KINCOPPAL–ROSE BAY
SCHOOL OF THE SACRED HEART

2009
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Start a new booklet for each question

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

Total Marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1. (15 marks) Start a new page **Marks**

(a) (i) Show that $y = x\sqrt{4-x^2}$ is an odd function. **1**

(ii) Hence without finding the integral evaluate **2**

$\int_{-2}^{2} \left(x\sqrt{4-x^2} - \sqrt{4-x^2}\right)$, giving reasons.

(b) By using the table of standard integrals, find $\int \frac{dx}{\sqrt{4x^2+36}}$ **2**

(c) Use partial fractions to evaluate $\int_0^1 \frac{5\,dt}{(2t+1)(2-t)}$ **3**

(d) Find $\int \operatorname{cosec} x \; dx$ by using the substitution $t = \tan\frac{x}{2}$ **3**

(e) Find $\int \frac{\sqrt{x^2-16}}{x}\,dx$ using the substitution $x = 4\sec\theta$. **4**

End of Question 1

Question 2. (15 marks) Start a new page **Marks**

(a) Express $\dfrac{2-5i}{4-3i}$ in the form $x + iy$ where x and y are real. **2**

(b) Find all pairs of integers for a and b such that $(a-ib)^2 = -21-20i$ **3**

(c) Find the modulus and argument of $(\sin\theta + i\cos\theta)(\cos\theta - i\sin\theta)$ **3**

(d) (i) If $\left|\dfrac{z-1}{z+1}\right| = 2$, where $z = x + iy$, show that the locus of z is **2**

$$\left(x+\frac{5}{3}\right)^2 + y^2 = \frac{16}{9}$$

(ii) Represent this locus on an Argand Diagram and shade the region for which the inequalities $\left|\dfrac{z-1}{z+1}\right| \le 2$ and $0 \le \arg z \le \dfrac{3\pi}{4}$ are both satisfied. **3**

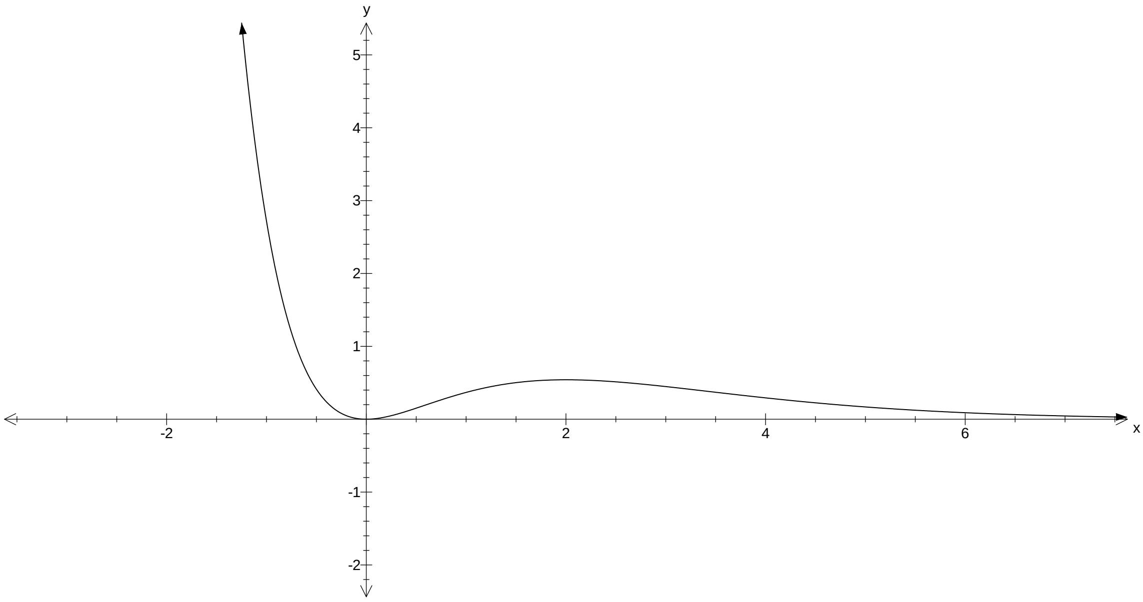
(e) z_1 and z_2 are two complex numbers such that $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$ **2**

On an Argand diagram show vectors representing $z_1, z_2, z_1 + z_2$ *and* $z_1 - z_2$

End of Question 2

Question 3. (15 marks) Start a new page **Marks**

(a)



The graph of $y = x^2 e^{-x}$ is sketched above. There is a stationary point at (0,0) and $\left(2, \frac{4}{e^2}\right)$

On separate diagrams, draw a neat sketch showing the main features of each of the following

(i) $y = f(x) + 1$ **1**

(ii) $y = f(|x|)$ **1**

(iii) $y = \{f(x)\}^2$ **2**

(iv) $y = \frac{1}{f(x)}$ **2**

(v) $y^2 = f(x)$ **2**

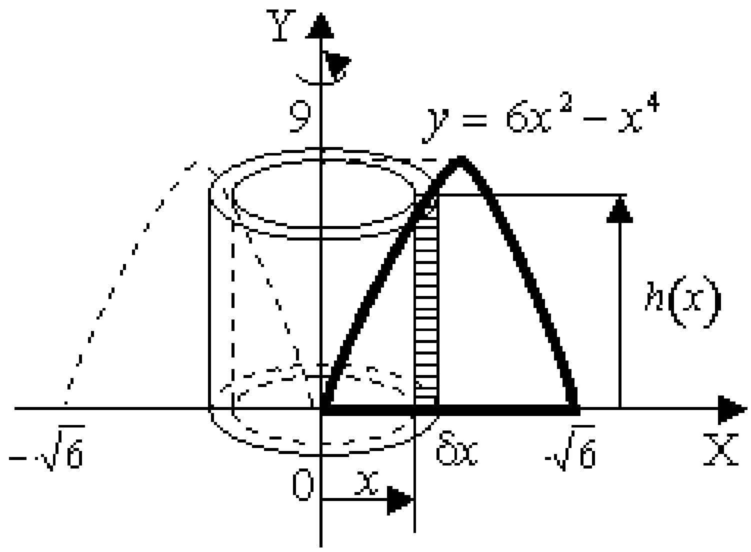
(vi) $y = \cos^{-1}(f(x))$ **2**

(b) If $x^m y^n = k$, where k is a constant, show that $\frac{dy}{dx} = -\frac{my}{nx}$ **2**

Question 3 continues on page 5

Question 3 continued **Marks**

(c) Using the method of cylindrical shells find the volume of the solid of revolution generated when the area enclosed by the curve $y = 6x^2 - x^4$ the x-axis and $0 \le x \le \sqrt{6}$ is rotated about the y- axis. **3**



End of Question 3

Question 4. (15 marks) Start a new page **Marks**

(a) If α, β, γ are the roots of the equation $x^3 - 4x^2 + 2x + 5 = 0$. Evaluate:

(i) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) $P(x)$ is a monic polynomial of degree 4 with integer coefficients and constant term 4. **3**
One zero is $\sqrt{2}$, another zero is rational and the sum of the zeros is positive.
Factorise $P(x)$ fully over **R**.

(c) (i) Use De Moivre's theorem to show $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ **3**

(ii) Hence solve $8x^3 - 6x - 1 = 0$ leaving answer in terms of $\cos\theta$ **3**

(d) For a real number *r*, the polynomial $8x^3 - 4x^2 - 42x + 45$ is divisible by $(x - r)^2$. **3**
Find the value of *r*.

End of Question 4

Question 5. (15 marks) Start a new page **Marks**

(a) Evaluate $\displaystyle\int_1^{\infty} \frac{1}{x+1} - \frac{1}{x+3}\, dx$ **2**

(b) (i) Use integration by parts to show that a reduction (recurrence) formula **3**

for $\mathrm{I}_n = \displaystyle\int \sin^n x\, dx$ is $\mathrm{I}_n = -\dfrac{1}{n}\cos x \sin^{n-1} x + \dfrac{n-1}{n}\mathrm{I}_{n-2}$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 x\, dx$ **2**

(c) The hyperbola H has equation $xy = 4$.

(i) Sketch the hyperbola and indicate on your diagram the position and coordinates of all points at which H intersects the axes of symmetry. **1**

(ii) Show that the equation of the tangent at $P\left(2t, \dfrac{2}{t}\right)$ where $t \neq 0$, is $x + t^2 y = 4t$ **2**

(iii) If $s \neq 0$ and $s^2 \neq t^2$, show that the tangents to H at P and $Q\left(2s, \dfrac{2}{s}\right)$ intersect at **2**

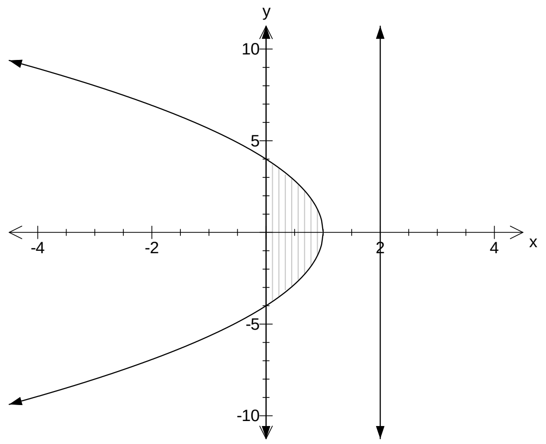
$M\left(\dfrac{4st}{s+t}, \dfrac{4}{s+t}\right)$

(iv) Suppose that in (iii) the parameter $s = -\dfrac{1}{t}$. Show that the locus of M is a straight line through, but excluding the origin. **3**

End of Question 5

Question 6. (15 marks) Start a new page **Marks**

(a)



A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y – axis around the line $x = 2$. **4**
By using the method of slices find the exact volume of S.

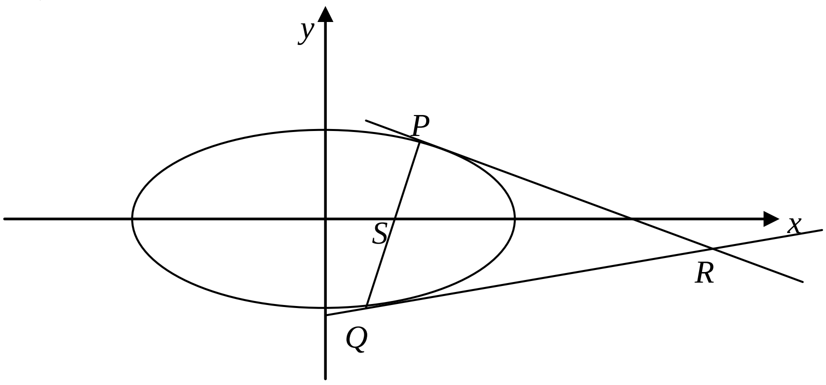
(b) A hyperbola has foci $(\pm 10, 0)$ and asymptotes $y = \pm \dfrac{4x}{3}$.

(i) Find the eccentricity. **1**

(ii) State the equation of the hyperbola. **1**

(iii) Sketch the hyperbola indicating important features such as vertices, foci, directrices and asymptotes **2**

(c) Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be points on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the extremities of a focal chord PQ. The tangents drawn from the extremities intersect at a point R.



(i) Show that the tangent at P is given by $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$. **2**

Question 6 continues on page 9

Question 6 continued **Marks**

(ii) Use simultaneous equations to show that the x coordinate of the point R is given by $x = \dfrac{a(\sin\phi - \sin\theta)}{\cos\theta\sin\phi - \sin\theta\cos\phi}$ **2**

(iii) Use the fact that the gradient of PS = gradient of SQ to show that $\dfrac{\sin\phi - \sin\theta}{\cos\theta\sin\phi - \sin\theta\cos\phi} = \dfrac{1}{e}$ **2**

(iv) Hence or otherwise show that R lies on the directrix of the ellipse. **1**

End of Question 6

Question 7. (15 marks) Start a new page **Marks**

(a) Let α, β, γ be the roots of the equation $x^3 + qx + r = 0$. **2**
Write down the cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$.

(b) Let ω be a non-real root of $z^7 - 1 = 0$.

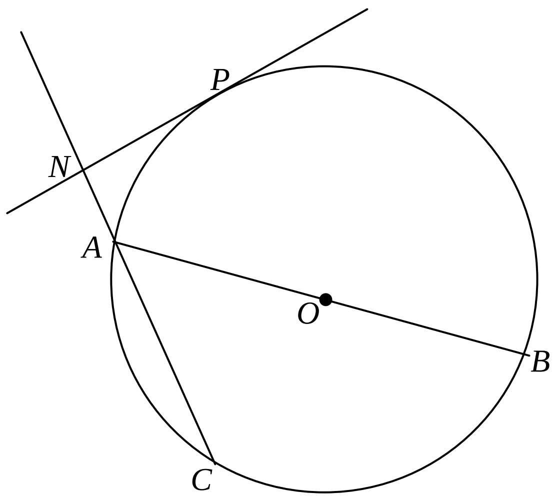
(i) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$. **1**

(ii) Show that $(1+\omega)(1+\omega^2)(1+\omega^4) = 1$. **1**

(iii) Simplify $(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3)$ **2**

(iv) Sketch on the Argand diagram all seven roots of $z^7 - 1 = 0$ **1**

(c) In a circle centre *O*, a diameter *AB* and a chord *AC* are drawn.
P is the point on the circumference on the side of *AB* opposite to *C*, such that the tangent at *P* is perpendicular to *CA* produced.
The tangent at *P* and the line *CA* produced intersect at the point *N*.



Copy this diagram into your examination booklet.

Prove that:

(i) $PC = PB$ **3**

(ii) $\angle APC + 2\angle ACP = 90^\circ$ **3**

(iii) $\angle PAB = \angle NPC$ **2**

End of Question 7

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Question 8. (15 marks) Start a new page **Marks**

(a) (i) Sketch $y = \sec x$ in the domain $-2\pi \le x \le 2\pi$ **1**

(ii) Using a suitable domain sketch $y = \sec^{-1} x$. **2**

(b) For all integers $n \ge 1$, let

$$t_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n-1} + \frac{1}{2n}$$

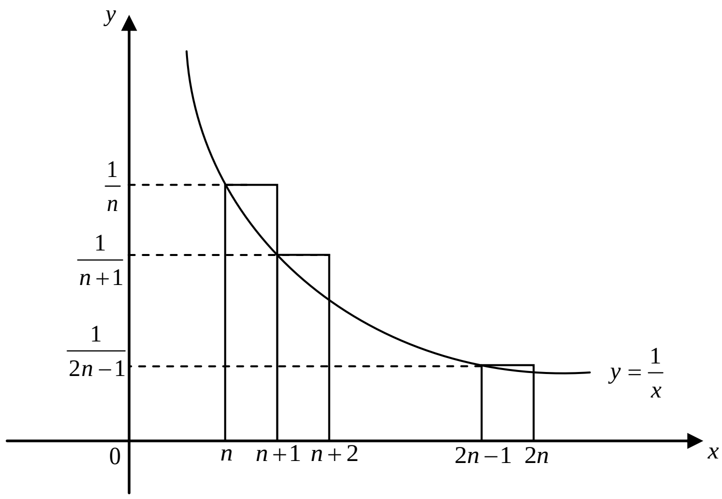
That is: $t_1 = \frac{1}{2}$

$t_2 = \frac{1}{3} + \frac{1}{4}$

$t_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

......

(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n-1}$ **2**



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \le x \le 2n$.

(ii) By using the diagram and the area of upper rectangles, show that $t_n + \frac{1}{2n} > \ln 2$ **3**

[Note that it can similarly be shown that $t_n < \ln 2$]

Questions 8 continued on page 13

Question 8 continued **Marks**

For all integers $n \geq 1$ let

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2n-1} - \frac{1}{2n}$$

That is:

$$s_1 = 1 - \frac{1}{2}$$
$$s_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$
$$s_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$$
.....

(iii) Prove by mathematical induction that $s_n = t_n$ **4**

(iv) Hence find, to three decimal places, the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{9999} - \frac{1}{10\,000}$ **3**

End of Test

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ n \neq -1; \ x \neq 0, \ if \ n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \ a \neq 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln(x + \sqrt{x^2 - a^2}), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \ x > 0$

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Extension 2 Mathematics (Solutions)

General Instructions

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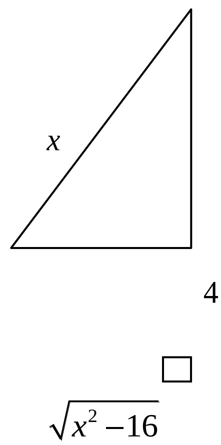
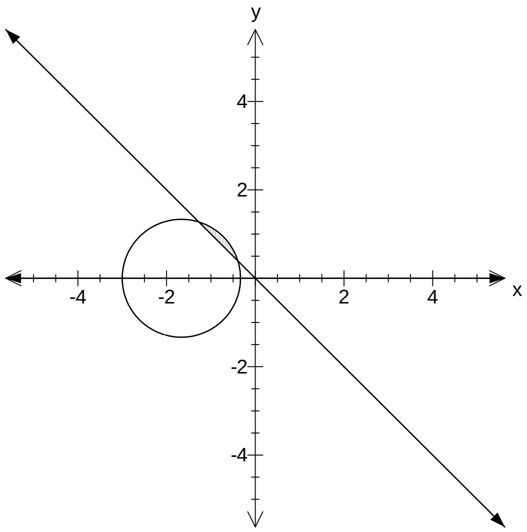
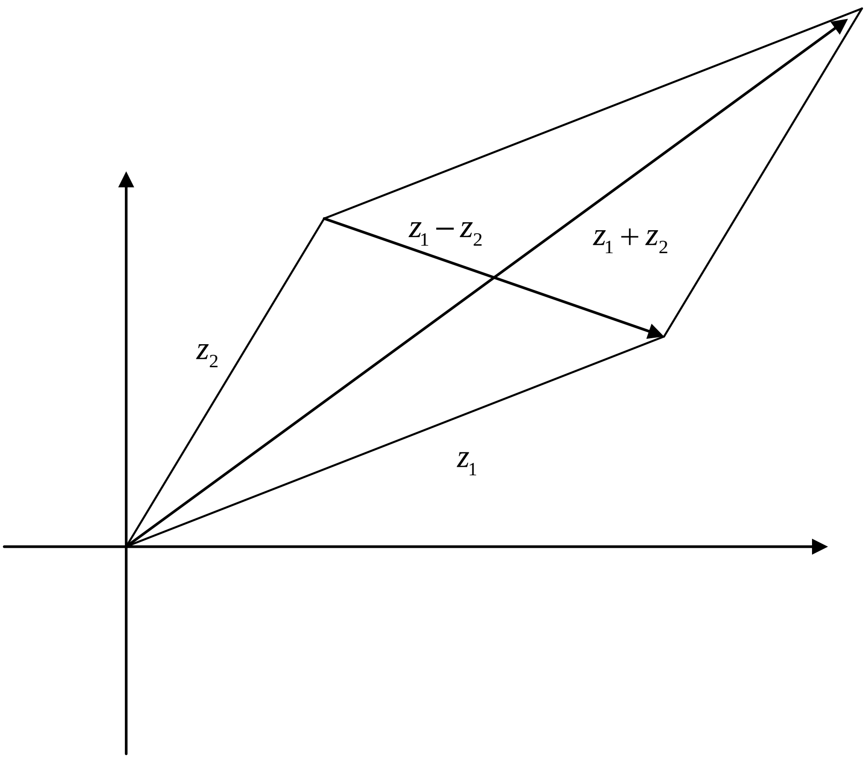
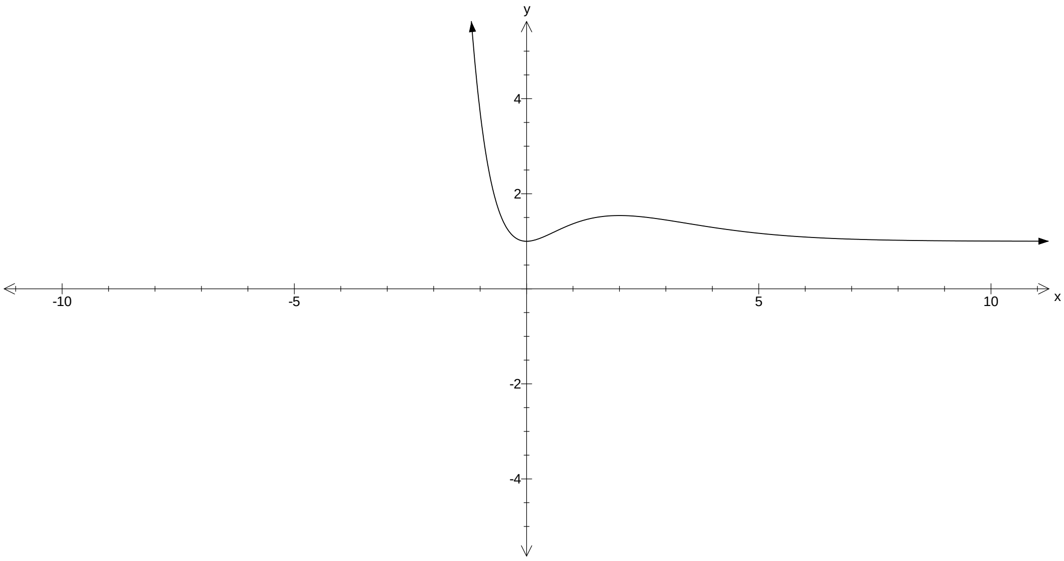
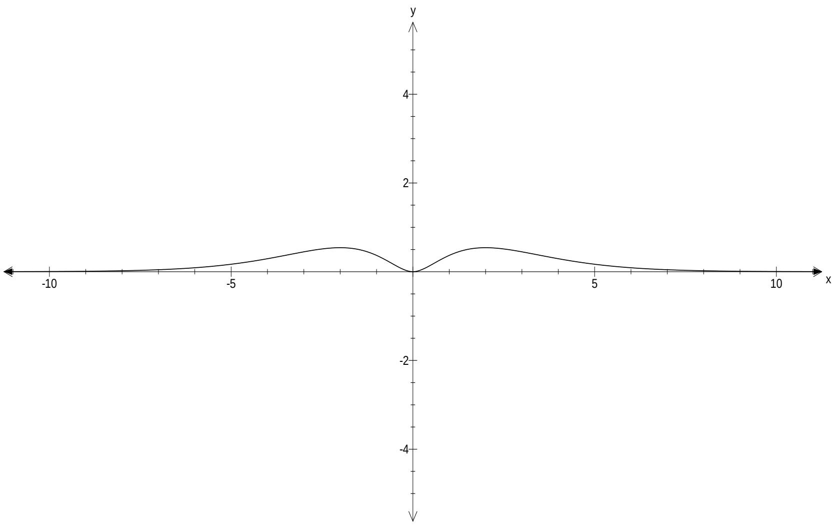
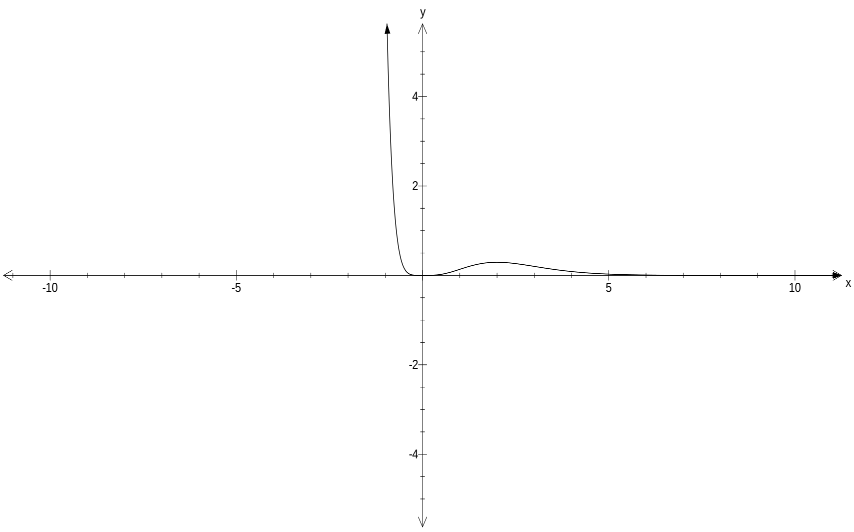
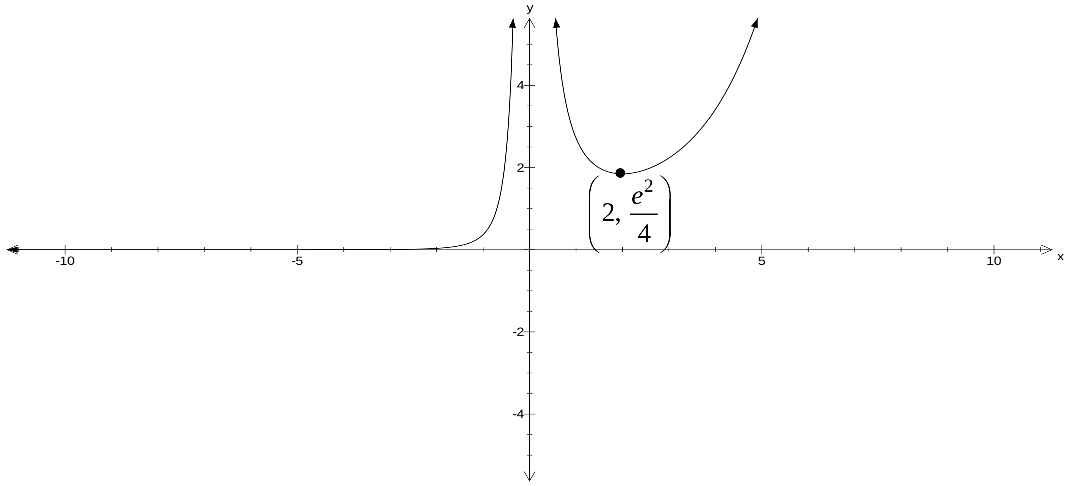
Question	Criteria	Marks
1(a)(i)	$f(x)=x\sqrt{4-x^2}$ $f(-x)=-x\sqrt{4-(-x)^2}=-x\sqrt{4-x^2}$ $-f(x)=-x\sqrt{4-x^2}$ $\therefore f(-x)=-f(x)$ *an odd function* ☑	**1**
1(a)(ii)	$\int_{-2}^{2}\left(x\sqrt{4-x^2}-\sqrt{4-x^2}\right)=\int_{-2}^{2}\left(x\sqrt{4-x^2}\right)-\int_{-2}^{2}\left(\sqrt{4-x^2}\right)$ $= odd\ \ function - semi\ circle$ ☑ $= 0-\frac{\pi\times 2^2}{2}$ $= 0-2\pi$ $= -2\pi$ ☑	**2**
(1)(b)	$\int \frac{dx}{\sqrt{4x^2+36}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+9}}$ ☑ $=\frac{1}{2}\int \frac{dx}{\sqrt{x^2+9}}$ $=\frac{1}{2}\ln\left(x+\sqrt{x^2+9}\right)+c \quad or \quad \ln(2x+\sqrt{4x^2+36})+C$ ☑	**2**
(1)(c)	Let $\frac{A}{2t+1}+\frac{B}{2-t}=\frac{5}{(2t+1)(2-t)}$ $\therefore A(2-t)+B(2t+1)=5$ If $t=2$, then $5B=5\rightarrow B=1$ $t=-\frac{1}{2}$, then $\frac{5}{2}A=5\rightarrow A=2$ $\therefore \int_0^1 \frac{5dt}{(2t+1)(2-t)}=\int_0^1\left(\frac{2}{2t+1}+\frac{1}{2-t}\right)dt$ ☑ $=\left[\ln(2t+1)-\ln(2-t)\right]_0^1$ $=\left[\ln\left(\frac{2t+1}{2-t}\right)\right]_0^1$ ☑ $=\ln 3-\ln\left(\frac{1}{2}\right)$ $=\ln 6$ ☑	**3**

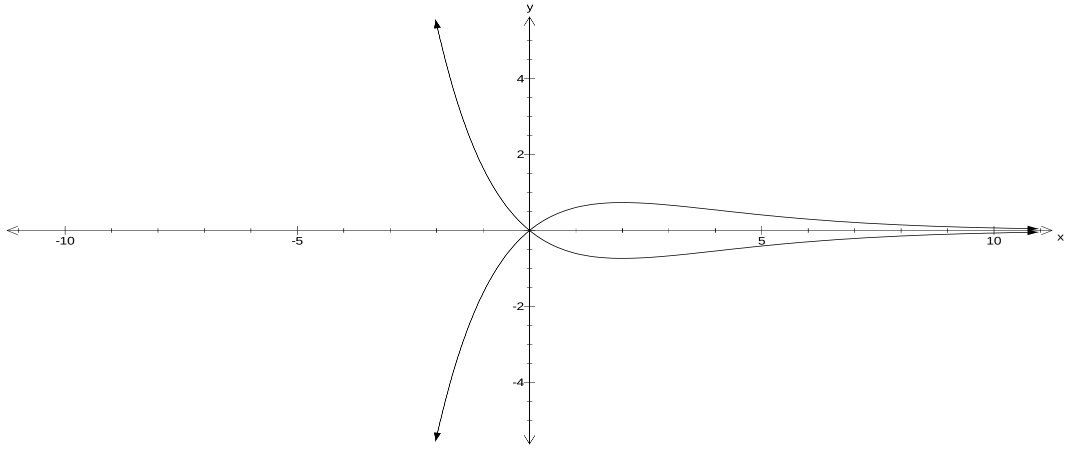
(1)(d)	$t = \tan \frac{x}{2}$ $\frac{dt}{dx} = \frac{1}{2}\sec^2 \frac{x}{2}$ $2\cos^2 \frac{x}{2}\, dt = dx$ since $\cos^2 \frac{x}{2} = \frac{1}{1+t^2}$ $\therefore dx = \frac{2}{1+t^2} dt$ ✓	$\int \cos ecx\ dx$ $= \int \frac{1}{\sin x} \times \frac{2}{1+t^2} dt$ $= \int \frac{1}{\frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt$ $= \int \frac{1+t^2}{2t} \times \frac{2}{1+t^2} dt$ $= \int \frac{1}{t}\, dt$ ✓ $= \ln(t) + C$ $= \ln\left(\tan \frac{x}{2}\right) + C$ ✓	**3**
(1)(e)	$(e)\ \ x = 4\sec\theta$ $x = \frac{4}{\cos\theta}$ $\frac{dx}{d\theta} = \frac{\cos\theta \times 0 - 4 \times -\sin\theta}{\cos^2\theta}$ $\frac{dx}{d\theta} = \frac{4\sin\theta}{\cos^2\theta}$ $\frac{dx}{d\theta} = 4\tan\theta\sec\theta$ ✓  $x = 4\sec\theta$ $\frac{x}{4} = \sec\theta$ $\frac{4}{x} = \cos\theta$ $\therefore \tan\theta = \frac{\sqrt{x^2-16}}{4}$	$\int \frac{\sqrt{x^2-16}}{x}\ dx$ $\int \frac{\sqrt{4^2\sec^2\theta - 16}}{4\sec\theta} \times 4\tan\theta\sec\theta\, d\theta$ $\int \frac{\sqrt{16(\sec^2\theta - 1)}}{4\sec\theta} \times 4\tan\theta\sec\theta\, d\theta$ $\int 4\tan^2\theta\ \ d\theta$ ✓ $\int 4(\sec^2\theta - 1)\ \ d\theta$ $\int 4\sec^2\theta - 4\ d\theta$ $= 4\tan\theta - 4\theta + C$ ✓ $= \frac{4\sqrt{16-x^2}}{4} - 4\cos^{-1}\left(\frac{4}{x}\right) + C$ $= \sqrt{16-x^2} -- 4\cos^{-1}\left(\frac{4}{x}\right) + C$ ✓	**4**

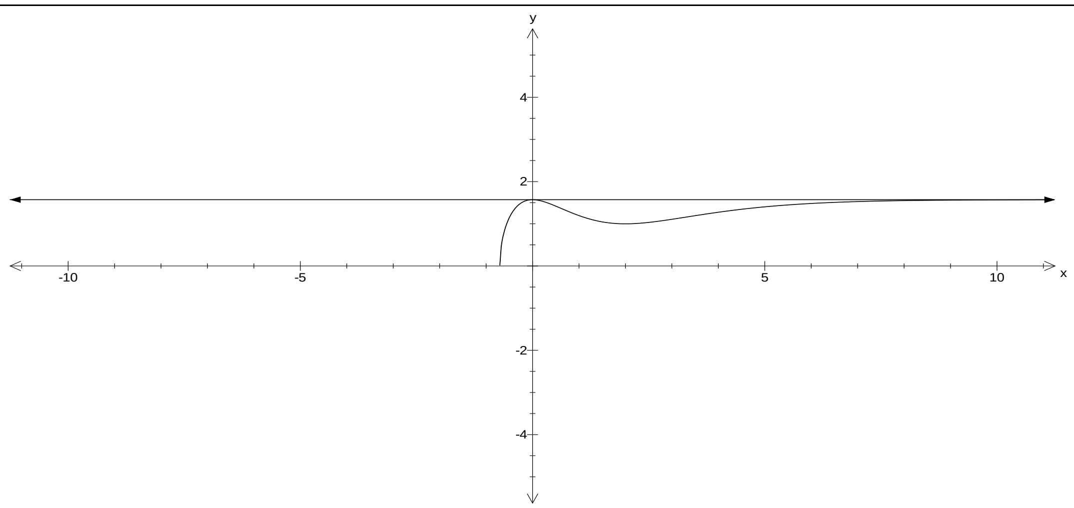
Question	Criteria	Marks
2(a)	$\frac{2-5i}{4-3i}\times\frac{4+3i}{4+3i}$ ☑ $=\frac{8+6i-20i-15i^2}{16-9i^2}$ $=\frac{23-14i}{25}$ ☑	**2**
2(b)	$(a-ib)^2=-21-20i$ $a^2-2aib+i^2b^2=-21-20i$ $a^2-b^2=-21 \quad and \quad -2aib=-20i$ ☑ $\therefore\ a=\frac{10}{b} \quad \Rightarrow \left(\frac{10}{b}\right)^2-b^2=-21$ $\frac{100}{b^2}-b^2=-21$ $b^4-21b^2-100=0$ $(b^2-25)(b^2+4)=0$ $\therefore\ b=\pm5 \quad and \quad a=\frac{10}{\pm5}=\pm2$ ☑☑	**3**
2(c)	$(\sin\theta+i\cos\theta)(\cos\theta-i\sin\theta)$ $=(\sin\theta+i\cos\theta)(\cos(-\theta)+i\sin(-\theta))$ ☑ $\lvert z\rvert=\sqrt{\sin^2\theta+\cos^2\theta}=1$ ☑ $\arg\ z=-\theta$ ☑	**3**
2(d)(i)	$(d)\ (i)\ \left\lvert\frac{z-1}{z+1}\right\rvert=2$ $\left\lvert\frac{x+iy-1}{x+iy+1}\right\rvert=2$ $\frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x+1)^2+y^2}}=2$ $\sqrt{(x-1)^2+y^2}=2\sqrt{(x+1)^2+y^2}$ $(x-1)^2+y^2=4\left[(x+1)^2+y^2\right]$ $x^2-2x+1+y^2=4x^2+8x+4+4y^2$ $3x^2+10x+3y^2+3=0$ ☑ $x^2+\frac{10}{3}x+y^2+1=0$ $x^2+\frac{10}{3}x+\left(\frac{5}{3}\right)^2+y^2=-1+\left(\frac{5}{3}\right)^2$ $\left(x+\frac{5}{3}\right)^2+y^2=\frac{16}{9}$ ☑	**2**

2(d)(ii)	(*ii*) $\left(x+\frac{5}{3}\right)^2+y^2\le\frac{16}{9}$ *centre* $\left(-\frac{5}{3},0\right)$ *radius* $=\frac{4}{3}$ ☑ $0\le\arg z\le\frac{3\pi}{4}$ $0\le y\le -x$ ☑  ☑ *shaded region*	**3**
2(e)	 *Since* $\frac{z_1+z_2}{z_1-z_2}=2i$ *then* $\arg(z_1+z_2)-\arg(z_1-z_2)=\frac{\pi}{2}$ $\therefore\overrightarrow{z_1+z_2}\perp\overrightarrow{z_1-z_2}$ ☑ *vectors* $z_1,\ z_2$ *and* z_1+z_2 ☑ *vectors* z_1-z_2	**2**

Question	Criteria	Marks
3(a)(i)	 ✓ *shift up of* 1 *unit, asymptote to* $y = 1$	**1**
3(a)(ii)	 ✓ *reflection in* y *axis*	**1**
(3)(a)(iii)	 ✓ x<0 steeper ✓ $x > 0$ graph $0 < y < 1$ (less steep gradient) and graph $y > 1$ steeper gradient	**2**

<table>
<tr>
<td>(3)(a)(iv)</td>
<td>


☑ asymptote $at\ x = 0$
☑ min $TP\ \ at\ \ x = 2\ \ and$ increasing gradient $x < 2$
</td>
<td>2</td>
</tr>
<tr>
<td>3(a)(v)</td>
<td>


☑ sketch only +ve section of orginal graph and reflect in x - axis
☑ for $-1 < y < 1$ the sketch is less steep and more steep after $y < -1$ and $y > 1$
</td>
<td>2</td>
</tr>
<tr>
<td>(3)(a)(vi)</td>
<td>


☑ range $0 \le y \le \frac{\pi}{2}$
☑ asymptote to $\frac{\pi}{2}$, $x \ge 2$
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</table>

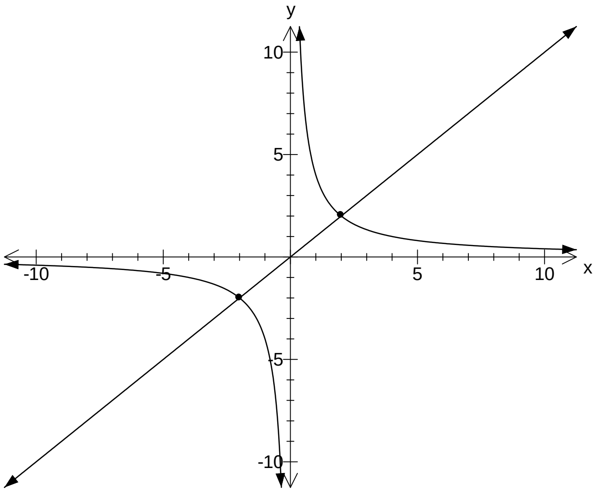
Question	Criteria	Marks
(3)(b)	$x^m y^n = k$ $u = x^m \qquad v = y^n$ $u' = mx^{m-1} \qquad v' = ny^{n-1}\dfrac{dy}{dx}$ $my^n x^{m-1} + nx^m y^{n-1}\dfrac{dy}{dx} = 0$ ✓ $nx^m y^{n-1}\dfrac{dy}{dx} = -my^n x^{m-1}$ $\dfrac{dy}{dx} = \dfrac{-my^n x^{m-1}}{nx^m y^{n-1}}$ $\dfrac{dy}{dx} = \dfrac{-my^n x^m \times x^{-1}}{nx^m y^n \times y^{-1}}$ $\dfrac{dy}{dx} = \dfrac{-my^n x^m \times y}{nx^m y^n \times x}$ $\dfrac{dy}{dx} = \dfrac{-my}{nx}$ ✓	**2**
(3)(c)	$V = 2\pi r \times thickness \times height$ $V = 2\pi \times x \times y \times dx$ $V = 2\pi\int_0^{\sqrt{6}} x\ y\ dx =$ ✓ $V = 2\pi\int_0^{\sqrt{6}} x\left(6x^2 - x^4\right) dx$ ✓ $V = 2\pi\int_0^{\sqrt{6}} 6x^3 - x^5\ \ dx$ $V = 2\pi\left[\dfrac{6x^4}{4} - \dfrac{x^6}{6}\right]_0^{\sqrt{6}}$ $V = 36\pi$ ✓	**3**

Question	Criteria	Marks

4(a)(i)	$x^3-4x^2+2x+5=0$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$ $=(4)^2-2(2)$ $=12$ ✓	**1**
4(a)(ii)	$x^3-4x^2+2x+5=0$ $\alpha^3-4\alpha^2+2\alpha+5=0$ $\beta^3-4\beta^2+2\beta+5=0$ $\underline{\gamma^3-4\gamma^2+2\gamma+5=0}$ $\therefore\ \alpha^3+\beta^3+\gamma^3-4(\alpha^2+\beta^2+\gamma^2)+2(\alpha+\beta+\gamma)+15=0$ ✓ $\alpha^3+\beta^3+\gamma^3-4(12)+2(4)+15=0$ $\alpha^3+\beta^3+\gamma^3=48-8-15$ $=25$ ✓	**2**
4(b)	$Monic\ \ a=1\ \ and\ \ \text{int}eger\ \ solutions\ \ \therefore (x-\sqrt{2})(x+\sqrt{2})(x-a)(x-b)$ ✓ since sum or roots is positive then $-\sqrt{2}+\sqrt{2}+a+b>0$ $product\ \ roots=4\quad \therefore \sqrt{2}\times\sqrt{2}\times a\times b=4$ $\therefore a=-2\quad b=1$ } ✓ $Hence\quad P(x)=(x-\sqrt{2})(x+\sqrt{2})(x-2)(x+1)$ ✓	**3**
4(c)(i)	$(\cos\theta+i\sin\theta)^3={}^3C_0(\cos\theta)^3(i\sin\theta)^0+{}^3C_1(\cos\theta)^2(i\sin\theta)^1+{}^3C_2(\cos\theta)^1(i\sin\theta)^2$ $+{}^3C_3(\cos\theta)^3(i\sin\theta)^3$ $=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$ ✓ $DeMoivres\ theorem:$ $(\cos\theta+i\sin\theta)^3=(\cos 3\theta+i\sin 3\theta)$ ✓ $Equating\ \ real\ \ parts:$ $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$ $=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$ $=\cos^3\theta-3\cos\theta+3\cos^3\theta$ ✓ $=4\cos^3\theta-3\cos\theta$	**3**

4(c)(ii)	*Let* $x = \cos\theta$ $8\cos^3 x - 6\cos x - 1 = 0$ $8\cos^3 x - 6\cos x = 1$ $2(4\cos^3\theta - 3\cos\theta) = 1$ $2\cos 3\theta = 1$ $\cos 3\theta = \frac{1}{2}$ ☑ $3\theta = \cos^{-1}\frac{1}{2} \pm 2k\pi$, where k is an integer $\theta = \frac{\pi}{9}(6k \pm 1)$ ☑ *These values of* θ *give exactly three distinct values of* $\cos\theta$, *namely* $\therefore\ (k=0) \quad \sin ce\ x = \cos\theta \Rightarrow x = \cos\left(\frac{\pi}{9}\right)$ $(k=1) \quad \sin ce\ x = \cos\theta \Rightarrow x = \cos\left(\frac{5\pi}{9}\right) = -\cos\left(\frac{4\pi}{9}\right)$ $(k=1) \quad \sin ce\ x = \cos\theta \Rightarrow x = \cos\left(\frac{7\pi}{9}\right) = -\cos\left(\frac{2\pi}{9}\right)$ ☑	
4(d)	$P(r) = 8r^3 - 4r^2 - 42r + 45 = 0$ $P'(r) = 24r^2 - 8r - 42 = 0 \quad (double\ root)$ ☑ $\therefore 24r^2 - 8r - 42 = 2(6r+7)(2r-3) = 0$ $r = \frac{-7}{6} \quad or \quad \frac{3}{2}$ ☑ $sub\ P\left(\frac{-7}{6}\right) = 8r^3 - 4r^2 - 42r + 45 \neq 0$ $P\left(\frac{3}{2}\right) = 8r^3 - 4r^2 - 42r + 45 = 0 \quad hence\ r = \frac{3}{2}$ ☑	

Question	Criteria	Marks
5(a)	$\int_1^\infty \frac{1}{x+1}-\frac{1}{x+3} = \left[\ln(x+1)-\ln(x+3)\right]_1^\infty$ $= \ln\left[\frac{x+1}{x+3}\right]_1^\infty$ ✓ $= \ln 1 - \ln\left(\frac{1}{2}\right)$ $= 0 - (\ln(1)-\ln(2))$ $= 0 - 0 + \ln 2$ $= \ln 2 \quad \left(or\ \ln\left(\frac{1}{2}\right)\right)$ ✓	**2**
5(b)(i)	$I_n = \int \sin^n x\ dx$ $= \int \sin x.\sin^{n-1}x\ dx \qquad u = \sin^{n-1}x \qquad \frac{dv}{dx} = \sin x$ $\frac{du}{dx} = (n-1)\sin^{n-2}x\cos x \qquad v = -\cos x$ $\therefore \int u\,dv = uv - \int v\,du$ $= -\cos x\sin^{n-1}x - \int -\cos x(n-1)\sin^{n-2}x.\cos x dx$ ✓ $= -\cos x\sin^{n-1}x + (n-1)\int \cos^2 x.\sin^{n-2} x\,dx$ $= -\cos x\sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,dx$ $= -\cos x\sin^{n-1}x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x dx$ ✓ $= -\cos x\sin^{n-1}x + (n-1)\mathrm{I}_{n-2} - (n-1)\mathrm{I}_n$ $\therefore n\mathrm{I}_n = -\cos x\sin^{n-1}x + (n-1)\mathrm{I}_{n-2}$ ✓ $\therefore \mathrm{I}_n = -\frac{1}{n}\cos x\sin^{n-1}x + \frac{n-1}{n}\mathrm{I}_{n-2}$	**3**
5b(ii)	$\int_0^{\frac{\pi}{2}} \sin^4 x dx = \left[-\frac{1}{4}\cos x\sin^3 x\right]_0^{\frac{\pi}{2}} + \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2 x dx$ $= 0 + \frac{3}{4}\left[-\frac{1}{2}\cos x\sin x\right]_0^{\frac{\pi}{2}} + \frac{3}{4}.\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^0 x dx$ ✓ $= 0 + 0 + \frac{3}{8}\int_0^{\frac{\pi}{2}} 1.dx$ $= \frac{3}{8}\left[\ x\ \right]_0^{\frac{\pi}{2}} = \frac{3}{8}\left[\frac{\pi}{2}-0\right]$ $\frac{3\pi}{16}$ ✓	**2**

<table>
<tr><td>5(c)(i)</td><td>

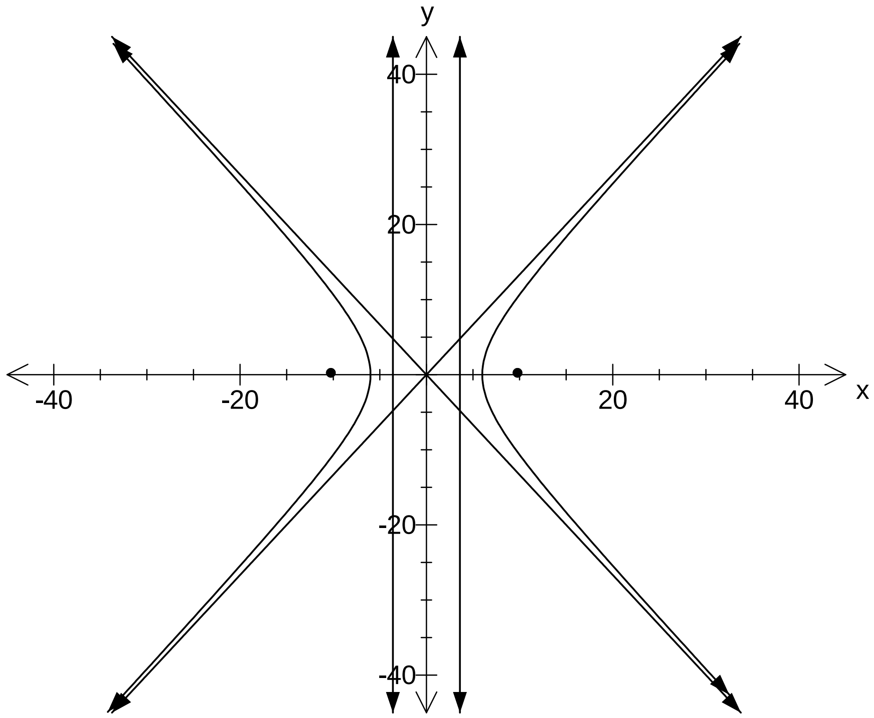
☑ shape and where it cuts $y = x \quad ie: (2,2) \; and \; (-2,2)$</td><td>1</td></tr>
<tr><td>5(c)(ii)</td><td>$xy = 4 \quad \therefore y = \frac{4}{x}$
$\frac{dy}{dx} = \frac{-4}{x^2}$
$\frac{dy}{dx} = \frac{-4}{4t^2} = -\frac{1}{t^2} \quad at\ P\left(2t, \frac{2}{t}\right)$ ☑
$\therefore y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$ ☑
$t^2 y - 2t = -x + 2t$
$x + t^2 y = 4t$</td><td>2</td></tr>
<tr><td>5(c)(iii)</td><td>$Tangent \;\; P: \;\; x + t^2 y = 4t$
$Tangent \;\; Q: \;\; x + s^2 y = 4s$

$x = 4t - t^2 y \quad and \quad x = 4s - s^2 y$
$\therefore 4t - t^2 y = 4s - s^2 y$
$4t - 4s = t^2 y - s^2 y$
$4(t - s) = y(t^2 - s^2)$
$y = \frac{4}{t + s}$ ☑

$x = 4t - t^2 y$
$= 4t - t^2\left(\frac{4}{t + s}\right)$
$= \frac{4t(t + s) - 4t^2}{t + s}$
$= \frac{4ts}{t + s}$ ☑ $\quad \therefore M\left(\frac{4st}{t + s}, \frac{4}{t + s}\right)$</td><td>2</td></tr>
</table>

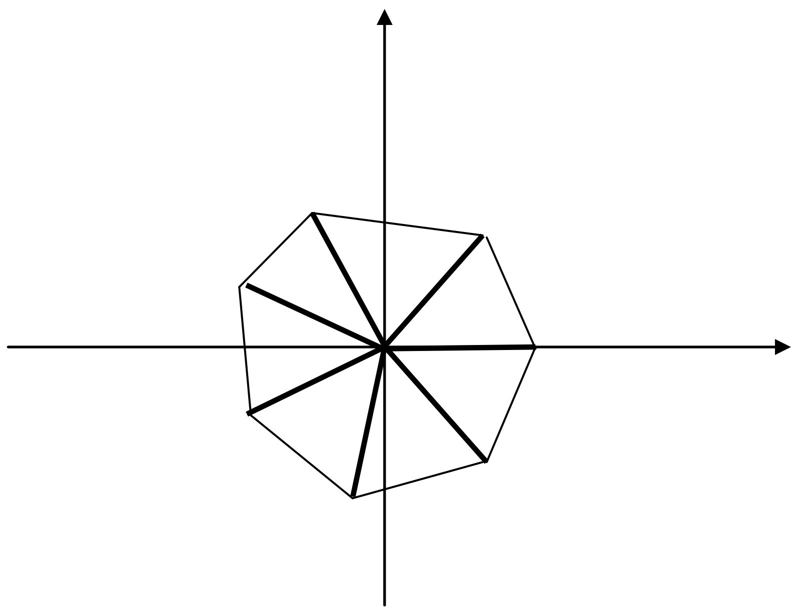
5(c)(iv)	$M\left(\frac{4st}{t+s}, \frac{4}{t+s}\right)$ $\therefore x = \frac{4st}{t+s}$ *and* $y = \frac{4}{t+s}$ $t+s = \frac{4st}{x}$ *and* $t+s = \frac{4}{y}$ ☑ *hence* $\frac{4st}{x} = \frac{4}{y} \Rightarrow y = \frac{x}{st}$ *and since* $s = -\frac{1}{t}$ $\therefore\ y = -x$ (*straight line locus through* $(0,0)$) ☑ *However it cannot pass through the origin as* $t, s \neq 0$ ☑	**3**

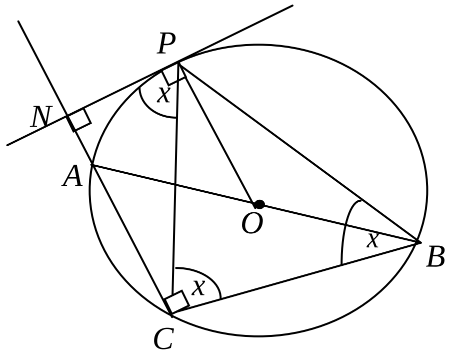
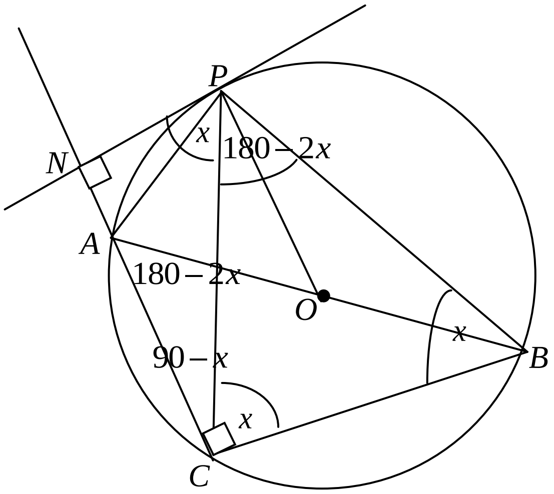
Question	Criteria	Marks
6(a)	$y^2 = 16(1-x) \quad \Rightarrow \quad x = 1 - \frac{y^2}{16}$ $A = \pi(R^2 - r^2)$ $= \pi(2^2 - (2-x)^2)$ $= \pi\left(4x - x^2\right)$ ☑ $V = \pi\int_{-4}^{4}\left(4x - x^2\right)\ dy$ ☑ $V = 2\pi\int_0^4 \left(4\left(1-\frac{y^2}{16}\right) - \left(1-\frac{y^2}{16}\right)^2\right)dy$ $V = 2\pi\int_0^4 \left(4 - \frac{y^2}{4} - 1 + \frac{y^2}{8} - \frac{y^4}{256}\right)dy$ $V = 2\pi\int_0^4 \left(3 - \frac{y^2}{8} - \frac{y^4}{256}\right)dy$ ☑ $A = 2\pi\left[3y - \frac{y^3}{24} - \frac{y^5}{1280}\right]_0^4$ $A = 2\pi\left[\left(12 - \frac{64}{24} - \frac{1024}{1280}\right) - (0)\right]$ $A = 2\pi\left(\frac{128}{15}\right)$ $A = \frac{256\pi}{15}$ ☑	**4**
6(b)(i)	$ae = 10 \ \ and \ \ \frac{b}{a} = \frac{4}{3}$ $\therefore a = \frac{10}{e} \ \ and \ \ b = \frac{4a}{3} \ \ and$ since $b^2 = a^2(e^2 - 1)$ $then \left(\frac{4a}{3}\right)^2 = \left(\frac{10}{e}\right)^2 (e^2 - 1)$ $\left(\frac{4\left(\frac{10}{e}\right)}{3}\right)^2 = \left(\frac{10}{e}\right)^2 (e^2 - 1)$ $\frac{1600}{9e^2} = \frac{100}{e^2}\left((e^2 - 1)\right)$ $\frac{16}{9} = e^2 - 1$ $e = \sqrt{\frac{16}{9} + 1} \qquad as\ e > 1$ $e = \frac{5}{3}$ ☑	**1**

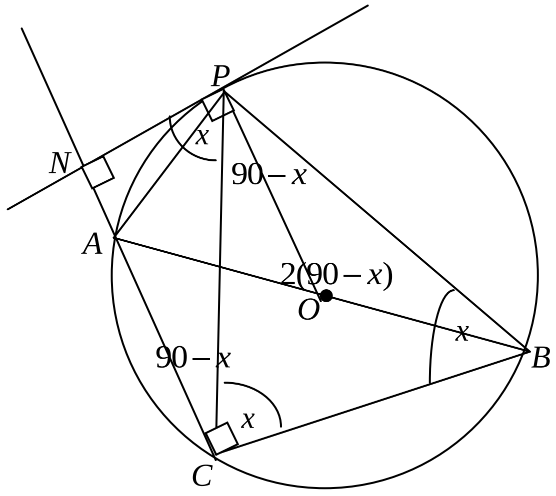
6(b)(ii)	since $e=\frac{5}{3}$ and $a=\frac{10}{e}=6$ and $b=\frac{4a}{3}=8$ $\therefore$ Equation of hyperbola is $\frac{x^2}{36}-\frac{y^2}{64}=1$ ☑	**1**
6(b)(iii)	 ☑ shape and asymptotes $y=\pm\frac{4x}{3}$ ☑ directrices $x=\pm\frac{18}{5}$ and foci $(\pm 6, 0)$	**2**

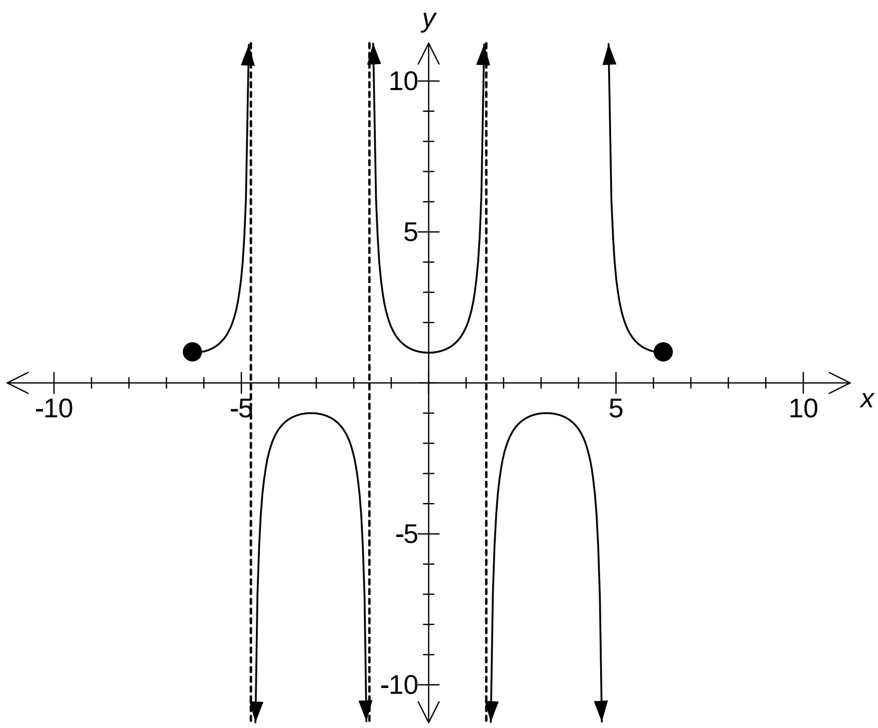
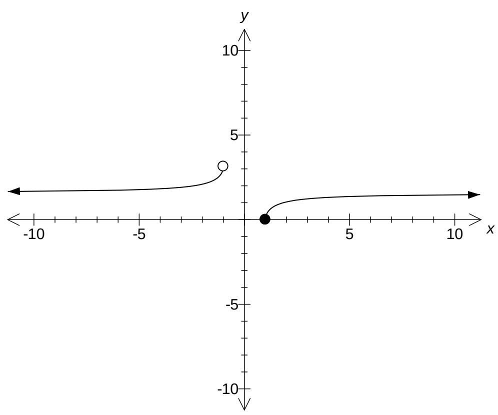
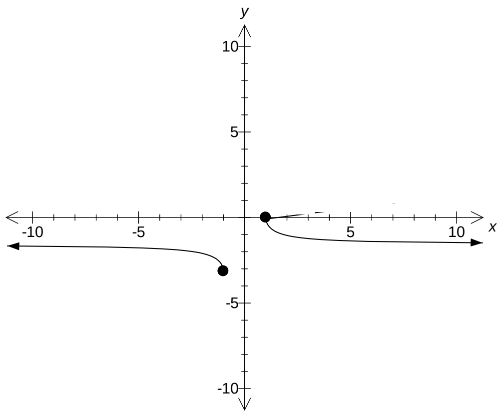
6(c)(i)	$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \qquad \therefore \frac{dy}{dx}=\frac{\frac{-2x}{a^2}}{\frac{2y}{b^2}}=\frac{-b^2x}{a^2y}=\frac{-b^2(a\cos\theta)}{a^2(b\sin\theta)}=-\frac{b\cos\theta}{a\sin\theta}$ ☑ *Equation of Tangent:* $y-b\sin\theta=\frac{b\cos\theta}{-a\sin\theta}(x-a\cos\theta)$ $-a\sin\theta y+ab\sin\theta\sin\theta=b\cos\theta x-ab\cos\theta\cos\theta$ $-a\sin\theta y-b\cos\theta x=-ab\sin\theta\sin\theta-ab\cos\theta\cos\theta$ $a\sin\theta y+b\cos\theta x=ab\sin\theta\sin\theta+ab\cos\theta\cos\theta$ $a\sin\theta y+b\cos\theta x=ab\sin^2\theta+ab\cos^2\theta$ $\frac{a\sin\theta y}{ab}+\frac{b\cos\theta x}{ab}=\frac{ab\sin^2\theta}{ab}+\frac{ab\cos^2\theta}{ab}$ $\frac{\sin\theta y}{b}+\frac{\cos\theta x}{a}=\sin^2\theta+\cos^2\theta$ ☑ $\frac{y\sin\theta}{b}+\frac{x\cos\theta}{a}=1$ $\therefore$ equation of tangent at $P(a\cos\theta, b\sin\theta)$ *is* $\boxed{\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1}$	**2**
6(c)(ii)	$\therefore$ equation of tangent at $P(a\cos\theta, b\sin\theta)$ *is* $\boxed{\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1}$ .... $\boxed{1}$ *similarly the equation at* $P(a\cos\phi, b\sin\phi)$ *is* $\boxed{\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1}$ .... $\boxed{2}$ $y=\frac{ab-xb\cos\theta}{a\sin\theta}$ *and* $y=\frac{ab-xb\cos\phi}{a\sin\phi}$ ☑ $\therefore \frac{ab-xb\cos\theta}{a\sin\theta}=\frac{ab-xb\cos\phi}{a\sin\phi}$ $a^2b\sin\phi-xab\cos\theta\sin\phi=a^2b\sin\theta-xab\cos\phi\sin\theta$ $a^2b\sin\phi-a^2b\sin\theta=xab\cos\theta\sin\phi-xab\cos\phi\sin\theta$ ☑ $a^2b(\sin\phi-\sin\theta)=xab(\cos\theta\sin\phi-\cos\phi\sin\theta)$ $\therefore x=\frac{a(\sin\phi-\sin\theta)}{(\cos\theta\sin\phi-\cos\phi\sin\theta)}$	**2**

6(c)(iii)	$m_{PS} = \dfrac{b\sin\theta}{a\cos\theta - ae}$ $m_{QS} = \dfrac{b\sin\phi}{a\cos\phi - ae}$ $\sin ce\ m_{PS} = m_{QS} \quad then\ \dfrac{b\sin\theta}{a\cos\theta - ae} = \dfrac{b\sin\phi}{a\cos\phi - ae}$ ☑ $ab\sin\theta\cos\phi - aeb\sin\theta = ab\sin\phi\cos\theta - aeb\sin\phi$ $aeb\sin\phi - aeb\sin\theta = ab\sin\phi\cos\theta - ab\sin\theta\cos\phi$ $aeb\left(\sin\phi - \sin\theta\right) = ab\left(\sin\phi\cos\theta - \sin\theta\cos\phi\right)$ $e = \dfrac{\sin\phi\cos\theta - \sin\theta\cos\phi}{\sin\phi - \sin\theta}$ $\therefore \dfrac{\sin\phi - \sin\theta}{\sin\phi\cos\theta - \sin\theta\cos\phi} = \dfrac{1}{e}$ ☑	**2**
6(c)(iv)	Directrix of the ellipse is $x = \dfrac{a}{e}$ $R\ has\ x - coordinates \quad x = \dfrac{a(\sin\phi - \sin\theta)}{(\cos\theta\sin\phi - \cos\phi\sin\theta)}$ $\Rightarrow \dfrac{x}{a} = \dfrac{a(\sin\phi - \sin\theta)}{(\cos\theta\sin\phi - \cos\phi\sin\theta)}$ $from\ (iii) \quad \dfrac{\cos\phi - \cos\theta}{\sin\theta\cos\phi - \cos\theta b\sin\phi} = \dfrac{b}{e}$ $\therefore\ \dfrac{x}{a} = \dfrac{b}{e}$ $x = \dfrac{a}{e}$ (lies on the discriminant of the ellipse) ☑	**1**

Question	Criteria	Marks
7(a)	If α, β, γ are the roots of $x^3 + qx + r = 0$ *then* $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ *or* $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ *satisfy* $\left(\frac{1}{x}\right)^3 + q\left(\frac{1}{x}\right) + r = 0$ ✓ $\therefore \; 1 + qx^2 + rx^3 = 0$ ✓	**2**
7(b)(i)	ω *is a root or* $z^7 - 1$ *hence* $\omega^7 - 1 = 0$ $\omega^7 - 1 = (\omega - 1)\left(\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1\right) = 0$ $\omega = 1$ *or* $\omega^6 + \omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$ ✓	**1**
7(b)(ii)	*since* $\omega = 1$ *and* $\omega^7 = 1$ *then* $(1+\omega)(1+\omega^2)(1+\omega^4) = \left(1 + \omega^2 + \omega + \omega^3\right)(1+\omega^4)$ $= 1 + \omega^4 + \omega^2 + \omega^6 + \omega + \omega^5 + \omega^3 + \omega^7$ $= 1 - 1 + 1$ ✓ $= 1$	**1**
7(b)(iii)	$(\omega + \omega^2 + \omega^4)(\omega^6 + \omega^5 + \omega^3) = \omega^7 + \omega^6 + \omega^4 + \omega^8 + \omega^7 + \omega^5 + \omega^{10} + \omega^9 + \omega^7$ $= 1 + \underbrace{\omega^6 + \omega^4 + \omega + 1 + \omega^5 + \omega^3 + \omega^2}_{\text{using (i)}} + 1$ ✓ $= 1 - 0 + 1$ $= 2$ ✓	**2**
7(b)(iv)	 ✓ $\left(\textit{angle of rotation of } \frac{2\pi}{7} \textit{ and } z = 1\right)$	**1**

7(c)(i)	 $Let\ \angle NPC = x$ $\angle NPC = \angle PBC = x$ (*angles in alternate segment are equal*) ✓ $\sin ce\ \angle ACB = 90$ (*angles in a semi circle*) $and\ \angle CNP = 90$ ($NP \perp NC$) $\therefore\ NP \parallel BC$ ✓ $\therefore\ \angle NPC = \angle PCB = x$ (*alternate* $\angle$*'s are equal*) $\therefore \angle PCB = \angle PBC = x$ ✓ $Hence\ PC = PB$ (*base angles of isoceles* Δ *are equal*)	**3**
7(c)(ii)	 $\angle NPC = \angle PCB = \angle PBC = x$ $\angle CPB = 180 - 2x$ (*angle sum of a triangle is* sup*plementary*) $\angle APB = 90$ (*angles in a semi circle are right angles*) $\therefore \angle APC = 90 - (180 - 2x)$ $= 2x - 90$ (*angles in a semi circle are right angles*) ✓ $\angle ACP = 90 - x$ (*angle in semi circle are right angles*) ✓ $\angle APC + 2\angle ACP = 2x - 90 + 2(90 - x)$ $= 2x - 90 + 180 - 2x$ $= 90$ ✓	**3**

7(c)(iii)	 $\angle NPC = \angle PCB = x$ (*proven* (*i*)) $\angle PCB = \angle PAB = x$ (angles standing on the same arc are equal) ✓ $\therefore \angle NPC = \angle PAB$ ✓	**2**

Question	Criteria	Marks
8(a)(i)	 ☑ *shape and asymptotes at* $x=\pm\frac{\pi}{2}$ *and* $x=\pm\frac{3\pi}{2}$	**1**
8(a)(ii)	  *domain* $0\le x\le 1$ ☑ *domain* ☑ *shape and features* *domain* $-1\le x\le 0$ ☑ *domain* ☑ *shape and features*	**2**

8(b)(i)	$t_n = \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+.....+\frac{1}{2n-1}+\frac{1}{2n}$ $\therefore t_n+\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+.....+\frac{1}{2n-1}+\frac{1}{2n}+\frac{1}{2n}$ ☑ $t_n+\frac{1}{2n}=\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+.....+\frac{1}{2n-1}\right)+\frac{1}{n}$ $=\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+.....+\frac{1}{2n-1}$ ☑	**2**
8(b)(ii)	$t_n+\frac{1}{2n}=\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+.....+\frac{1}{2n-1}$ =sum of upper rectangles drawn on the graph ☑ $> \int_n^{2n} \frac{1}{x}\,dx = [\ln x]_n^{2n}$ ☑ $\therefore t_n+\frac{1}{2n} > \ln 2n - \ln n = \ln\left(\frac{2n}{n}\right)=\ln 2$ ☑	**3**
8(b)(iii)	Prove true for $n=1$ $\therefore\ t_1=\frac{1}{2}$ $s_1=1-\frac{1}{2}=\frac{1}{2}$ *Hence true for* $n=1$ ☑ *Assume true for* $n=k$ $\therefore t_k = s_k$ Prove true for $n=k+1$ *Now* $s_{k+1}=1-\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2k}+\frac{1}{2(k+1)-1}+\frac{1}{2(k+1)}$ $=1-\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}$ $=s_k+\frac{1}{2k+1}+\frac{1}{2k+2}$ ☑ *from* (i) *for* $n=k+1$ $t_{k+1}+\frac{1}{2(k+1)}=\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\frac{1}{k+4}+.....+\frac{1}{2(k+1)-1}$ $=\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\frac{1}{k+4}+.....+\frac{1}{2k+1}$ $=t_{k+1}+\frac{1}{2k+1}$ $\therefore t_{k+1}=t_k+\frac{1}{2k+1}-\frac{1}{2k+2}$ ☑ since $s_k=t_k$ *and* $s_{k+1}=s_k+\frac{1}{2k+1}+\frac{1}{2k+2}$ *and* $t_{k+1}=t_k+\frac{1}{2k+1}-\frac{1}{2k+2}$ *then* $s_{k+1}=t_{k+1}$ ☑ $\therefore$ *By principles of mathematical induction* $s_b=t_n$ *for* $n=1,2,3,....$	**4**

8(b)(iv)	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+....+\frac{1}{9999}-\frac{1}{10\,000}=s_{5000}$ $=t_{5000}\quad by\ (iii)$ ☑ and $\ln 2-\frac{1}{2n}<t_n<\ln 2\qquad (by\ (ii))$ ☑ $\therefore\ \ln 2-\frac{1}{10000}<t_{5000}<\ln 2$ $\ln 2-0.0001<t_{5000}<\ln 2$ $0.693147-0.0001<t_{5000}<0.693147$ $\therefore t_{5000}=0.693\ (3\ dec\ \ pl)$ ☑	**3**