THE KING'S SCHOOL

2003
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Attempt Questions 1-8
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\displaystyle\int \frac{1+x+x^2}{1+x^2}\,dx$ **2**

(ii) $\displaystyle\int \frac{x^2}{1+x^2}\,dx$ **2**

(b) Use integration by parts to evaluate

$\displaystyle\int_0^1 2x\tan^{-1}x\,dx$ **3**

(c) Find $\displaystyle\int_0^1 \frac{x-3}{(x^2+1)(3x+1)}\,dx$, giving your answer in simplest exact form. **4**

(d) $\displaystyle u_n = \int_0^1 \frac{x^n}{1+x^2}\,dx,\ n \ge 0$

(i) Show that $u_{n+2} + u_n = \dfrac{1}{n+1}$ **2**

(ii) Hence, evaluate $\displaystyle\int_0^1 \frac{x^3}{1+x^2}\,dx$ **2**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) $u = 2+ai$, $v = a+2i$, where a is a real number.

Find in the form $x+iy$,

(i) uv **2**

(ii) $(uv)^{-1}$ **1**

(b) (i) Express $z = -2\sqrt{3}+2i$ in modulus-argument form **2**

(ii) Hence, find z^3 in the form $x+iy$ **2**

(c) Sketch the region in the complex plane where

$|z-i| \le |z+1|$ **3**

(d) Consider the equation $(a+ib)^2 = 1+2i$, a,b real

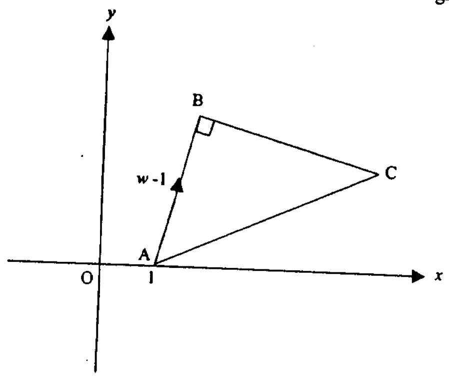
(i) Show that $a^2+b^2 = \sqrt{1^2+2^2}$ **1**

(ii) Hence, or otherwise, find the value of a^2 **2**

Question 2 continues on next page

Question 2 (continued) **Marks**

(e) In the complex plane, A is the point (1,0) and the complex number $\overrightarrow{AB}$ is $w-1$. ΔABC is isosceles and right-angled at B. O is the origin.



Find, in terms of w, the complex numbers

(i) $\overrightarrow{CB}$ 1

(ii) $\overrightarrow{OC}$ 1

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch on the same axes the graphs of

$y=|x-1|$ and $y=2x-x^2$ 2

(ii) Use (i) to show on separate diagrams, the graphs of

(α) $y=\dfrac{|x-1|}{2x-x^2}$, showing any asymptotes 3

(β) $y=\dfrac{2x-x^2}{|x-1|}$, showing any asymptotes 3

(b) Consider the function $f(x)=\tan^{-1}x-\dfrac{x}{1+x^2}$

(i) Show that f is an odd function. 1

(ii) Find $f'(x)$ 2

(iii) Show that $f(x)>0$ if $x>0$ 2

(iv) Sketch the graph of $y=\tan^{-1}x-\dfrac{x}{1+x^2}$ 2

End of Question 3

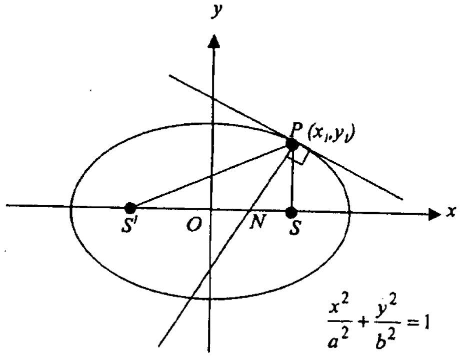
Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Find the gradient of the tangent to the curve $x^3 + y^2 + xy = 0$ at the point $(-2,4)$ **3**

(b) $P(x_1, y_1)$ is a point in the first quadrant on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $a>b>0$

S and S' are the foci of the ellipse. O is the origin.



(i) Show that the equation of the normal at $P(x_1, y_1)$ is
$a^2 y_1(x-x_1) = b^2 x_1 (y-y_1)$ **2**

(ii) The normal at P meets the major axis at N.

Prove that the x coordinate at N is $e^2 x_1$, where e is the eccentricity of the ellipse. **2**

(iii) Deduce that N lies between O and S. **2**

(iv) Show that $NS = eSP$ and $NS' = eS'P$ **3**

(v) Using the sine rule in ΔPSN and $\Delta PS'N$, or otherwise, prove that PN bisects $\angle SPS'$ **3**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Four married couples are to be seated at a circular table.

(i) How many arrangements are possible if the men and women are to be separated? **2**

(ii) For the arrangements in (i), find the probability that no woman is sitting next to her husband. **2**

(b) The equation $x^3 + ax^2 + bx + c = 0$ has one root the sum of the other two roots.

Prove that $a^3 - 4ab + 8c = 0$ **4**

(c) (i) By considering the circle $x^2 + y^2 = a^2$, or otherwise, find

$$\int_0^a \sqrt{a^2 - x^2}\, dx$$ **2**

(ii) The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $a>b>0$, is revolved about the line $y=a$.

By considering slices perpendicular to the line $y=a$, find the volume of the solid of revolution generated. **5**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle of mass m kg falls vertically from rest from point O in a medium whose resistance is mkv, where k is a positive constant and v is its velocity in m/s. After t seconds the particle has fallen x metres.

g m/s^2 is the acceleration due to gravity.

(i) Show that $\frac{dv}{dt} = g - kv$ **1**

(ii) Find the terminal velocity, V m/s, of the particle. **1**

(iii) Use integration to prove that $v = \frac{g}{k}\left(1 - e^{-kt}\right)$ **3**

(iv) Find the distance the particle has fallen when its velocity is one half of its terminal velocity. **4**

(b) α, β are the two complex roots of the equation $x^3 + 5x + 1 = 0$

(i) Explain why α, β are complex conjugates. **1**

(ii) Show that the real root is $\frac{-1}{|\alpha|^2}$ **2**

(iii) Show that $\alpha\beta$ is a root of the equation $x^3 - 5x^2 - 1 = 0$ **3**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) By mathematical induction it is easy to show that

$$1^2 - 2^2 + 3^2 - \ \ldots\ - (2n)^2 = -n(2n+1)$$

If, further, it is known that

$$1^2 + 2^2 + 3^2 + \ \ldots\ + (2n)^2 = \frac{n}{3}(2n+1)(4n+1),$$

deduce that

$$1^2 + 2^2 + 3^2 + \ \ldots\ + n^2 = \frac{n}{6}(n+1)(2n+1)$$

(Do not use induction) **3**

(b) (i) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$$ **3**

(ii) Let $F(x)$ be a primitive function of $f(x)$.

Using this, or otherwise, show that

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x) + f(2a - x)\,dx$$ **2**

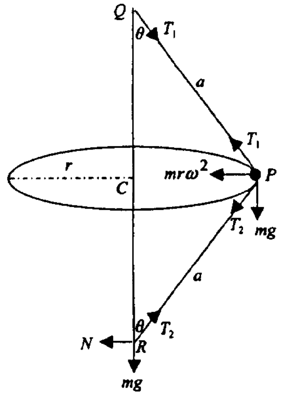
(iii) Deduce $\int_0^{\pi} \frac{x}{1 + \sin x}\,dx$ **3**

Question 7 continues next page

Marks

Question 7 (continued)

(c) A mass m at P is freely joined to two equal light rods PQ and PR of length a. The end Q of PQ is pivoted to a fixed point Q and the end R of PR is freely joined to a ring of mass m which slides on a smooth vertical pole. If P rotates in a horizontal circle with uniform angular velocity ω, show the angle of inclination of the rods PQ and PR to the vertical is $\tan^{-1}\left(\dfrac{rw^2}{3g}\right)$. T_1, T_2 are tensions in the rods, N is the normal reaction of QR on the ring R. **4**



End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $z^n = 1$, n a positive integer, are

$$z_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 1, 2, \ldots, n$$

(i) Show that $z_k^n = z_1^{kp}$, p a positive integer **2**

(ii) If z_k is such that $z_k, z_k^2, z_k^3, \ldots, z_k^n$ generates all the roots of $z^n = 1$, then z_k is called a primitive root of $z^n = 1$

(α) Show that z_1 is a primitive root of $z^n = 1$ **1**

(β) Show that z_5 is a primitive root of $z^6 = 1$ **2**

(γ) Suppose the highest common factor of n and k is h, ie, $n = ph$ and $k = qh$, p, q integers.

Show that for z_k to be a primitive root of $z^n = 1$, then $h = 1$ **2**

(b) (i) Show that $\displaystyle\sum_{k=0}^{n-1}(1-x)^k = \frac{1-(1-x)^n}{x}$, $x \neq 0$ **2**

(ii) Deduce that $\displaystyle\sum_{k=0}^{n-1}(1-x)^k = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}x^{k-1}$ **2**

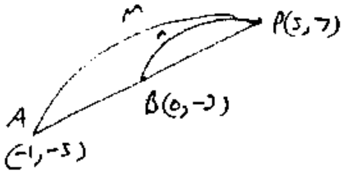
(iii) Explain or show why $\displaystyle\int \sum_{k=0}^{n-1}(1-x)^k\,dx = \sum_{k=0}^{n-1}\int(1-x)^k\,dx$ **1**

(iv) Deduce that $\displaystyle\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\ \ldots\ +\frac{1}{n}$ **3**

SOLUTIONS TKS TRIAL HSC 2003 EXTENSION 1

Qu 1

(a)



$\Rightarrow \boxed{m:n = 6:5}$

(b) $\dfrac{d}{dx} \tan^{-1}(1+x^2) = \dfrac{1}{1+(1+x^2)^2} \times 2x = \boxed{\dfrac{2x}{1+(1+x^2)^2}}$

(c) $\binom{8}{5}$ or, of course, $\binom{8}{3} = \boxed{56}$

(d) gradients of lines are 2 and -3

$\therefore \tan\alpha = \left| \dfrac{2-(-3)}{1+2(-3)} \right| = \dfrac{5}{5} = 1$

$\therefore$ acute angle is $\boxed{45^0}$

(e) $P(-1) = 0 \Rightarrow (-1)^{2n+1} - (-1)^{2n} + b = 0$

ie $-1 - 1 + b = 0 \quad \therefore \boxed{b = 2}$

(f) $\displaystyle\sum_{n=1}^{9} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \dots + \left(\cancel{\tfrac{1}{8}} - \cancel{\tfrac{1}{9}}\right) + \left(\cancel{\tfrac{1}{9}} - \tfrac{1}{10}\right)$

$= 1 - \dfrac{1}{10}$

$= \boxed{\dfrac{9}{10}}$

Qu 2

(a) $f'(x) = 5 - 4\cos 4x$

$\geq 5 - 4(1)$ since $-1 \leq \cos 4x \leq 1$

$\Rightarrow f'(x) \geq 1 > 0 \quad \forall x$

$\therefore f(x)$ increases $\forall x$

(b) (i) $R\sin(x-\alpha) = R\cos\alpha \sin x - R\sin\alpha\cos x$

$\equiv \sin x - \sqrt{3}\cos x$

$\Rightarrow$ $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$ $\quad \therefore \tan\alpha = \sqrt{3}$, $\boxed{\alpha = \frac{\pi}{3}}$

and $R = \sqrt{1^2 + \sqrt{3}^2} = \boxed{2}$

(ii) From (i), $2\sin\left(x - \frac{\pi}{3}\right) = \sqrt{2}$

$\therefore \sin\left(x - \frac{\pi}{3}\right) = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

$\therefore x - \dfrac{\pi}{3} = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$

ie $x = \boxed{\dfrac{7\pi}{12} \text{ or } \dfrac{13\pi}{12}}$

(c) (i) $u = 4 - x^2$ $\qquad x = 0, u = 4$

$\dfrac{du}{dx} = -2x$ or $du = -2x\,dx$ $\qquad x = \sqrt{3}, u = 1$

$\therefore I = -\dfrac{1}{2}\displaystyle\int_4^1 \frac{du}{\sqrt{u}} = \frac{1}{2}\int_1^4 u^{-\frac{1}{2}}\,du$

$= \dfrac{1}{2} \cdot 2\left[u^{\frac{1}{2}}\right]_1^4$

$= 2 - 1 = 1$

(ii) $I = \displaystyle\int_0^{\sqrt{3}} \frac{4}{\sqrt{4-x^2}} - \frac{x}{\sqrt{4-x^2}}\,dx = 4.\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}} - 1$, from (i)

$= 4\,\dfrac{\pi}{3} - 1$

Qn 3

(a) $f'(x) = 2x e^{x^2} - 1$

$\therefore x_1 = 1{\cdot}2 - \dfrac{e^{1{\cdot}44} - 1{\cdot}2 - 3}{2{\cdot}4 e^{1{\cdot}44} - 1} = 1{\cdot}1977\ldots$

$\therefore$ two decimal approx^n = $\boxed{1{\cdot}20}$

(b) (i) $\sin 2A = \dfrac{2t}{1+t^2}$

(ii) put $t = \tan A$,

then $\operatorname{cosec} 2A - 3\cot 2A = \dfrac{1+t^2}{2t} - 3 \cdot \dfrac{1-t^2}{2t}$

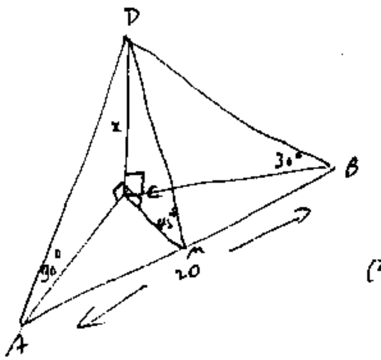
$= \dfrac{1+t^2 - 3 + 3t^2}{2t}$

$= \dfrac{4t^2 - 2}{2t}$

$= 2t - \dfrac{1}{t}$

$= 2\tan A - \cot A$

(c) (i)

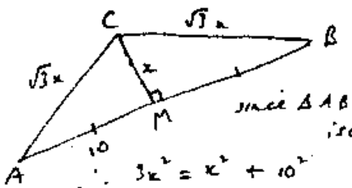


(ii) In ΔACD,

$\tan 30° = \dfrac{x}{AC} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow AC = \sqrt{3}\, x$

(iii) From (ii), we have



since ΔABC is isosceles

$\therefore 3x^2 = x^2 + 10^2$

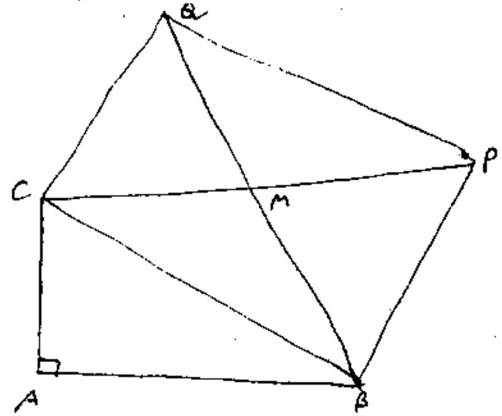
$2x^2 = 100$

$x^2 = 50$

$\therefore x = \sqrt{50}$ or $5\sqrt{2}$

Qn 4

(i)



(ii) $\angle CMB = 90°$, the diagonals of a square meet at right angles

$\therefore$ In $\Diamond ABMC$, $\angle A + \angle M = 180°$

$\Rightarrow \Diamond ABMC$ is cyclic, opposite angles are supplementary

(iii) $MC = MB$, equal diagonals in a square bisect each other.

$\therefore \angle CAM = \angle BAM$, $\angle$s at the circumference of a circle standing on equal arcs.

$\therefore MA$ bisects $\angle BAC$

(b) (i) $t = 0$, $x = 10\cos 0 = 10$

$\dot{x} = -10n \sin nt = -10n \sin 0$ at $t = 0$

$= 0$

$\therefore$ particle is initially at rest at $x = 10$

(ii) $T = \dfrac{2\pi}{n}$ $\therefore n = \dfrac{2\pi}{T}$

$\therefore b = 10\cos\left(\dfrac{2\pi}{T} \cdot \dfrac{T}{3}\right) = 10\cos\left(\dfrac{2\pi}{3}\right)$

ie $b = 10\left(-\dfrac{1}{2}\right) = -5$

(iii) $\dot{x} = -10n \sin nt = -10 \cdot \frac{2\pi}{T} \sin\left(\frac{2\pi t}{T}\right)$

$\therefore -20\sqrt{3} = -\frac{20\pi}{T} \sin\left(\frac{2\pi}{3}\right) = -\frac{20\pi}{T} \cdot \frac{\sqrt{3}}{2}$

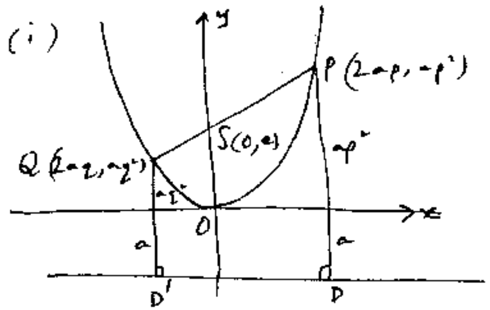
$\therefore T = \frac{\pi}{2}$ (seconds) ie period is $\frac{\pi}{2}$ s

(c) $\binom{n}{3} \div \binom{n-1}{2} = \frac{n!}{(n-3)!\,3!} \div \frac{(n-1)!}{(n-3)!\,2!}$

$= \frac{n!}{(n-3)!\,3!} \times \frac{(n-3)!\,2!}{(n-1)!} = \frac{n}{3}$

Qn 5

(a) (i)



gradient PQ

$= \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p-q)(p+q)}{2a(p-q)}$

$= \frac{p+q}{2}$

$\therefore$ chord PQ is $y - ap^2 = \frac{1}{2}(p+q)(x - 2ap)$

or $y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$

$\Rightarrow y - \frac{1}{2}(p+q)x + apq = 0$

(ii) Since $S(0,a)$ is on PQ, then

$a - 0 + apq = 0$

ie. $pq = -1$ or $q = -\frac{1}{p}$

(iii) $PQ = PS + QS = PD + QD'$, focus-directrix defn (see diagram)

$= a + ap^2 + a + aq^2, \quad q = -\frac{1}{p}$

$= 2a + a\left(p^2 + \frac{1}{p^2}\right)$

(iv) If PQ is a diameter, the radius is $a + \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)$, from (iii)

& the centre is $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a\left(p - \frac{1}{p}\right), \frac{a}{2}\left(p^2 + \frac{1}{p^2}\right)\right), \quad q = -\frac{1}{p}$

$\therefore$ distance from centre to directrix $y = -a$ is

$\frac{a}{2}\left(p^2 + \frac{1}{p^2}\right) + a$ = radius

$\therefore$ directrix is a tangent to the circle

(b) (i) $A = 6\times 10^2 + 12\cdot 6t = 600 + 12\cdot 6t$

(ii) If an edge is x, $A = 6x^2$, $V = x^3$

$\therefore$ From (i), $6x^2 = 600 + 12\cdot 6t$

$x^2 = 100 + 2\cdot 1t$

$\therefore V = (100 + 2\cdot 1t)^{3/2}$

& so, $\dfrac{dV}{dt} = \dfrac{3}{2}(100 + 2\cdot 1t)^{\frac{1}{2}}(2\cdot 1)$

$= 3\cdot 15 \times \sqrt{121}$ cm^3/s when $t = 10$

$= 34\cdot 65$ cm^3/s

Alternatively, using $A = 6x^2$, $V = x^3$, $\dfrac{dA}{dt} = 12\cdot 6$,

we have $\dfrac{dV}{dt} = \dfrac{dV}{dx}\cdot\dfrac{dx}{dt}$

$= \dfrac{dV}{dx}\cdot\dfrac{dx}{dA}\cdot\dfrac{dA}{dt}$

$= 3x^2\cdot\dfrac{1}{12x}\cdot(12\cdot 6)$

$= 3\cdot 15x$

But, when $t = 10$, $A = 6\times 10^2 + 12\cdot 6\times 10 = 726$

$\therefore 6x^2 = 726 \Rightarrow x = 11$

$\therefore \dfrac{dV}{dt} = 3\cdot 15\times 11$ cm^3/s $= 34\cdot 65$ cm^3/s

Qu 6

(a) $E(0) = 9^2 - 4^0 = 80$ is a multiple of 5

$\therefore$ assume $E(n) = 9^{n+2} - 4^n = 5q$, q an integer, $n \geq 0$

Then, $E(n+1) = 9^{n+3} - 4^{n+1}$

$= 9(9^{n+2}) - 4^{n+1}$

$= 9(5q + 4^n) - 4^{n+1}$, using the assumption

$= 5(9q) + 4^n(9-4)$

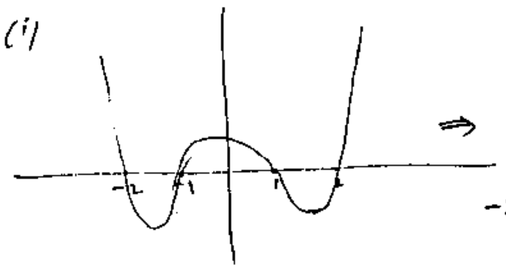
$= 5(9q + 4^n)$ is a multiple of 5 since $9q + 4^n$ is an integer

$\therefore$ if $E(n)$ is a multiple of 5, so is $E(n+1)$

But, $E(0)$ is a multiple of 5

$\therefore E(n)$ is a multiple of 5 by induction

(b) (i)



$\Rightarrow (x^2-1)(x^2-4) \leq 0$ has solutions $-2 \leq x \leq 1$ or $1 \leq x \leq 2$

(ii) (a) $\dfrac{d(\frac{1}{2}v^2)}{dx} = 10x - 4x^3$

$\therefore \frac{1}{2}v^2 = 5x^2 - x^4 + c$, c a constant

ie. $\frac{1}{2}v^2 + x^4 - 5x^2 = c$

When $x = \sqrt{2}$, $v = 2$

$\therefore 2 + 4 - 10 = c = -4$

(β) we have $\frac{1}{2}v^2 = 5x^2 - x^4 - 4$

or $\frac{1}{2}v^2 = -(x^4 - 5x^2 + 4)$

$= -(x^2-1)(x^2-4)$

Now, $\frac{1}{2}v^2 \geq 0 \Rightarrow (x^2-1)(x^2-4) \leq 0$

∴ Using (b)(i) and when $x=\sqrt{2}$, $v=2$, we have the particle oscillates between $x=1$ and $x=2$

(c) (i) $P(5 \text{ males}, 5 \text{ females}) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5$

$= 0.246$, 3 d.p.

(ii) $P(\text{more females}) = P(\text{more males})$, since $P(M) = P(F) = \frac{1}{2}$

∴ using (i), $P(\text{more females}) = \dfrac{1-0.246}{2} = 0.377$

Qn 7

(a) if $x+1>0$, then $-2x>0$

i.e. $x>-1$ i.e. $x<0$

∴ solution is $-1<x<0$

if $x<-1$, we'd have $x>0 \Rightarrow$ no further solutions

∴ $-1<x<0$

(b)(i) We need $\dfrac{-2x}{x+1}>0$ and $-2x>0$ and $x+1>0$.

All 3 inequalities are "true" from (a)

(ii) From (i), $y = \ln(-2x) - \ln(x+1)$

$\therefore \dfrac{dy}{dx} = \dfrac{-2}{-2x} - \dfrac{1}{x+1}$

$= \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{x+1-x}{x(x+1)}$

$= \dfrac{1}{x(x+1)} \neq 0$ for any x

∴ there are no stationary points

(iii) Since $-1<x<0$, then $x(x+1)<0$

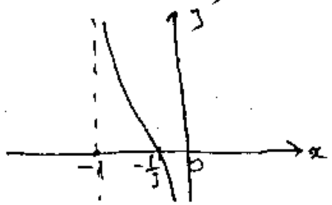
∴ From (ii) $\dfrac{dy}{dx}<0$ for $-1<x<0$

i.e. curve is decreasing for $-1<x<0$

When $y=0$, $\dfrac{-2x}{x+1} = 1$

$\therefore -2x = x+1$

$\Rightarrow x$ intercept is $-\frac{1}{3}$



(iv) since curve is decreasing, the inverse function

is $x = \ln\left(\frac{-2y}{y+1}\right)$

$$\therefore \quad \frac{-2y}{y+1} = e^{x}$$

$$-2y = e^{x}y + e^{x}$$

$$\text{or} \quad y(e^{x}+2) = -e^{x}$$

ie inverse function is $y = -\frac{e^{x}}{e^{x}+2}$

(v) $A = \left| \int_0^2 x \, dy \right| = \int_0^2 \frac{e^{y}}{e^{y}+2} \, dy$, from (iv)

$$= \Big[\ln(e^{y}+2)\Big]_0^2$$

$$= \ln(e^{2}+2) - \ln 3 \quad u^2$$