dx=\frac{du}{2x}$ ☑ $\therefore \int \frac{2x}{\sqrt{1-x^4}}\,dx = \int \frac{2x}{\sqrt{1-u^2}}\cdot\frac{du}{2x}$ $= \int \frac{1}{\sqrt{1-u^2}}\,du$ $=\sin^{-1} u + C$ $=\sin^{-1} x^2 + C$ ☑	**2**	
1(b)	$\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ *Let* $t=\tan\frac{x}{2}$ $\therefore \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}\left[1+\tan^2\frac{x}{2}\right]=\frac{1}{2}\left[1+t^2\right]$ *or* $dx=\frac{2\,dt}{1+t^2}$ *and* $\cos\theta=\frac{1-t^2}{1+t^2}$ ☑ $\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$ since $t=\tan\frac{\frac{\pi}{2}}{2}=1$ *and* $t=\tan\frac{0}{2}=0$ $= \int_0^1 \frac{1}{\frac{2(1+t^2)}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$ $= \int_0^1 \frac{1+t^2}{3+t^2}\cdot\frac{2\,dt}{1+t^2}$ ☑ $= \int_0^1 \frac{2}{3+t^2}\,dt$ $= 2\int_0^1 \frac{1}{3+t^2}\,dt$ $=2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$ ☑ $=\frac{\pi}{3\sqrt{3}}$	**3**	

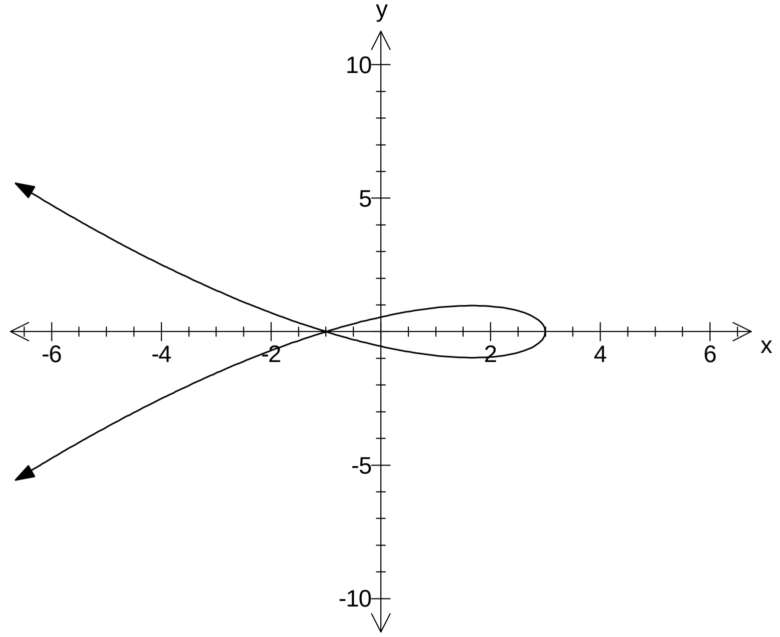
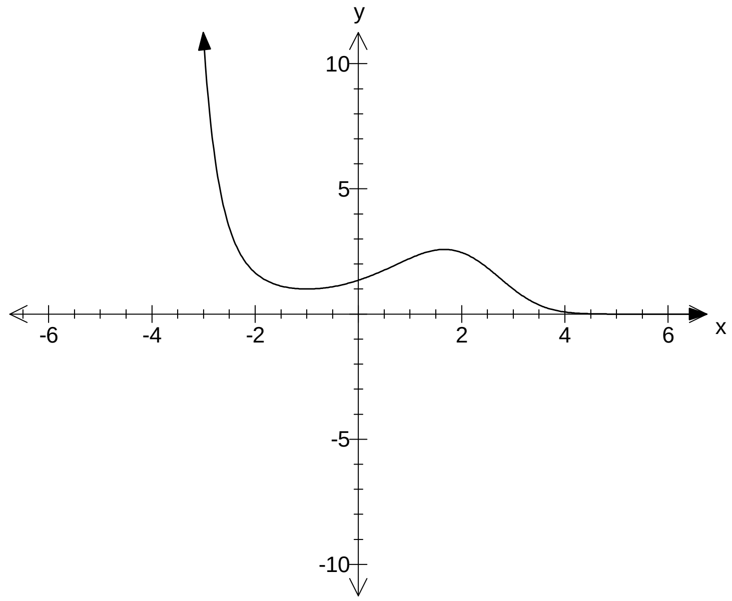
1(c)	$\int_1^{e^2} 3x^2 \ln x \; dx$ *Let* $u = \ln x$ $\frac{dv}{dx} = 3x^2$ $\frac{du}{dx} = \frac{1}{x}$ $v = x^3$ $\therefore \int_1^{e^2} 3x^2 \ln x \; dx = uv - \int v \; du$ ☑ $= \left[x^3 lnx\right]_1^{e^2} - \int x^3 \frac{dx}{x}$ $= \left[x^3 lnx\right]_1^{e^2} - \int x^2 \; dx$ $= \left[x^3 lnx\right]_1^{e^2} - \left[\frac{x^3}{3}\right]_1^{e^2}$ ☑ $= \left[e^6 \ln e^2 - 1^3 \ln 1\right] - \left[\frac{e^6}{3} - \frac{1}{3}\right]_1^{e^2}$ $= 2e^6 - \frac{e^6}{3} + \frac{1}{3}$ $= \frac{5e^6 + 1}{3}$ ☑	**3**	
1(d)	$\int \frac{dx}{\sqrt{x^2 - x + 1}} = \int \frac{1}{\sqrt{x^2 - x + (\frac{1}{2})^2 + 1 - (\frac{1}{2})^2}} \; dx$ ☑ $= \int \frac{1}{\sqrt{(x - \frac{1}{2})^2 + \frac{3}{4}}} \; dx$ $= \log \lvert (x - \frac{1}{2}) + \sqrt{(x - \frac{1}{2})^2 + \frac{3}{4}} \rvert + C$ ☑	**2**	
1(e)(i)	$\int_0^a f(a-x) \; dx$ *Let* $u = a - x$ $\therefore \frac{du}{dx} = -1$ *If* $x = a$ *then* $u = a - a =$ $x = 0$ *then* $u = a - 0 = a$ ☑ $\int_0^a f(u) \; .-du = -\int_a^0 f(u) \; du$ $= \int_0^a f(u) \; du$ ☑	**2**	
1(e)(ii)	$\int_0^1 x^3(1-x)^6 \; dx = \int_0^1 (1-x)^3(1-(1-x))^6 \; dx$ ☑ $= \int_0^1 (1 - 3x + 3x^2 - x^3)x^6 \; dx$ $= \int_0^1 x^6 - 3x^7 + 3x^8 - x^9 \; dx$ ☑ $= \left[\frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{3} - \frac{x^{10}}{10}\right]_0^1$ ☑ $= \frac{1}{840}$	**3**	

Question	Criteria	Marks	Bands
2(a)(i)	$z\bar{z} = \left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right) = 4$ $\therefore z\bar{z}$ is real.	2	
2(a)(ii)	$\frac{1}{z} = \frac{1}{\left(\sqrt{3}+i\right)} \cdot \frac{\left(\sqrt{3}-i\right)}{\left(\sqrt{3}-i\right)} = \frac{\left(\sqrt{3}-i\right)}{4}$	1	
2(b) (i)-(iii)		3	
2(c)	$i-1 = \sqrt{2}cis\frac{3\pi}{4}$ $(i-1)^5 = \left(\sqrt{2}\right)^5 cis\frac{15\pi}{4} = 4\sqrt{2}cis\frac{7\pi}{4}$ $= 4\sqrt{2}\left\{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right\}$ $= 4-4i$	1 1	
2(d)(i)	 $y = x$	1 1	

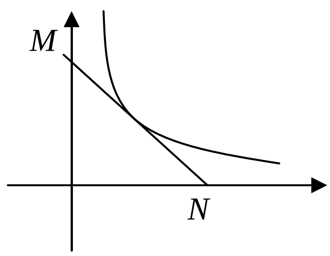
2(d)(ii)	 $(x-3)^2 + y^2 \le 9$	1 1	
2(e)	$\angle AOR = \angle AOQ + \angle QOR$ $= \angle AOQ + \angle AOP$ *i.e.* $\arg z_3 = \arg z_2 + \arg z_1$ the triangles ORQ and OPA are equiangular and hence similar $\therefore$ their sides are proportional $\frac{OR}{OP} = \frac{OQ}{OA} = \frac{\lvert z_3\rvert}{\lvert z_1\rvert} = \frac{\lvert z_2\rvert}{\lvert 1\rvert}$ $\lvert z_3\rvert = \frac{\lvert z_2\rvert\lvert z_1\rvert}{1}$ $= \lvert z_2\rvert\lvert z_1\rvert$ $\therefore z_3 = z_2 . z$ *i.e* R represents the complex number $z_2 z_1$	1 1 1	

Question	Criteria	Marks	Bands
3(a)(i)	 ☑ *shift graph* 1 *place to the right*	**1**	
3(a)(ii)	 ☑ *reflection in y axis*	**1**	

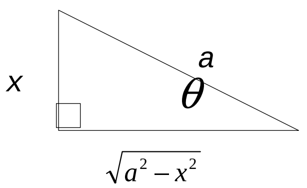
3(a)(iii)	 ☑ graph above x-axis ☑ graph $0 < y < 1$ (less steep gradient) and graph $y > 1$ steeper gradient	**2**	
3(a)(iv)	 ☑ 3 roots at $-1, 0$ and 3 ☑ 2 stationary points	**2**	

3(a)(v)	 ☑ sketch only +ve section of orginal graph and reflect in x - axis ☑ *for* $-1 < y < 1$ *the sketch is less steep and more steep after* $y < -1$ *and* $y > 1$	**2**	
3(a)(vi)	 ☑ *where cuts* $x-$*axis now becomes* $(-1,1)$ *and* $(3,1)$ ☑ anything that was negative now becomes an asymptote above the $x-axis$.	**2**	

3(b)(i)	$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \; dx$ $\int \sec^n x \; dx = \int \sec^{n-2} x.\ \sec^2 x \, dx$ ✓ $where \quad u = \sec^{n-2} x \qquad \frac{dv}{dx} = \sec^2 x$ $\frac{du}{dx} = (n-2)\sec^{n-3} x \sec x \tan x \qquad v = \tan x$ $\int \sec^{n-2} x.\ \sec^2 x \, dx = uv - \int v \; du$ $= \sec^{n-2} x \tan x - \int (n-2)\sec^{n-3} x \sec x \tan^2 x \; dx$ $= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \; dx$ $= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1) \; dx$ $= \sec^{n-2} x \tan x - (n-2)\int \sec^n x - \sec^{n-2} \; dx$ ✓ $\therefore \int \sec^n x \; dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x - (n-2)\int \sec^{n-2} dx$ $\int \sec^n x \; dx + (n-2)\int \sec^n x = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} dx$ $(n-1)\int \sec^n x \; dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} dx$ $\int \sec^n x \; dx = \frac{1}{n-1}\sec^{n-2} x \tan x - \frac{n-2}{n-1}\int \sec^{n-2} dx$ ✓ $I_n = \frac{1}{n-1}\left[\left[\sec^{n-2} x \tan x\right]_0^{\frac{\pi}{4}} - (n-2)I_{n-2}\right]$ $I_n = \frac{1}{n-1}\left[\left[\sec^{n-2}(\frac{\pi}{4})\tan(\frac{\pi}{4}) - \sec^{n-2}(0)\tan(0)\right] - (n-2)I_{n-2}\right]$ ✓ $I_n = \frac{1}{n-1}\left[\left(\sqrt{2}\right)^{n-2} - (n-2)I_{n-2}\right]$	**4**	
3(b)(ii)	$I_4 = \frac{1}{3}\left[\left(\sqrt{2}\right)^2 - 2I_2\right]$ $I_2 = \frac{1}{2}\left[\left(\sqrt{2}\right)^0 - 0I_0\right] = \frac{1}{2}$ $\therefore \; I_4 = \frac{1}{3}\left[\left(\sqrt{2}\right)^2 - 2\left(\frac{1}{2}\right)\right]$ $= \frac{2}{3} - \frac{1}{3}$ $= \frac{1}{3}$ ✓	**1**	

Question	Criteria	Marks	Bands
4(a)(i)	$P'(x)=4x^3+3x^2-6x-5$ $P''(x)=12x^2+6x-6$ $12x^2+6x-6=0 \Rightarrow x=-1,-\frac{1}{2}$ $P'(-1)=0$ $\therefore x=-1$ is a root of multiplicity 3	1 1 1	
4(a)(ii)	$x^4+x^3-3x^2-5x-2=(x+1)^3(x-2)$	1	
4(b)(i)	$\left(\sqrt{x}\right)^3+2\sqrt{x}-1=0$ $x\sqrt{x}+2\sqrt{x}=1$ $\left(\sqrt{x}(x+2)\right)^2=1$ $x\left(x^2+4x+4\right)=1$ $x^3+4x^2+4x-1=0$	1 1	
4(b)(ii)	$\alpha\beta,\alpha\gamma,\beta\gamma=\frac{\alpha\beta\gamma}{\gamma},\frac{\alpha\beta\gamma}{\beta},\frac{\alpha\beta\gamma}{\alpha}=\frac{1}{\gamma},\frac{1}{\beta},\frac{1}{\alpha}$ $\left(\frac{1}{x}\right)^3+2\left(\frac{1}{x}\right)-1=0$ $\frac{1}{x^3}+\frac{2}{x}-1=0$ $x^3-2x^2-1=0$	1 1 1	
4(b)(iii)	$\alpha^3+2\alpha-1=0$ $\beta^3+2\beta-1=0$ $\gamma^3+2\gamma-1=0$ $\left(\alpha^3+\beta^3+\gamma^3\right)+2(\alpha+\beta+\gamma)-3=0$ $\therefore \alpha^3+\beta^3+\gamma^3=3-2(0)$ $=3$	1 1	
4(c)(i)	$y'=-\frac{c}{x^2}$ at $\left(ct,\frac{c}{t}\right)$ $y'=-\frac{1}{t^2}$ $y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$ $yt^2-ct=-x+ct$ $yt^2+x=2ct$	1 1	
4(c)(ii)	$M: x=0 \Rightarrow y=\frac{2c}{t}$ $N: y=0 \Rightarrow x=2ct$ area = $\frac{1}{2}(2ct)\left(\frac{2c}{t}\right)$ $=2c^2$ which is a constant  	1 1	

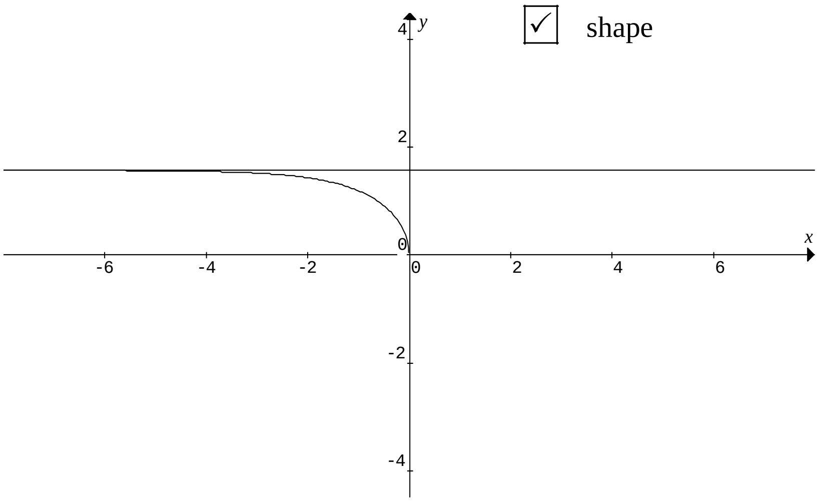
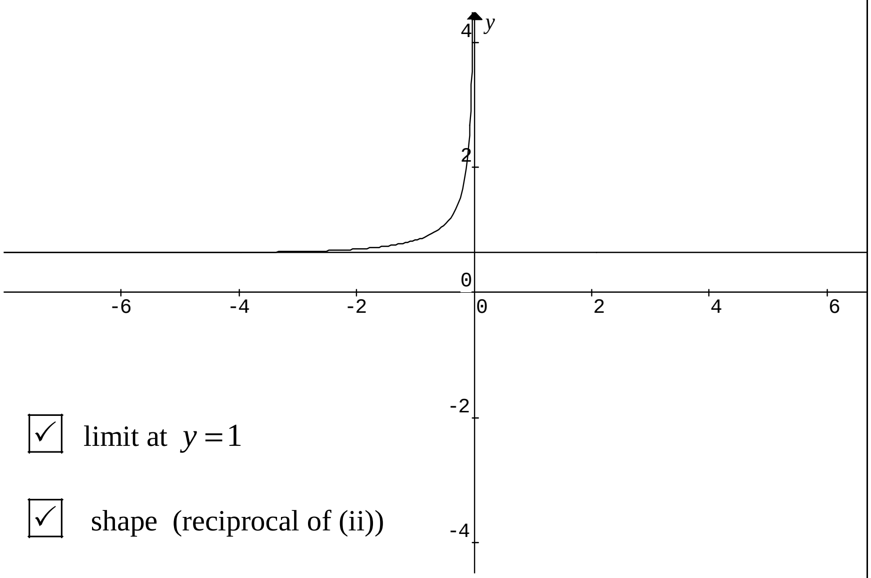
Question	Criteria	Marks	Bands
5(a)(i)	$\angle BAC = 60$ (ΔABC *is equliateral* Δ) $\angle BXC = 120$ (opposite angles in cyclic quad ABXC are supplementary) $\angle AXC = \angle ABC = 60$ ($\angle$'s *on smae arc at circumference are equal*) $\therefore \angle AXC = \angle AXB = 60$ *in* ΔBDX *and* ΔACX $\angle DXC = \angle AXB = 60$ (*proved above*) ☑ $\angle DXB = \angle CAX$ ($\angle$'s *at circumference on same arc*) ☑ $\therefore \Delta BDX \parallel\vert \Delta ACX$ (*eqiangular*) ☑	**3**	
5(a)(ii)	$\Delta CDX \parallel\vert \Delta ABX$ (*as proved in* (*i*) *above*) $\sin ce\, \Delta BDX \parallel\vert \Delta ACX$ $\therefore \frac{BD}{AC} = \frac{BX}{AX} = \frac{DX}{CX}$ *and* $\frac{CD}{AB} = \frac{CX}{AX} = \frac{DX}{BX}$ $\therefore BD = \frac{BX.AC}{AX}$ *and* $CD = \frac{CX.AB}{AX}$ ☑ $\sin ce \quad BC = BD + DC$ *hence* $BC = \frac{BX.AC}{AX} + \frac{CX.AB}{AX}$ ☑ $BC.AX = (BX.AC) + (CX.AB)$ *and as* $\boxed{BC = AC = AB \text{ (equaliteral } \Delta)}$ $\therefore \div$ *LHS and RHS by BC* ☑ $\therefore \quad AX = BX + CX$	**3**	
5(b)(i)	$y = \sin 9x$ *when* $0 < x < \pi$ *it has* $4\frac{1}{2}$ *cycles,* *more area above* $x-$*axis then below* $\therefore \int_0^{\pi} \sin 9x\, dx \neq 0$ (*false*) ☑	**1**	
5(b)(ii)	$x \sin x$ *is an even function* ☑ $\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x\, dx = 2\int_0^{\frac{\pi}{2}} x \sin x\, dx \neq 0$ ☑ (*false*)	**2**	

5(c)	slope tangent $=\frac{dy}{dx}$: (differentiating implicitly) $-2\sin 2x+\cos y\frac{dy}{dx}=0$ $\cos y\frac{dy}{dx}=2\sin 2x$ $\frac{dy}{dx}=\frac{2\sin 2x}{\cos y}$ ✓ At $(\frac{\pi}{6},\frac{\pi}{6})$, slope of tangent = 2 ✓ Equation of tangent is $y-\frac{\pi}{6}=2(x-\frac{\pi}{6})$ $y=2x-\frac{\pi}{6}$ ✓	**3**	
5(d)	Let $x=a\sin\theta \quad dx=a\sin\theta\, d\theta$ $\int\sqrt{a^2-x^2}\,dx=\int\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta d\theta=a^2\int\cos^2\theta\, d\theta$ ✓ $=\frac{a^2}{2}\int\cos 2\theta+1\, d\theta=$ $\frac{a^2}{2}\left[\frac{1}{2}\sin 2\theta+\theta\right]+C=\frac{a^2}{2}[\sin\theta\cos\theta+\theta]+C$ ✓ From this triangle :  $\frac{x}{a}=\sin\theta$ $\cos\theta=\frac{\sqrt{a^2-x^2}}{a}$ $\theta=\sin^{-1}a$ $\frac{a^2}{2}[\sin\theta\cos\theta+\theta]+C=$ ✓ $\frac{a^2}{2}\left[\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}+\sin^{-1}\frac{x}{a}\right]+C=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$	**3**	

Question	Criteria	Marks	Bands
6(a)(i)	$\overline{a\alpha^2+b\alpha+c}=\overline{0}$ $\overline{a\alpha^2}+\overline{b\alpha}+\overline{c}=0$ $a\overline{\alpha^2}+b\overline{\alpha}+c=0$ $a\overline{\alpha}^2+b\overline{\alpha}+c=0$ $\therefore \overline{\alpha}$ is a solution	1 1	
6(a)(ii)	$(x-3i)(x+3i)(x-(1+2i))(x-(1-2i))$ $(x^2+9)(x^2-2x+5)$ $x^4-2x^3+14x^2-18x+45$	1 1 1	
6(b)	$\cos 4\theta+i\sin 4\theta=(\cos\theta+i\sin\theta)^4$ $=\cos^4\theta+4i\cos^3\theta\sin\theta-6\cos^2\theta\sin^2\theta-4i\cos\theta\sin^3\theta+\sin^4\theta$ equate real part: $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$ $=\cos^4\theta-6\cos^2\theta(1-\cos^2\theta)+(1-\cos^2\theta)^2$ $=8\cos^4\theta-8\cos^2\theta+1$	1 1 1	
6(c)(i)	$z^n=\cos n\theta+i\sin n\theta$ $z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$ $=\cos n\theta-i\sin n\theta$ $z^n+z^{-n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$ $=2\cos n\theta$	1 1	
6(c)(ii)	$\left(z+\frac{1}{z}\right)^4=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$ $=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$ $=2\cos 4\theta+4(2\cos 2\theta)+6$ $\left(z+\frac{1}{z}\right)^4=(2\cos\theta)^4=16\cos^4\theta$ $\therefore 16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$ $\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$	1 1 1	
6(c)(iii)	$\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}d\theta$ $=\left[\frac{1}{8}\frac{\sin 4\theta}{4}+\frac{1}{2}\frac{\sin 2\theta}{2}+\frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$ $=\frac{1}{32}\sin 2\pi+\frac{1}{4}\sin\pi+\frac{3\pi}{16}-0$ $=\frac{3\pi}{16}$	1 1	

Question	Criteria	Marks	Bands
7(a)	Let roots be: $\alpha - d, \alpha, \alpha + d$ sum of roots: $3\alpha = -a \quad \Rightarrow \alpha = -\frac{a}{3}$ $\alpha = -\frac{a}{3}$ is a root to: $x^3 + ax^2 + bx + c = 0$ $\left(-\frac{a}{3}\right)^3 + a\left(-\frac{a}{3}\right)^2 + b\left(-\frac{a}{3}\right) + c = 0$ $-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c = 0$ $-a^3 + 3a^3 - 9ab + 27c = 0$ $2a^3 - 9ab + 27c = 0$	**1** **1** **1** **1**	
7(b)(i)	Consider the intersection of the two curves: $y = \frac{c}{x^2}$ $\frac{x^2}{a^2} + \frac{c^4}{x^2b^2} = 1 \quad x^4b^2 - x^2a^2b^2 + a^2c^4 = 0$ Solving for x^2 : $\Delta = a^4b^4 - 4a^2b^2c^4$ for the roots to be real and distinct: $\Delta > 0$ $a^4b^4 - 4a^2b^2c^4 > 0$ $a^2b^2 > 4c^2 \quad$ *or* $\quad ab > 2c^2$ If $ab > 2c^2$, x^2 has two distinct values and hence x has 4 values corresponding to 4 points of intersection.	**1** **1** **1**	
7(b)(ii)	$y = \frac{c^2}{x} \quad xa\sin\theta - \frac{c^2}{x}b\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$ $x^2a\sin\theta - c^2b\cos\theta = \left(a^2 - b^2\right)x\sin\theta\cos\theta$ $x^2a\sin\theta - \left(a^2 - b^2\right)x\sin\theta\cos\theta - c^2b\cos\theta = 0$ $\Delta = \left[\left(a^2 - b^2\right)^2 \sin\theta cos\theta\right]^2 + 4ac^2b\cos\theta\sin\theta$ If $0 < \theta < \frac{\pi}{2}$, $0 < \sin\theta < 1, 0 < \cos\theta < 1, \quad a, b > 0$ $\therefore \Delta > 0$ and this gives two values for x.	**1** **1** **1**	

7(b)(iii)	$x_1 + x_2 = \frac{2\left(a^2 - b^2\right)\sin\theta\cos\theta}{2a\sin\theta} = \frac{\left(a^2 - b^2\right)\sin\theta\cos\theta}{a\sin\theta}$ $\frac{x_1 + x_2}{2} = \frac{\left(a^2 - b^2\right)\sin\theta\cos\theta}{2a\sin\theta}$ $x = \frac{\left(a^2 - b^2\right)\cos\theta}{2a} \qquad (1)$ sub into normal to find y: $a\frac{\left(a^2 - b^2\right)\cos\theta}{2a}\sin\theta - yb\cos\theta = \left(a^2 - b^2\right)\sin\theta\cos\theta$ $\frac{\left(a^2 - b^2\right)}{2}\sin\theta - yb = \left(a^2 - b^2\right)\sin\theta \qquad \cos\theta \neq 0$ $y = \frac{-\left(a^2 - b^2\right)}{2b}\sin\theta \qquad (2)$	**1** **1**	
7(b)(iv)	Eliminate θ: From (1) $\qquad \cos\theta = \frac{2ax}{a^2 - b^2}$ From (2) $\qquad \sin\theta = -\frac{2by}{a^2 - b^2}$ $\sin^2\theta + \cos^2\theta = 1$ $\frac{4a^2x^2}{\left(a^2 - b^2\right)^2} + \frac{4b^2y^2}{\left(a^2 - b^2\right)^2} = 1$ $\frac{x^2}{\left(\frac{a^2 - b^2}{2a}\right)^2} + \frac{y^2}{\left(\frac{a^2 - b^2}{2b}\right)^2} = 1$	**1** **1** **1**	

Question	Criteria	Marks	Bands
8(a)(i)	Domain: $e^x > 0 \quad for\ all\ x$ for $\cos^{-1}(e^x): -1 \le e^x \le 1$ only if $0 \le e^x \le 1$, i.e. if $\boldsymbol{x \le 0}$. ☑ Range: For this domain range will be : $0 \le y \le \frac{\pi}{2}$ ☑	**2**	
8(a)(ii)	☑ limit at $y = 1$ ☑ shape  	**2**	
8(a)(iii)	 ☑ limit at $y = 1$ ☑ shape (reciprocal of (ii))	**2**	

8(b)(i)	*prove true for* $n=1$ $\therefore (1+\sqrt{2})^1 = 1+\sqrt{2}$ *true where* $p_n = 1$ *and* $q_n = 1$ ☑ *assume true for* $n=k$ $\therefore (1+\sqrt{2})^k = p_k + q_k\sqrt{2}$ *prove true for* $n=k+1$ $\therefore (1+\sqrt{2})^{k+1} = (1+\sqrt{2})^k(1+\sqrt{2})^1$ $=(p_k+q_k\sqrt{2})(1+\sqrt{2})$ (*by assumption above*) $= p_k + p_k\sqrt{2} + q_k\sqrt{2} + 2q_k$ $=(p_k+2q_k)+(p_k+q_k)\sqrt{2}$ ☑ since p_k and q_k are integers $\therefore p_k + 2q_k$ is an integer $= p_{k+1}$ $\therefore p_k + q_k$ is an integer $= q_{k+1}$ *hence* $(1+\sqrt{2})^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$ ☑ If true for $n=k$ and $n=k+1$ and since true for $n=1,2,3...$ $\therefore$ *true for* $\forall\, n$ positive integers	**3**	
8(b)(ii)	$p_1^2 - 2q_1^2 = 1 - 2\times 1^2 = -1 = (-1)^1$ *If* $p_k^2 - 2q_k^2 = (-1)^k$ *then when* $n=k+1$ $(p_{k+1})^2 - (q_{k+1})^2 = (p_k+2q_k)^2 - 2(p_k+q_k)^2$ (*from above*) ☑ $= p_k^2 + 4p_kq_k + 4q_k^2 - 2p_k^2 - 4p_kq_k - 2q_k^2$ $= 2q_k^2 - p_k^2$ $= -1(p_k^2 - 2q_k^2)$ $= -1\times(-1)^k$ $= (-1)^{k+1}$ ☑ if true for $n=k$ and $n=k+1$ and since true for $n=1,2,3...$ $\therefore$ *true for* $\forall\, n$ positive integers	**2**	
8(c)(i)	$f(xa) = f(x) + f(a)$ *if* $x=1$ *then* $f(a) = f(1) + f(a) \Rightarrow f(1) = 0$ *if* $x=a$ *then* $f(a^2) = f(a) + f(a)$ $f(a^2) = 2f(a)$ ☑ $\therefore 2f(-1) = f(-1^2) = f(1) = 0$ ☑	**2**	
8(c)(ii)	since $2f(a) = f(a^2)$ $\therefore 2f(-a)) = f((-a))^2$ $= f(a^2)$ $= 2f(a)$ ☑ sin*ce* $2f(a) = 2f(-a))$ $\therefore f(a) = f(-a)$ *even function* ☑	**2**	