THE KING'S SCHOOL

2008
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

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Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \tan^2 x \, dx$ **2**

(b) Find $\int \frac{x}{x+1} \, dx$ **2**

(c) (i) Let $F(x)$ be a primitive function of $f(x)$. Hence, or otherwise, show that

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

(ii) Show that $\int_0^\pi x \sin x \, dx = \frac{\pi}{2}\int_0^\pi \sin x \, dx$ **2**

(iii) Use integration by parts to evaluate $\int_0^\pi x^2 \cos x \, dx$ **3**

(d) (i) Find $\int \frac{dt}{(2t+1)^2+1}$ **1**

(ii) Use the substitution $t = \tan\frac{\theta}{2}$ to show that

$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta - \cos\theta + 3} = \tan^{-1}\left(\frac{1}{2}\right)$ **4**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find $|\sqrt{7} + \sqrt{33}\, i\,|$ **1**

(ii) $x + iy = \dfrac{\sqrt{7} + \sqrt{33}\, i}{3 - i}$

Find the value of $x^2 + y^2$ **2**

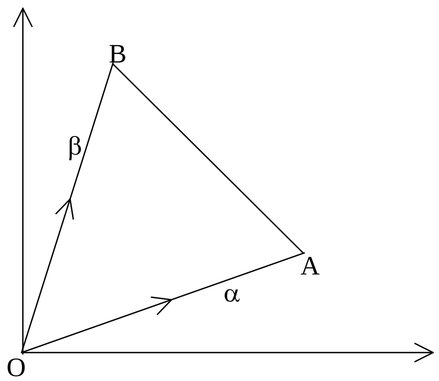
(b) Precisely show on the Argand diagram the locus of the complex numbers z such that $|z - i| = 1$ and $|z| \leq 1$ hold simultaneously. **3**

(c) Let $z = 1 - \cos 2\theta + i\sin 2\theta,\ 0 < \theta < \frac{\pi}{2}$

(i) Show that $z = 2\sin\theta\,(\sin\theta + i\cos\theta)$ **2**

(ii) Hence find $|z|$ and $\arg z$ **2**

(d)



The diagram shows the equilateral triangle OAB in the complex plane.

O is the origin and points A, B represent the complex numbers $\alpha,\ \beta$, respectively.

Let $v = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

(i) Write down the complex number $\overrightarrow{BA}$ **1**

(ii) Show that $\alpha = v(\alpha - \beta)$ **2**

(iii) Prove that $\alpha^2 + \beta^2 = \alpha\beta$ **2**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the hyperbola $(y+1)^2 - x^2 = 1$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{1}{(y+1)^3}$ **4**

(b) Let $P(x) = x^4 - 2Ax^3 + B$, where $A \neq 0$

$P(x) = 0$ has the roots α, β, γ and $\alpha + \beta + \gamma$

(i) Deduce that $B = A^4$ **3**

(ii) Find, in simplest form, $\alpha^2 + \beta^2 + \gamma^2$ **3**

(c) Let $(1 + x)^{2008} = u_1 + u_2 + \ldots + u_k + u_{k+1} + \ldots + u_{2009}$, $x > 0$

(i) Show that $\dfrac{u_{k+1}}{u_k} = \dfrac{2009 - k}{k} . \ x$ **2**

(ii) The middle term in the expansion of $(1 + x)^{2008}$ is the greatest term.

Deduce that $\dfrac{1004}{1005} < x < \dfrac{1005}{1004}$ **3**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ showing its foci, directrices and asymptotes. **4**

(ii) A particular solid has as its base the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and the line $x = 4$.

Cross-sections perpendicular to this base and the *x* axis are equilateral triangles.

Find the volume of this solid. **4**

(b) A particle moves on the *x* axis according to the acceleration equation of motion $\ddot{x} = x$. Initially the particle is at the origin with velocity $v = 2$.

(i) Explain why the velocity will always increase. **1**

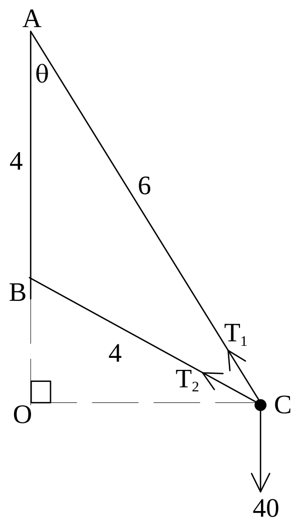
(ii) By integration, prove that $v = \sqrt{x^2 + 4}$ **2**

(iii) By using the table of standard integrals, or otherwise, find the displacement *x* as a function of time *t*. **4**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



Two pieces of light inextensible string AC of length 6 metres and BC of length 4 metres are attached at two points A and B, respectively. B is 4 metres vertically below A.

At C a mass of 4 kg is attached to the strings and this mass rotates in uniform circular motion of 3 rad/*s* about a point O which is vertically below B. Take 10 m/s^2 as the acceleration due to gravity.

Let the tensions in the strings AC and BC be T_1 Newtons and T_2 Newtons, respectively, and let $\angle$ BAC = θ

(i) Show that $\cos\theta = \dfrac{3}{4}$ **1**

(ii) Be resolving forces at C in the vertical direction, show that $6T_1 + T_2 = 320$ **3**

(iii) Find the tensions in the strings. **3**

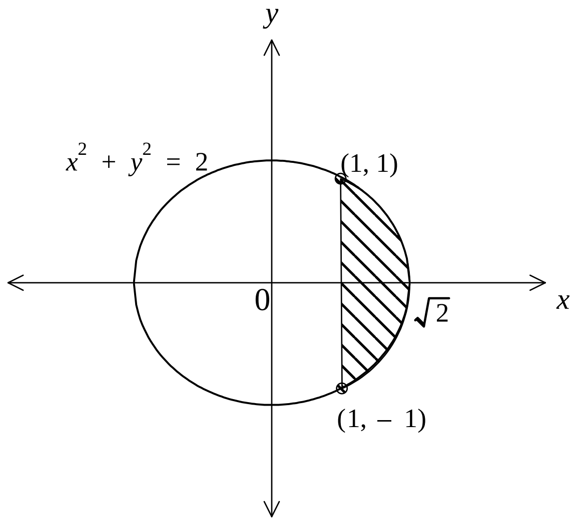
Question 5 continues on the next page

(b) (i) Show that $\int_1^{\sqrt{2}} x\sqrt{2 - x^2}\; dx = \frac{1}{3}$ **2**

(ii) By considering the circle $x^2 + y^2 = 2$, or otherwise, show that

$\int_1^{\sqrt{2}} \sqrt{2 - x^2}\; dx = \frac{\pi}{4} - \frac{1}{2}$ **2**

(iii)



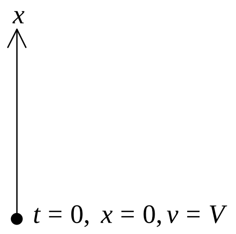
The minor segment of the circle $x^2 + y^2 = 2$ bounded by the chord $x = 1$ is revolved about that chord.

Use the method of cylindrical shells to find the volume of the solid generated. **4**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A particle of mass m is projected vertically upwards with speed V in a medium where there is a resistance mgk^2v^2 when v is its speed. g is the acceleration due to gravity and k is a positive constant.

Take $x = 0$ and $v = V$ when $t = 0$

The particle reaches a maximum height X when the time is T.

(i) Show that the equation of motion is given by $\ddot{x} = -g(1 + k^2v^2)$ **1**

(ii) Show that $X = \dfrac{1}{2gk^2}\ln(1 + k^2V^2)$ **4**

(iii) Show that $T = \dfrac{1}{gk}\tan^{-1}(kV)$ **3**

(iv) If the only force acting on the particle is due to gravity the equations of motion are:

$$\ddot{x} = -g$$

$$\dot{x} = -gt + V$$

$$x = -\frac{gt^2}{2} + Vt$$

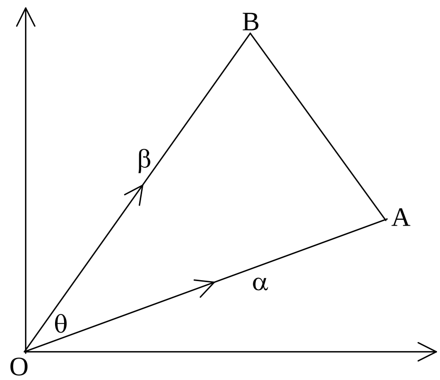
[DO NOT SHOW THESE]

Deduce that $\displaystyle\lim_{k \to 0} \frac{\ln(1 + k^2V^2)}{k^2} = V^2$ **2**

Question 6 continues next page

(b) (i) Use the results $z + \bar{z} = 2Re(z)$ and $|z|^2 = z\bar{z}$ for complex numbers z to show that $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2Re\left(\alpha\bar{\beta}\right)$ **3**

(ii)



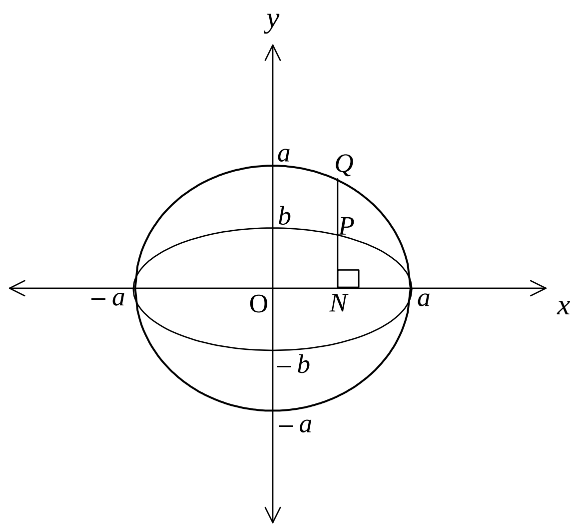
The diagram shows the angle θ between the complex numbers α and β.

Prove that $|\alpha|\ |\beta| \cos\theta = Re\left(\alpha\bar{\beta}\right)$ **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the circle $x^2 + y^2 = a^2$ and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, \quad a > b > 0$

$P(a\cos\theta,\ b\sin\theta), \quad \theta \neq -\dfrac{\pi}{2}, \dfrac{\pi}{2}$, is a point on the ellipse. PN is perpendicular to the x axis at N and meets the circle at Q in the same quadrant.

O is the origin.

(i) Write down the coordinates of Q. **1**

(ii) Show that the equation of the tangent at $P(a\cos\theta,\ b\sin\theta)$ on the ellipse is $\dfrac{\cos\theta}{a}x + \dfrac{\sin\theta}{b}y = 1$ **2**

(iii) Hence, or otherwise, find the equation of the tangent at the point Q on the circle. **1**

(iv) The tangents at P and Q meet at T. Prove that $ON.OT = a^2$. **2**

Question 7 continues on the next page

(b) Let $u_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta, \quad n = 0, 1, 2, \ldots$

(i) Explain why $u_n < u_{n-1} < u_{n-2}$ **1**

(ii) Prove that $u_n = \dfrac{n-1}{n} u_{n-2}, \quad n = 2, 3, 4, \ldots$ **3**

(iii) Deduce that $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} u_{n-1}$ **1**

(iv) Use (ii) to show that $n \, u_n \, u_{n-1} = \dfrac{\pi}{2}, \quad n = 1, 2, 3, \ldots$ **2**

(v) Given that $\int_0^{\frac{\pi}{2}} \cos^{11} \theta \, d\theta = \dfrac{256}{693}$,

evaluate $\int_0^{\frac{\pi}{2}} \cos^{10} \theta \, d\theta$ **1**

(vi) Find an approximate value of $\int_0^{\frac{\pi}{2}} \cos^{2008} \theta \, d\theta$ **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $\dfrac{(x+a)(x+b)(x+c)}{(x-a)(x-b)(x-c)} \equiv k + \dfrac{p}{x-a} + \dfrac{q}{x-b} + \dfrac{t}{x-c}$

(i) Explain why $k = 1$ **1**

(ii) Show that $p = \dfrac{2a(a+b)(a+c)}{(a-b)(a-c)}$ and write down expressions for q and t. **3**

(iii) Hence, or otherwise, prove that

$$\frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$ **2**

Question 8 continues on the next page

(b) The roots of $x^4 + x^3 + 2x^2 + 3x + 1 = 0$ are $\alpha,\ \beta,\ \gamma,\ \delta$

(i) Find a polynomial with the roots $\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma},\ \frac{1}{\delta}$. **2**

(ii) Hence, or otherwise, show that

$$\left(\alpha + \frac{1}{\alpha}\right) + \left(\beta + \frac{1}{\beta}\right) + \left(\gamma + \frac{1}{\gamma}\right) + \left(\delta + \frac{1}{\delta}\right) = -4$$ **1**

(iii) Explain why the equation $x^4 + 4x^3 + Ax^2 + Bx + C = 0$ for some $A,\ B,\ C$ has the roots $\alpha + \frac{1}{\alpha},\ \beta + \frac{1}{\beta},\ \gamma + \frac{1}{\gamma},\ \delta + \frac{1}{\delta}$ **1**

(iv) Hence state the eight roots of the equation

$$\left(x + \frac{1}{x}\right)^4 + 4\left(x + \frac{1}{x}\right)^3 + A\left(x + \frac{1}{x}\right)^2 + B\left(x + \frac{1}{x}\right) + C = 0$$ **1**

(v) Use the equation in (iii) to state the four roots of the equation $Cx^4 + Bx^3 + Ax^2 + 4x + 1 = 0$ **1**

(vi) By multiplying both sides by x^4, the equation in (iv) could be expressed as $(x^2 + 1)^4 + 4x\,(x^2 + 1)^3 + Ax^2(x^2 + 1)^2 + Bx^3(x^2 + 1) + Cx^4 = 0$

[DO NOT SHOW THIS]

Hence, by using the polynomial found in (i) and another suitable equation, prove that $B = 0$. **2**

(vii) Evaluate $\left(\alpha + \frac{1}{\alpha}\right)^{-1} + \left(\beta + \frac{1}{\beta}\right)^{-1} + \left(\gamma + \frac{1}{\gamma}\right)^{-1} + \left(\delta + \frac{1}{\delta}\right)^{-1}$ **1**

End of Examination Paper

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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Question (Marks)	Complex Numbers	Functions	Integration	Conics	Mechanics
1 (15)			15		
2 (15)	15				
3 (15)		15			
4 (15)			(a)(ii) 4	(a)(i) 4	(b) 7
5 (15)			(b) 8		(a) 7
6 (15)	(b) 5				(a) 10
7 (15)			(b) 9	(a) 6	
8 (15)		15			
Total (120)	20	30	36	10	24

TKS EXTENSION 2 SOLUTIONS TRIAL 2008

Question 1

(a) $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x$

(b) $\int \frac{x}{x+1} \, dx = \int \frac{x+1-1}{x+1} \, dx = \int 1 - \frac{1}{1+x} \, dx = x - \ln(1+x)$

(c) (i) $\int_0^a f(a-x) \, dx = \Big[-F(a-x)\Big]_0^a = -F(0) + F(a)$

$= \int_0^a f(x) \, dx$

(ii) $\int_0^\pi x \sin x \, dx = \int_0^\pi (\pi - x) \sin(\pi - x) \, dx$

$= \int_0^\pi (\pi - x) \sin x \, dx$

$= \pi \int_0^\pi \sin x \, dx - \int_0^\pi x \sin x \, dx$

$\therefore 2\int_0^\pi x \sin x \, dx = \pi \int_0^\pi \sin x \, dx \Rightarrow \int_0^\pi x \sin x \, dx = \frac{\pi}{2} \int_0^\pi \sin x \, dx$

(iii) $\int_0^\pi x^2 \cos x \, dx = \int_0^\pi x^2 \frac{d \sin x}{dx} \, dx$

$= \Big[x^2 \sin x\Big]_0^\pi - \int_0^\pi 2x \sin x \, dx$

$= 0 - \pi \int_0^\pi \sin x \, dx$ from (ii)

$= \pi \Big[\cos x\Big]_0^\pi = \pi(-1-1) = -2\pi$

(d) (i) $\frac{1}{2}\tan^{-1}(2t+1)$

(ii) $t = \tan\frac{\theta}{2}$ $\quad : \theta = 0,\ t = 0$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(1+t^2)$ $\quad \theta = \frac{\pi}{2},\ t = 1$

$$\therefore I = \int_0^1 \frac{2\,dt}{(1+t^2)\left(\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 3\right)}$$

$$= \int_0^1 \frac{2\,dt}{4t - 1 + t^2 + 3 + 3t^2}$$

$$= \int_0^1 \frac{2\,dt}{4t^2 + 4t + 2}$$

$$= \int_0^1 \frac{2\,dt}{(2t+1)^2 + 1}$$

$$= \left[\tan^{-1}(2t+1)\right]_0^1$$

$$= \tan^{-1}3 - \tan^{-1}1$$

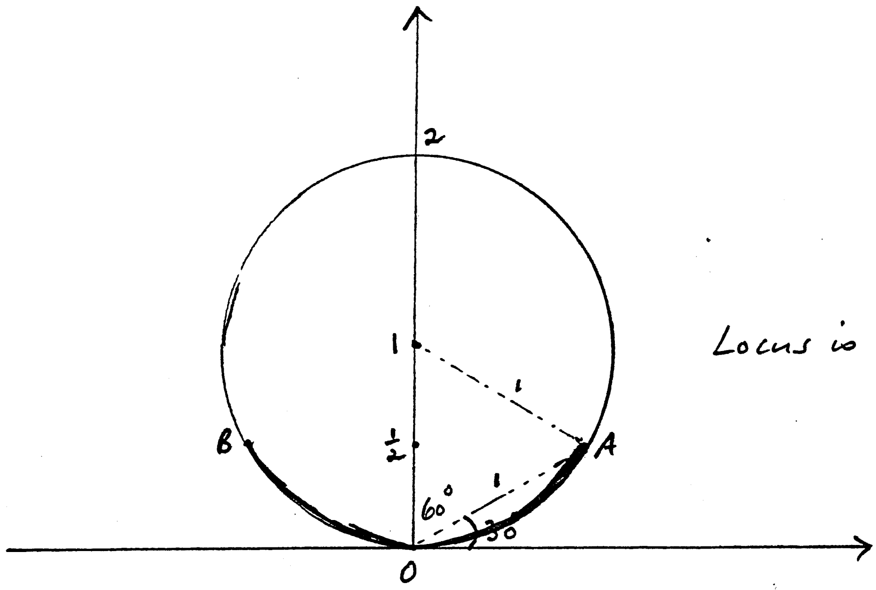
$$= \tan^{-1}\left(\frac{3-1}{1+3.1}\right) = \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$$

Question 2

(a) (i) $= \sqrt{7+33} = \sqrt{40}$

(ii) $x^2 + y^2 = |x+iy|^2 = \dfrac{|\sqrt{7} + \sqrt{33}\,i|^2}{|3-i|^2} = \dfrac{40}{10} = 4$

(b)



Locus is arc AB, $y \le \frac{1}{2}$

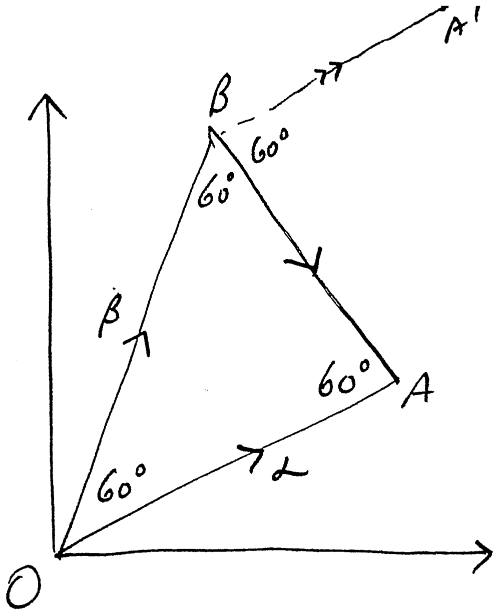
(c) (i) $z = 2\sin^2\theta + i\,(2\sin\theta\cos\theta)$

$= 2\sin\theta\,(\sin\theta + i\cos\theta)$

(ii) $z = 2\sin\theta\left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)$

$\Rightarrow |z| = 2\sin\theta$, $\arg z = \frac{\pi}{2} - \theta$ since $0 < \theta < \frac{\pi}{2}$

(d)



(i) $\overrightarrow{BA} = \alpha - \beta$

(ii) $v\,\overrightarrow{BA} = \overrightarrow{BA'} = \alpha$, see diagram

ie. $\alpha = v(\alpha - \beta)$

(iii) Now, $\beta = v\alpha$

$\therefore \dfrac{\alpha}{\beta} = \dfrac{\alpha - \beta}{\alpha}$

or $\alpha^2 = \alpha\beta - \beta^2$

ie $\alpha^2 + \beta^2 = \alpha\beta$

Question 3

(a) $2(y+1)\,y' - 2x = 0$

$\therefore\ y' = \dfrac{x}{y+1}$

$\therefore\ y'' = \dfrac{y+1 - x y'}{(y+1)^2} = \dfrac{y+1 - \dfrac{x^2}{y+1}}{(y+1)^2}$

$= \dfrac{(y+1)^2 - x^2}{(y+1)^3} = \dfrac{1}{(y+1)^3}$

(b) (i) $\Sigma\alpha = 2(\alpha+\beta+\gamma) = 2A \Rightarrow \alpha+\beta+\gamma = A$

But, $\alpha+\beta+\gamma$ is a root

$\therefore\ A^4 - 2A(A^3) + B = 0 \Rightarrow B = A^4$

(ii) $\alpha^2+\beta^2+\gamma^2 + (\alpha+\beta+\gamma)^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$\Rightarrow \alpha^2+\beta^2+\gamma^2 + A^2 = (2A)^2 - 2(0) = 4A^2$

$\therefore\ \alpha^2+\beta^2+\gamma^2 = 3A^2$

(c) (i) $\dfrac{u_{k+1}}{u_k} = \dfrac{\binom{2008}{k}x^k}{\binom{2008}{k-1}x^{k-1}} = \dfrac{2008!\,(2009-k)!\,(k-1)!}{(2008-k)!\,k!\,2008!}\cdot x$

$= \dfrac{2009-k}{k}\cdot x$

(ii) The middle term is u_{1005}

$\therefore u_{1005} > u_{1004}$

or $\dfrac{u_{1005}}{u_{1004}} > 1 \Rightarrow \dfrac{2009-1004}{1004} x > 1$ from (i)

i.e. $x > \dfrac{1004}{1005}$

Also $u_{1005} > u_{1006}$

$\Rightarrow \dfrac{2009-1005}{1005} x < 1$ from (i)

i.e. $x < \dfrac{1005}{1004}$

i.e. $\dfrac{1004}{1005} < x < \dfrac{1005}{1004}$

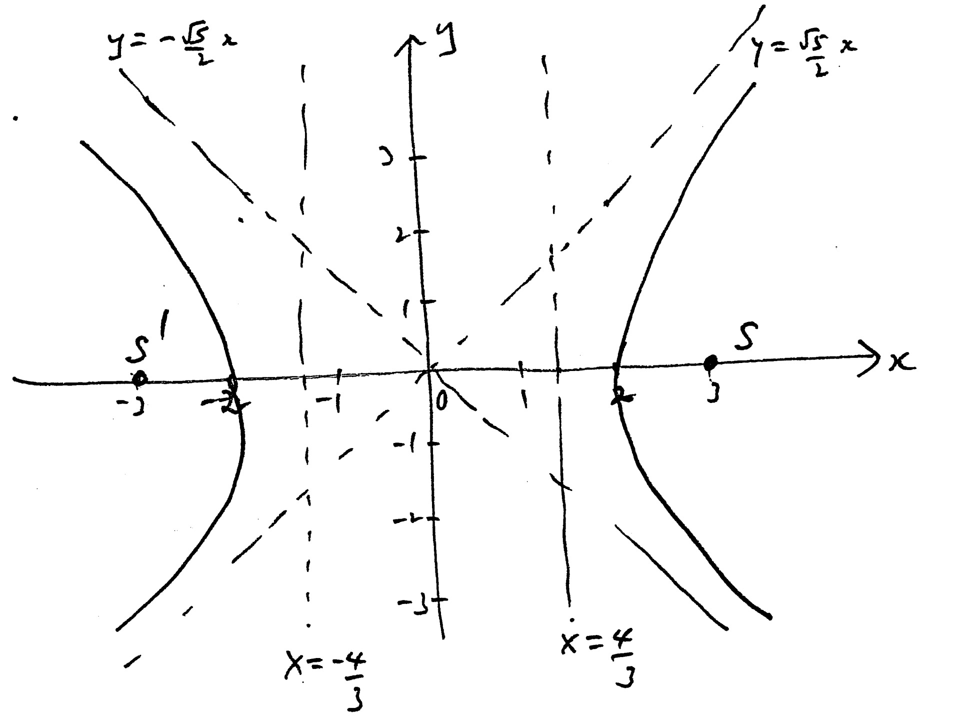
Question 4

(a) (i) $c^2 = a^2 + b^2 \Rightarrow c^2 = 4+5 = 9, \quad c = 3$

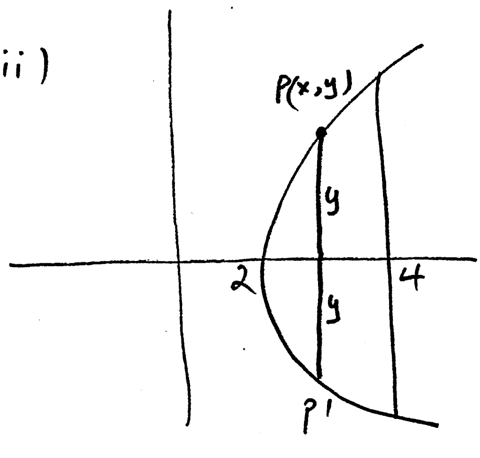
$e = \frac{3}{2}$

$\therefore$ foci $(\pm 3, 0)$, directrices $x = \pm \frac{4}{3}$,

asymptotes $\frac{x}{2} \pm \frac{y}{\sqrt{5}} = 0$ ie $y = \pm \frac{\sqrt{5}}{2} x$



(ii)



Take $P(x,y)$ on curve, $y \geq 0$

Then, area Δ with base PP'

$= \frac{1}{2}(2y)^2 \sin\frac{\pi}{3} = \sqrt{3}\,y^2$

$\therefore V = \int_2^4 \sqrt{3} y^2 \, dx$

$= \sqrt{3}.5 \int_2^4 \frac{x^2}{4} - 1 \, dx$

$= 5\sqrt{3} \left[\frac{x^3}{12} - x\right]_2^4$

$= 5\sqrt{3}\left(\frac{16}{3} - 4 - \frac{2}{3} + 2\right)$

$= \frac{40\sqrt{3}}{3}$

(b) (i) Initially $x=0$ and $v>0$

$\Rightarrow$ after $t=0$ then $x>0$ ie. $\ddot{x}>0$

$\Rightarrow v$ will increase for all t

(ii) $\dfrac{d(\frac{1}{2}v^2)}{dx} = x$

$\therefore \left[\frac{1}{2}v^2\right]_2^v = \left[\frac{x^2}{2}\right]_0^x$

ie $\frac{1}{2}v^2 - \frac{2^2}{2} = \frac{x^2}{2}$

or $v^2 = x^2 + 4$

$\therefore v = \sqrt{x^2+4}$ since $v>0$

(iii) $\dfrac{dx}{dt} = \sqrt{x^2+4}$

$\therefore \dfrac{dt}{dx} = \dfrac{1}{\sqrt{x^2+4}}$

$t = \displaystyle\int_0^x \frac{1}{\sqrt{x^2+4}}\,dx = \left[\ln(x+\sqrt{x^2+4})\right]_0^x$

ie. $t = \ln\left(\dfrac{x+\sqrt{x^2+4}}{2}\right)$

$\therefore \dfrac{x+\sqrt{x^2+4}}{2} = e^t$

or $\sqrt{x^2+4} = 2e^t - x$

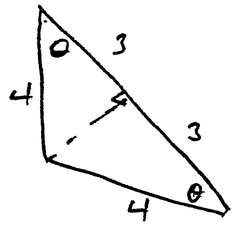
$\therefore x^2+4 = 4e^{2t} - 4e^t x + x^2$

$\Rightarrow 1 = e^{2t} - e^t x$

or $x = \dfrac{e^{2t}-1}{e^t} = e^t - e^{-t}$

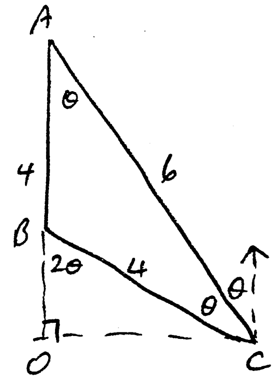
Question 5

(a) (i)



$\Rightarrow \cos\theta = \frac{3}{4}$

(ii) $T_1 \cos\theta + T_2 \cos 2\theta = 40$



$\therefore T_1 \cos\theta + T_2 (2\cos^2\theta - 1) = 40$

$\Rightarrow \frac{3}{4} T_1 + T_2 \left(\frac{9}{8} - 1 \right) = 40$

i.e. $6 T_1 + T_2 = 320$

(iii) Resolving in the direction CO,

$4 \cdot 3^2 \cdot OC = T_1 \cos(\frac{\pi}{2} - \theta) + T_2 \cos(\frac{\pi}{2} - 2\theta)$,

where $OC = 4 \sin 2\theta$

$\therefore 16 \cdot 9 \cdot \sin 2\theta = T_1 \sin\theta + T_2 \sin 2\theta$

$\Rightarrow 16 \cdot 9 \cdot 2\cos\theta = T_1 + T_2 \, 2\cos\theta$

or $216 = T_1 + T_2 \cdot \frac{3}{2}$

$\therefore$ From (ii), $6\left(216 - \frac{3T_2}{2}\right) + T_2 = 320$

$1296 - 9T_2 + T_2 = 320$

or $T_2 = \dfrac{1296 - 320}{8} = 122 \text{ N}$

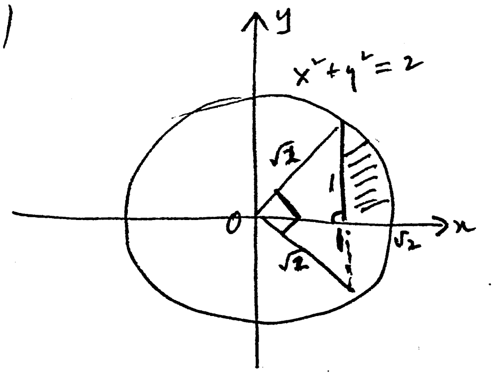
& $T_1 = 216 - \frac{3}{2} \times 122 \text{ N} = 33 \text{ N}$

(b) (i) put $u = 2 - x^2$: $x = 1,\ u = 1$

$\frac{du}{dx} = -2x$ $\quad x = \sqrt{2},\ u = 0$

$$\therefore\ I = \frac{1}{2}\int_0^1 \sqrt{u}\ du = \frac{1}{3}\left[u^{3/2}\right]_0^1 = \frac{1}{3}$$

(ii)



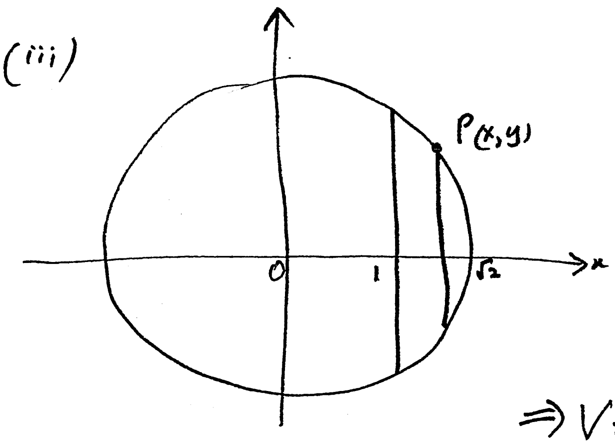
$\therefore$ Shaded area

$$= \int_1^{\sqrt{2}} \sqrt{2-x^2}\ dx$$

$$= \frac{1}{2}\left[\frac{\pi(\sqrt{2})^2}{4} - \frac{1}{2}(\sqrt{2})^2\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

(iii)



Take $P(x,y)$, $y \geq 0$, on circle

Then, $\delta V \simeq \pi\left((x+\delta x-1)^2 - (x-1)^2\right)2y$

$$\approx 2\pi y\ 2(x-1)\delta x$$

$$\Rightarrow V = 4\pi\int_1^{\sqrt{2}} (x-1)\,y\ dx$$

i.e. $V = 4\pi\int_1^{\sqrt{2}} (x-1)\sqrt{2-x^2}\ dx$

$$= 4\pi\int_1^{\sqrt{2}} x\sqrt{2-x^2} - \sqrt{2-x^2}\ dx$$

$$= 4\pi\left(\frac{1}{3} - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right) \quad \text{from (i) + (ii)}$$

$$= 4\pi\left(\frac{5}{6} - \frac{\pi}{4}\right) = \frac{\pi}{3}(10 - 3\pi)$$

Question 6

(a) (i) $m\ddot{x} = -mg - mgk^2v^2$

$$\Rightarrow \ddot{x} = -g(1 + k^2v^2)$$

(ii) $\ddot{x} = v\dfrac{dv}{dx} = -g(1+k^2v^2)$

$$\Rightarrow \frac{dv}{dx} = -g\left(\frac{1+k^2v^2}{v}\right)$$

$$\therefore -g\frac{dx}{dv} = \frac{v}{1+k^2v^2}$$

$$\therefore -g\,[x]_0^X = \int_V^0 \frac{v}{1+k^2v^2}\,dv$$

$$\text{i.e. } -gX = \frac{1}{2k^2}\left[\ln(1+k^2v^2)\right]_V^0$$

$$= \frac{1}{2k^2}\left(0 - \ln(1+k^2V^2)\right)$$

$$\therefore X = \frac{1}{2gk^2}\ln(1+k^2V^2)$$

(iii) $\ddot{x} = \dfrac{dv}{dt} = -g(1+k^2v^2)$

$$\therefore -g\frac{dt}{dv} = \frac{1}{1+k^2v^2}$$

$$\Rightarrow -g\,[t]_0^T = \frac{1}{k}\left[\tan^{-1}kv\right]_V^0$$

$$\text{ie } -gT = \frac{1}{k}\left(0 - \tan^{-1}kV\right)$$

$$\therefore T = \frac{1}{gk}\tan^{-1}(kV)$$

(iv) $\dot{x} = 0 \Rightarrow t = \frac{V}{g}$

$$\therefore x_{max} = -\frac{g}{2} \cdot \frac{V^2}{g^2} + V \cdot \frac{V}{g} = \frac{V^2}{2g}$$

$$\therefore \text{from (ii), } \lim_{k \to 0} \frac{1}{2gk^2} \ln(1+k^2V^2) = \frac{V^2}{2g}$$

$$\text{ie } \lim_{k \to 0} \frac{\ln(1+k^2V^2)}{k^2} = V^2$$

(b) (i) $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\overline{\alpha-\beta})$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\bar{\alpha}-\bar{\beta})$$

$$= \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha\bar{\alpha} - \alpha\bar{\beta} - \bar{\alpha}\beta + \beta\bar{\beta})$$

$$= \alpha\bar{\beta} + \bar{\alpha}\beta$$

$$= \alpha\bar{\beta} + \overline{(\alpha\bar{\beta})} = 2\,\mathrm{Re}(\alpha\bar{\beta})$$

(ii) $\cos\theta = \dfrac{|\alpha|^2 + |\beta|^2 - |\alpha-\beta|^2}{2|\alpha||\beta|}$ since $\overrightarrow{BA} = \alpha - \beta$

$$= \frac{2\,\mathrm{Re}(\alpha\bar{\beta})}{2|\alpha||\beta|}$$

$$\Rightarrow |\alpha||\beta|\cos\theta = \mathrm{Re}(\alpha\bar{\beta})$$

Question 7

(a) (i) $Q = (a\cos\theta, a\sin\theta)$

(ii) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

$$= -\frac{b^2}{a^2} \cdot \frac{a\cos\theta}{b\sin\theta} \quad \text{at } P$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ tangent at P is $y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

or $\frac{\sin\theta}{b} y - \sin^2\theta = -\frac{\cos\theta}{a} x + \cos^2\theta$

ie. $\frac{\cos\theta}{a} x + \frac{\sin\theta}{b} y = \cos^2\theta + \sin^2\theta = 1$

(iii) (ii) $\Rightarrow$ tangent at Q is $\frac{\cos\theta}{a} x + \frac{\sin\theta}{a} y = 1$ [ie $a = b$]

(iv) at T, $\left(\frac{\sin\theta}{b} - \frac{\sin\theta}{a}\right) y = 0 \Rightarrow y = 0$

$$\therefore x = \frac{a}{\cos\theta} = a\sec\theta$$

ie we have

O $N(a\cos\theta, 0)$ $T(a\sec\theta, 0)$

$\therefore ON \cdot OT = |a\cos\theta|\,|a\sec\theta| = a^2$

(b) (i) For $0 < x < \frac{\pi}{2}$, $0 < \cos x < 1$

$\therefore \cos^n x < \cos^{n-1} x < \cos^{n-2} x$

$\Rightarrow u_n < u_{n-1} < u_{n-2}$

(ii) $u_n = \int_0^{\frac{\pi}{2}} \cos\theta \cos^{n-1}\theta \, d\theta$

$= \int_0^{\pi/2} \cos^{n-1}\theta \frac{d\sin\theta}{d\theta} \, d\theta$ where $u = \cos^{n-1}\theta$

$\frac{du}{d\theta} = -(n-1)\cos^{n-2}\theta \sin\theta$

$= \left[\cos^{n-1}\theta \sin\theta\right]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\cos^{n-2}\theta \sin^2\theta \, d\theta$

$= 0 + (n-1)\int_0^{\pi/2} \cos^{n-2}\theta (1-\cos^2\theta) \, d\theta$

$= (n-1)(u_{n-2} - u_n)$

$\therefore u_n(1 + n - 1) = (n-1)u_{n-2}$ ie $u_n = \frac{n-1}{n} u_{n-2}$

(iii) $u_n = \left(1 - \frac{1}{n}\right) u_{n-2}$

$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n-2} \Rightarrow$ from (i),

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n-1}$

(iv) From (ii), $n u_n u_{n-1} = (n-1) u_{n-1} u_{n-2}$

$= (n-2) u_{n-2} u_{n-3}$

$= \cdots$

$= 1 \cdot u_1 u_0$

$= \int_0^{\pi/2} \cos\theta \, d\theta \int_0^{\pi/2} 1 \, d\theta$

$= \left[\sin\theta\right]_0^{\pi/2} \left[\theta\right]_0^{\pi/2}$

$= 1 \cdot \frac{\pi}{2} = \frac{\pi}{2}$

(v) From (iv), $11 u_{11} u_{10} = \frac{\pi}{2}$

$$\therefore \quad u_{10} = \frac{\pi}{22} \cdot \frac{693}{256} = \frac{63\pi}{512}$$

(vi) From (iii) and (iv), for large n,

$$n u_n u_{n-1} \approx n u_n^2$$

$$\therefore \quad 2008\, u_{2008}^2 \approx \frac{\pi}{2}$$

$$\therefore \quad u_{2008} \approx \sqrt{\frac{\pi}{4016}} \quad [\approx 0.028]$$

or, of course, $2009\, u_{2008}^2 \approx \frac{\pi}{2}$

$$\Rightarrow u_{2008} \approx \sqrt{\frac{\pi}{4018}}$$

Indeed, $\sqrt{\frac{\pi}{4018}} < u_{2008} < \sqrt{\frac{\pi}{4016}}$

Question 8

(a) (i) $(x+a)(x+b)(x+c)$ and $(x-a)(x-b)(x-c)$ both have the same leading term x^3 $\Rightarrow$ $k=1$

(ii) $$\frac{(x+a)(x+b)(x+c)}{(x-b)(x-c)} = x-a + p + \frac{q(x-a)}{x-b} + \frac{t(x-a)}{x-c}$$

For $x=a$, $$\frac{2a(a+b)(a+c)}{(a-b)(a-c)} = p$$

$$\therefore\ q = \frac{2b(b+a)(b+c)}{(b-a)(b-c)}$$

$$t = \frac{2c(c+a)(c+b)}{(c-a)(c-b)}$$, on symmetry.

(iii) Put $x=-a$, then,

$$0 = 1 - \frac{p}{2a} - \frac{q}{a+b} - \frac{t}{a+c}$$

$$\Rightarrow\ 0 = 1 - \frac{(a+b)(a+c)}{(a-b)(a-c)} - \frac{2b(b+c)}{(b-a)(b-c)} - \frac{2c(c+b)}{(c-a)(c-b)}$$

$$\therefore\ \frac{(a+b)(a+c)}{(a-b)(a-c)} + \frac{2b(b+c)}{(b-a)(b-c)} + \frac{2c(c+b)}{(c-a)(c-b)} = 1$$

(b) (i) $\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 + 3\left(\frac{1}{x}\right) + 1 = 0$

i.e. $x^4 + 3x^3 + 2x^2 + x + 1 = 0$

(ii) $\Sigma \alpha = -1$, $\Sigma \frac{1}{\alpha} = -3$

$\therefore \Sigma\left(\alpha + \frac{1}{\alpha}\right) = -4$

(iii) The sum of the roots of the quartic is -4

$\therefore$ result from (ii)

(iv) $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \gamma, \frac{1}{\gamma}, \delta, \frac{1}{\delta}$

(v) $\left(\alpha + \frac{1}{\alpha}\right)^{-1}, \left(\beta + \frac{1}{\beta}\right)^{-1}, \left(\gamma + \frac{1}{\gamma}\right)^{-1}, \left(\delta + \frac{1}{\delta}\right)^{-1}$

(vi) $(x^4 + x^3 + 2x^2 + 3x + 1)(x^4 + 3x^3 + 2x^2 + x + 1) = 0$ has the roots $\alpha, \frac{1}{\alpha}, \ldots, \delta, \frac{1}{\delta}$

$\equiv (x^2+1)^4 + 4x(x^2+1)^3 + Ax^2(x^2+1)^2 + Bx^3(x^2+1) + Cx^4$,

from (iv)

$\therefore$ Equating coefficients of x^5,

$1 + 2 + 6 + 3 = 12 + B$

$\therefore B = 0$

(vii) From (v), $\Sigma\left(\alpha + \frac{1}{\alpha}\right)^{-1} = -\frac{B}{C} = 0$ from (vi)

* Note $C \neq 0$ since $\alpha + \frac{1}{\alpha} \neq 0$