THE KING'S SCHOOL

2009
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\displaystyle\int \frac{x}{(x+1)^2}\,dx$ **2**

(b) (i) Express $\dfrac{2x+9}{(2x-1)(x+2)}$ in partial fractions. **2**

(ii) Find $\displaystyle\int \frac{2x+9}{(2x-1)(x+2)}\,dx$ **1**

(c) (i) Show that $\cos^3 x \sin^{12} x = \cos x \sin^{12} x - \cos x \sin^{14} x$ **1**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x \sin^{12} x\,dx$ **2**

(d) Evaluate $\displaystyle\int_1^3 \frac{dx}{(x+1)\sqrt{x}}$ by using the substitution $x = u^2$ or otherwise. **3**

(e) Use integration by parts to evaluate $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$ **4**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = \sqrt{2} + \sqrt{2}\, i$

(i) Find $|z|$ and $\arg z$ **2**

(ii) Find z^{12} **2**

(b) Find the square roots of $1 + 2\sqrt{2}\, i$ **3**

(c) (i) On the same Argand diagram carefully sketch the region where

$|z - 1| \le |z - 3|$ and $|z - 2| \le 1$ hold simultaneously. **3**

(ii) Find the greatest possible values for $|z|$ and $\arg z$ in this region. **2**

(d) Let $P(x) = x^4 - 4A^3x + 3$, A real

By considering $P'(x)$, or otherwise, find the values for A for which $P(x) = 0$ has 4 complex roots. **3**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of $x^3 + x + 1 = 0$ are α, β, γ.

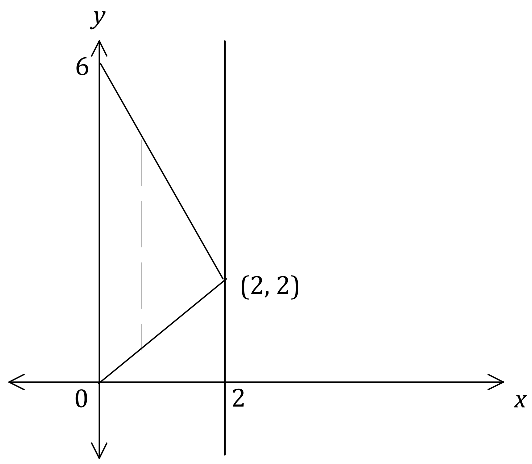
Find a cubic equation whose roots are:

$$\frac{1}{1-\alpha},\ \frac{1}{1-\beta},\ \frac{1}{1-\gamma}$$

Express your answer in the form $ax^3 + bx^2 + cx + d = 0$ **4**

(b) The triangular region bounded by the lines $y = x$, $y = 6 - 2x$ and the y axis is revolved about the line $x = 2$.

By considering slices of the region parallel to the line $x = 2$, find the volume of the solid of revolution.



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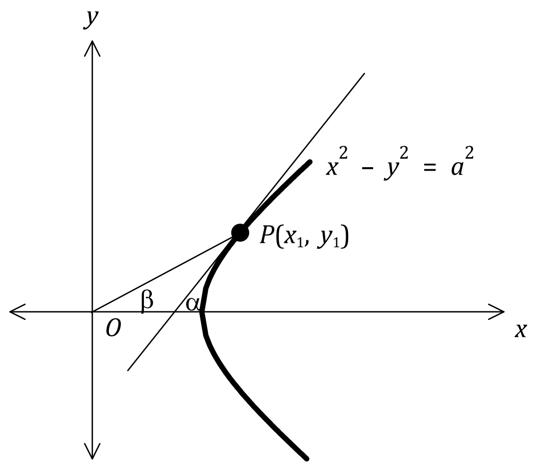
Question 3 continues on the next page

(c)



The tangent at $P(x_1, y_1)$ in the first quadrant on the hyperbola $x^2 - y^2 = a^2$ meets the x axis at an angle α. The line OP, where O is the origin, meets the x axis at an angle β.

(i) Prove that the product of the gradients of the line OP and the tangent at P is 1. **3**

(ii) Deduce that $\alpha + \beta = \dfrac{\pi}{2}$. **3**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $1 - x + x^2 - x^3 + \dots + x^{2n} = \dfrac{1 + x^{2n+1}}{1 + x}$ **1**

(ii) Let $J = \displaystyle\int_0^1 \frac{x^{2n+1}}{1 + x}\, dx$

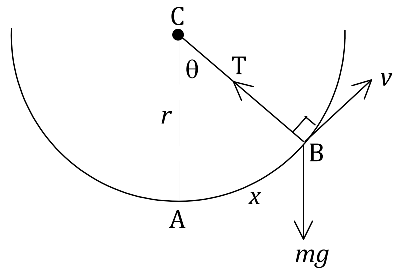
Deduce that $J = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dots + \dfrac{1}{2n+1} - \ln 2$ **2**

(iii) Show that $0 < J < \dfrac{1}{2n+2}$ by considering $\displaystyle\int_0^1 \frac{x^{2n+1}}{1 + x}\, dx$ **2**

(iv) Deduce that $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dots$ **1**

Question 4 continues on the next page

(b)



A simple pendulum consists of a small bob of mass m which is suspended from a fixed point C by a light inextensible string of length r.

The bob is initially at A, vertically below C. Then the bob is displaced through some angle and released from rest.

Suppose at time t the bob is at position B on the circle, as in the diagram.

Let $\angle ACB = \theta$, the arc length AB be x and the linear velocity be $v = \dfrac{dx}{dt}$

Let T be the tension in the string at time t.

(i) Show that $v = r\dfrac{d\theta}{dt}$ **2**

(ii) By resolving the forces at B in the tangential direction, show that $\dfrac{dv}{dt} = -g\sin\theta$ **2**

(iii) Deduce that $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{r}\sin\theta$ **1**

(iv) Suppose the initial angle of release from rest is small.

Deduce that the motion of the bob approximates simple harmonic motion and finds its period. **2**

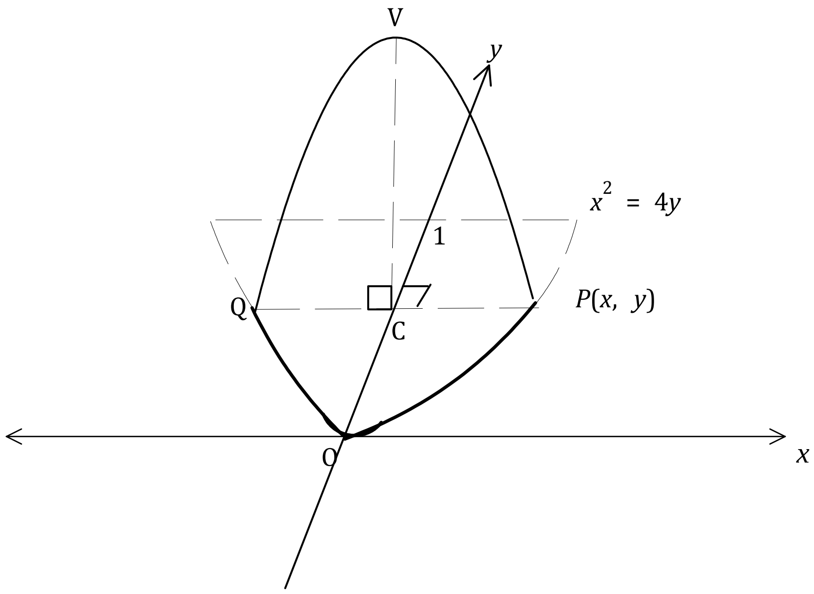
(v) If the initial release angle is small, by resolving forces at B in another suitable direction, show that the tension in the string is approximately

$$T = m\left(g + \frac{v^2}{r}\right)$$ **2**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The base of a solid is the region bounded by the parabola $x^2 = 4y$ and the line $y = 1$.

Cross-sections perpendicular to this base and the *y* axis are parabolic segments with their vertices V directly above the *y* axis. The diagram shows a typical segment PVQ. All the segments have the property that the vertical height VC is three times the base length PQ.

Let $P(x, y)$ where $x \geq 0$ be a point on the parabola $x^2 = 4y$.

(i) Show that the area of the segment PVQ is $8x^2$. **3**

(ii) Find the volume of the solid. **3**

Question 5 continues on the next page

(b) A particle of mass m falls vertically from rest from a point O in a medium whose resistance is mkv, *where* v is its velocity at any time t, and k is a positive constant.

g is the constant acceleration due to gravity.

Let x be the distance travelled from O by the particle.

(i) Show that the equation of motion is given by $\ddot{x} = g - kv$ **1**

(ii) Show that the terminal velocity $V = \dfrac{g}{k}$ **1**

(iii) Use integration to prove that $v = V(1 - e^{-kt})$ **3**

(iv) At the same time as the first particle is released from O another particle of mass m is projected vertically upward from O with initial velocity A.

Prove that when this second particle is momentarily at rest the velocity of the first particle is $\dfrac{\mathrm{AV}}{A + V}$ **4**

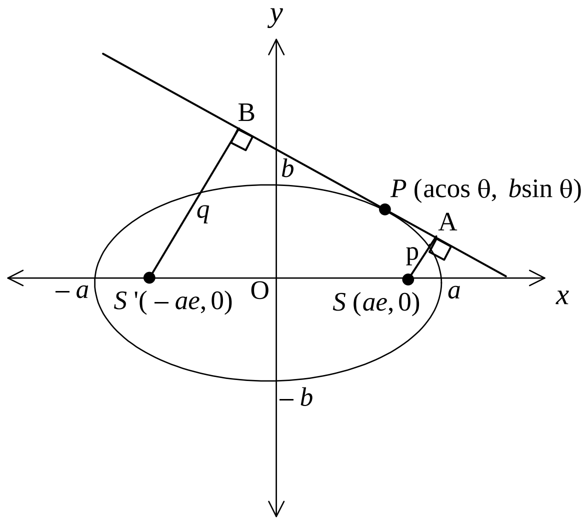
End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Sketch the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ clearly indicating its foci, directrices and asymptotes. Include on your sketch the points where the hyperbola meets the coordinates axes. **4**

(ii) $P(x_1, y_1)$, $x_1 > 0$, is a point on a branch of the hyperbola. Write down the distance from P to the focus of that branch. **1**

(b)



$P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$.

$S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the ellipse where e is the eccentricity.

(i) Prove that the equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is $bx\cos\theta + ay\sin\theta - ab = 0$. **3**

(ii) Perpendiculars of lengths p and q are drawn from the foci S and S' to meet the tangent at P at A and B respectively.

Prove that $pq = b^2$. **3**

Question 6 continues next page

(iii) Verify that $pq = b^2$ if P is the point $(a,\ 0)$. **1**

(iv) For a particular tangent it is found that $p^2 + q^2 = 6(a^2 - b^2)$ also.

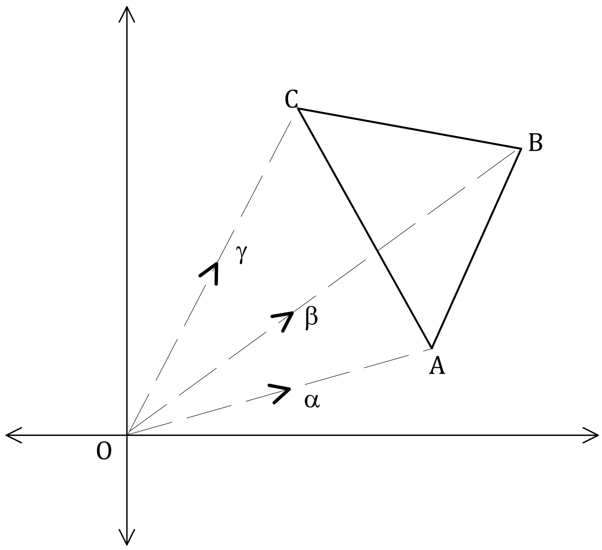
By considering $(p - q)^2$, or otherwise, prove that the ellipse must have an eccentricity $e \geq \frac{1}{2}$. **3**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $w = \cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}$

(i) Show that $w^3 = 1$ and $1 + w + w^2 = 0$ **3**



The points A, B, C in the Argand diagram represent the complex numbers α, β, γ, respectively.

ΔABC is equilateral.

(ii) Show that $\alpha - \gamma = w(\gamma - \beta)$ **2**

(iii) Deduce that $\alpha + w\beta + w^2\gamma = 0$ **1**

(iv) Explain why α, $w\beta$ and $w^2\gamma$ are the roots of a cubic equation $z^3 + pz + q = 0$. **1**

(v) Deduce that $q = -\alpha\beta\gamma$ **1**

(vi) Prove that $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$ **2**

Question 7 continues on the next page

(b) Let $u_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta}{\sin\theta}\, d\theta \quad , \quad n = 1, 2, 3, \ldots$

(i) Use the trigonometric relationship

$\sin 2n\theta - \sin 2(n-1)\theta = 2\cos(2n-1)\theta\sin\theta$ **[DO NOT PROVE THIS]**

to show that $u_n - u_{n-1} = (-1)^{n-1}\dfrac{2}{2n-1}\ , \quad n = 2, 3, 4, \ldots$ **2**

(ii) Deduce that $u_n = 2\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \ldots + \dfrac{(-1)^{n-1}}{2n-1}\right)$ **3**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A recurrence relationship is given by

$$u_{n+1} = \frac{u_n}{2} + \frac{1}{u_n}, \quad n = 1, 2, 3, \ldots \quad \text{where} \quad u_1 = 1$$

(i) Find u_3 **1**

(ii) It can be shown that $u_n = \sqrt{2}\left(\dfrac{1 + A}{1 - A}\right)$

where $A = (-1)^{2^{n-1}}(\sqrt{2}-1)^{2^n}$

[DO NOT PROVE THIS]

Show that $u_{n+1} = \sqrt{2}\left(\dfrac{1 + A^2}{1 - A^2}\right)$ **1**

(iii) Use mathematical induction to prove that

$$u_n = \sqrt{2}\;\frac{(1 + (-1)^{2^{n-1}}(\sqrt{2}-1)^{2^n})}{1 - (-1)^{2^{n-1}}(\sqrt{2}-1)^{2^n}}, \quad n \geq 1$$ **4**

(iv) Find $\lim\limits_{n \to \infty} u_n$ **1**

Question 8 continues on the next page

(b) Let $f(\theta) = \dfrac{14 - 12\sin\theta - 6\cos\theta}{9 - 8\sin\theta - 3\cos\theta}$

(i) Use the subsidiary angle method to show that

$9 - 8\sin\theta - 3\cos\theta > 0$ for all θ **2**

(ii) Alternative expressions for $f(\theta)$ are

$1 + \dfrac{5 - 4\sin\theta - 3\cos\theta}{9 - 8\sin\theta - 3\cos\theta}$ and $2 - \dfrac{4 - 4\sin\theta}{9 - 8\sin\theta - 3\cos\theta}$

[DO NOT VERIFY THESE] **2**

Deduce that $1 \leq f(\theta) \leq 2$ for all θ

(iii) Verify that $f(\theta) = 1$ when $\sin\theta = \dfrac{4}{5}$, $0 < \theta < \dfrac{\pi}{2}$ **1**

(iv) Sketch the graph of $y = f(\theta)$, $-\pi \leq \theta \leq \pi$, clearly indicating the y intercept. **3**

End of Examination Paper

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

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2008
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Mathematics Extension 2

Question (Marks)		Complex Numbers		Functions		Integration		Conics		Mechanics	
1	(15)			(b)(i), (c)(i)	3	(a), (b)(ii), (c)(ii), (d) (e)	12				
2	(15)	(a), (b), (c)	12	(d)	3						
3	(15)			(a), (c)	10	(b)	5				
4	(15)			(a)(i), (iv)	2	(a)(ii), (iii)	4			(b)	9
5	(15)					(a)	6			(b)	9
6	(15)							(a), (b)	15		
7	(15)	(a)	10	(b)(ii)	3	(b)(i)	2				
8	(15)			(a), (b)	15						
Total	(120)		22		36		29		15		18

TKS Extension 2 Solutions 2009

Question 1

(a) $I = \int \frac{x+1 \; -1}{(x+1)^2}\,dx = \int \frac{1}{x+1} - (x+1)^{-2}\,dx$

$= \ln(x+1) + \frac{1}{x+1} \quad (+c)$

(b) (i) Put $\frac{2x+9}{(2x-1)(x+2)} \equiv \frac{A}{2x-1} + \frac{B}{x+2}$

$: A(x+2) + B(2x-1) \equiv 2x+9$

$x = -2 \Rightarrow -5B = 5, \; B = -1 \quad \therefore A - 2 = 2, \; A = 4$

$\therefore \frac{4}{2x-1} - \frac{1}{x+2}$

(ii) From (i), $I = 2\ln(2x-1) - \ln(x+2) \quad (+c)$

(c) (i) $\cos^3 x \sin^{12} x = \cos x(1-\sin^2 x)\sin^{12} x$

$= \cos x \sin^{12} x - \cos x \sin^{14} x$

(ii) From (i), $I = \left[\frac{\sin^{13} x}{13} - \frac{\sin^{15} x}{15}\right]_0^{\pi/2}$

$= \frac{1}{13} - \frac{1}{15} - (0) = \frac{2}{195}$

(d) $x = u^2$ $\qquad x = 1, u = 1$

$\therefore \frac{dx}{du} = 2u$ $\qquad x = 3, u = \sqrt{3}$

$\therefore I = \int_1^{\sqrt{3}} \frac{2u}{(u^2+1)u}\,du = 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{6}$

(e) Put $u = \ln x$, $\frac{dv}{dx} = x^{-2}$

$\therefore \frac{du}{dx} = \frac{1}{x}$, $v = -\frac{1}{x}$

$$\therefore I = \left[-\frac{\ln x}{x}\right]_1^e + \int_1^e \frac{1}{x^2}\,dx$$

$$= -\frac{1}{e} - \left[\frac{1}{x}\right]_1^e = -\frac{1}{e} - \left(\frac{1}{e} - 1\right) = 1 - \frac{2}{e}$$

Question 2

(a) (i) $|z| = \sqrt{2+2} = 2$; $\arg z = \frac{\pi}{4}$

(ii) $z = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\therefore z^{12} = 2^{12}(\cos 3\pi + i\sin 3\pi) = -2^{12}$$

(b) Put $a + ib = \sqrt{1 + 2\sqrt{2}\,i}$

Then $(a+ib)^2 = a^2 - b^2 + 2abi = 1 + 2\sqrt{2}\,i$

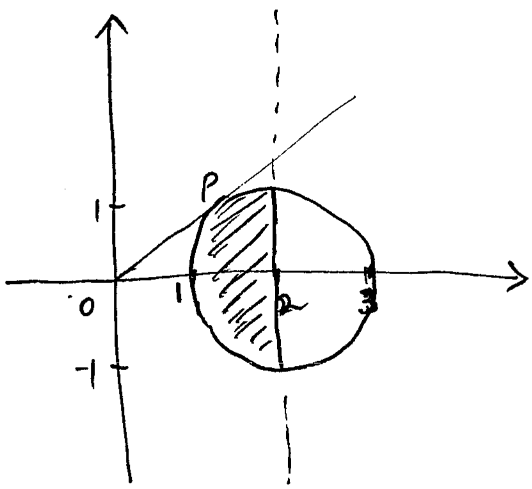
$\Rightarrow a^2 - b^2 = 1$

and $ab = \sqrt{2}$

$\therefore$ by inspection $a = \sqrt{2}$, $b = 1$

$\therefore \sqrt{1 + 2\sqrt{2}\,i} = \pm(\sqrt{2} + i)$

(c) (i)

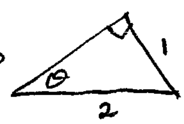


(ii) Max $|z|$ occurs at $(2,1)$ or $(2,-1)$

$\therefore$ Max $|z| = \sqrt{2^2+1^2} = \sqrt{5}$

Max $\arg z$ occurs at P on diagram

$\Rightarrow$



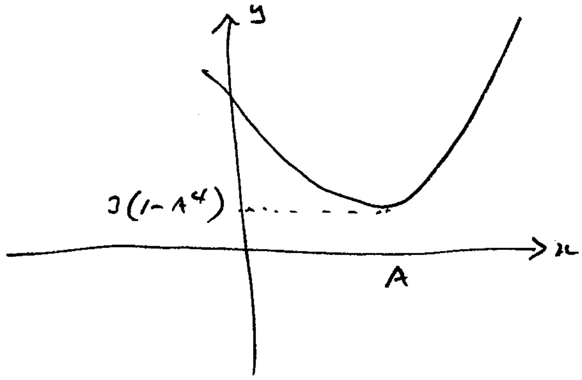
$\Rightarrow$ max $\arg z = \frac{\pi}{6}$

(d) $P'(x) = 4x^3 - 4A^3$

$= 4(x^3 - A^3) = 0$ if $x = A$, $P(A) = A^4 - 4A^4 + 3$

$= 3 - 3A^4$

$\Rightarrow$ for 4 complex roots of $P(x) = 0$ we must have



& so, $1 - A^4 > 0$ or $A^4 < 1$

$\Rightarrow -1 < A < 1$

Question 3

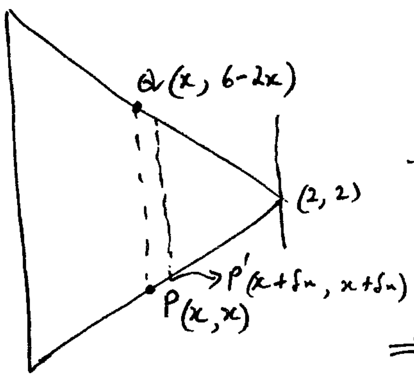
(a) Put $x = \frac{1}{1-\alpha}$ $\therefore 1 - \alpha = \frac{1}{x}$ or $\alpha = \frac{x-1}{x}$

$\therefore$ cubic is $\left(\frac{x-1}{x}\right)^3 + \frac{x-1}{x} + 1 = 0$

or $x^3 - 3x^2 + 3x - 1 + x^3 - x^2 + x^3 = 0$

i.e. $3x^3 - 4x^2 + 3x - 1 = 0$

(b)



$PQ = 6 - 2x - x = 3(2-x)$

$\therefore \delta V \simeq \pi\left((2-x)^2 - (2-x-\delta x)^2\right) 3(2-x)$

$\simeq \pi\left(2(2-x)\delta x\right) 3(2-x)$

$\Rightarrow V = 6\pi \int_0^2 (x-2)^2 \, dx$ for ease

$= \frac{6\pi}{3}\left[(x-2)^3\right]_0^2 = 16\pi$

(C) (i) For $x^2 - y^2 = a^2$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y} = \frac{x_1}{y_1} \text{ at } P$$

Gradient $OP = \frac{y_1}{x_1}$

$\therefore$ product of gradients is 1

(ii) Now $\tan\alpha = \frac{x_1}{y_1}$ and $\tan\beta = \frac{y_1}{x_1}$

$$\therefore \tan\alpha = \cot\beta = \tan\left(\frac{\pi}{2} - \beta\right)$$

$$\Rightarrow \alpha = \frac{\pi}{2} - \beta \text{ or } \alpha + \beta = \frac{\pi}{2}$$

[LOTS OF ALTERNATIVES]

Question 4

(a) (i) G.S, $r = -x$, $N = 2n+1$

$$\therefore 1 - x + \cdots + x^{2n} = \frac{1 - (-x)^{2n+1}}{1 - -x} = \frac{1 + x^{2n+1}}{1+x}$$ since $2n+1$ is odd

(ii) From (i), $J = \int_0^1 1 - x + x^2 - \cdots + x^{2n} - \frac{1}{1+x}\, dx$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n+1}}{2n+1} - \ln(1+x)\right]_0^1$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \ln 2$$

(iii) Since $0 \le x \le 1$, $\frac{x^{2n+1}}{1+x} \geq 0 \quad \therefore J > 0$

Also $\frac{x^{2n+1}}{1+x} < x^{2n+1}$

$$\therefore J < \int_0^1 x^{2n+1}\, dx = \left[\frac{x^{2n+2}}{2n+2}\right]_0^1 = \frac{1}{2n+2}$$

$$\therefore 0 < J < \frac{1}{2n+2}$$

(iv) From (ii) and (iii)

$$0 < 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \ln 2 < \frac{1}{2n+2}$$

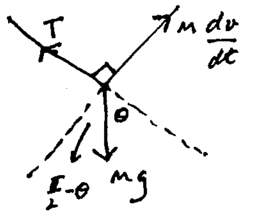
$\therefore$ since $\lim_{n\to\infty} \frac{1}{2n+2} = 0$,

$$\lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \ln 2\right) = 0$$

$$\Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \cdots \quad \text{since } \lim_{n\to\infty} \frac{1}{2n+1} = 0$$

(b) (i) $x = r\theta \quad \therefore \frac{dx}{dt} = r\frac{d\theta}{dt} \quad$ i.e. $v = r\frac{d\theta}{dt}$

(ii)



$$\therefore m\frac{dv}{dt} = -mg\cos\left(\frac{\pi}{2} - \theta\right)$$

$$\text{i.e. } \frac{dv}{dt} = -g\sin\theta$$

(iii) From (i), $\frac{dv}{dt} = r\frac{d^2\theta}{dt^2} = -g\sin\theta$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta$$

(iv) If θ is small, $\sin\theta \approx \theta$

$$\therefore \frac{d^2\theta}{dt^2} \approx -\frac{g}{r}\theta \quad \text{is of the form } -n^2\theta$$

$$\Rightarrow \text{SHM where } n = \sqrt{\frac{g}{r}}$$

$$\therefore \text{period} = 2\pi\sqrt{\frac{r}{g}}$$

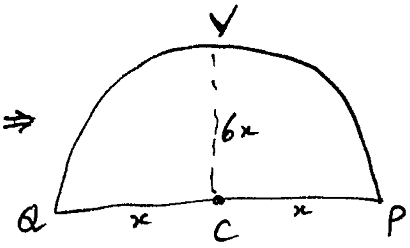
(v) Resolving in direction BC

$$m\frac{v^2}{r} = T - mg\cos\theta \approx T - mg \quad \text{for small } \theta$$

$$\therefore T \approx m\left(g + \frac{v^2}{r}\right)$$

Question 5

(a) (i) $\Rightarrow$



$\therefore$ Using Simpson's rule,

$$\text{Area} = \frac{1}{6} \cdot 2x\,(0 + 0 + 24x)$$
$$= 8x^2$$

(ii) $\therefore \quad \delta V \approx 8x^2\,\delta y = 32y\,\delta y$

$$\therefore \quad V = 32\int_0^1 y\,dy = 16\left[y^2\right]_0^1 = 16$$

(b) (i) $m\ddot{x} = mg - mkv \Rightarrow \ddot{x} = g - kv$

(ii) $\ddot{x} = 0 \Rightarrow g - kV = 0$ ie $V = \dfrac{g}{k}$

(iii) $\ddot{x} = \dfrac{dv}{dt} = k\left(\dfrac{g}{k} - v\right) = k\,(V - v)$

$$\therefore \quad k\,\frac{dt}{dv} = \frac{1}{V - v}$$

$$\Rightarrow \quad k\left[t\right]_0^t = -\left[\ln(V-v)\right]_0^v$$

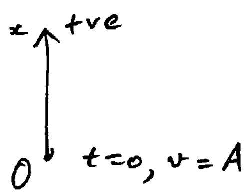
$$\therefore \quad kt = -\left(\ln(V-v) - \ln V\right)$$

$$\text{or} \quad \ln\left(\frac{V-v}{V}\right) = -kt$$

$$\therefore \quad \frac{V-v}{V} = e^{-kt}$$

$$\Rightarrow \quad v = V\left(1 - e^{-kt}\right)$$

(iv)



$$m\ddot{x} = -mg - mkv$$

$$\therefore \quad \ddot{x} = -g - kv = -k\,(V + v)$$

$$\therefore\ -k\frac{dt}{dv} = \frac{1}{V+v}$$

$$\Rightarrow\ -k\left[t\right]_0^T = \left[\ln(V+v)\right]_A^0$$

$$\Rightarrow\ -kT = \ln V - \ln(V+A) = \ln\left(\frac{V}{V+A}\right)$$

$\therefore$ From (iv), $\ln\left(\dfrac{V-v}{V}\right) = \ln\left(\dfrac{V}{V+A}\right)$ where v is the velocity of the first particle

$$\Rightarrow\ \frac{V-v}{V} = \frac{V}{V+A}$$

$$\therefore\ \frac{v}{V} = 1 - \frac{V}{V+A} = \frac{A}{V+A}$$

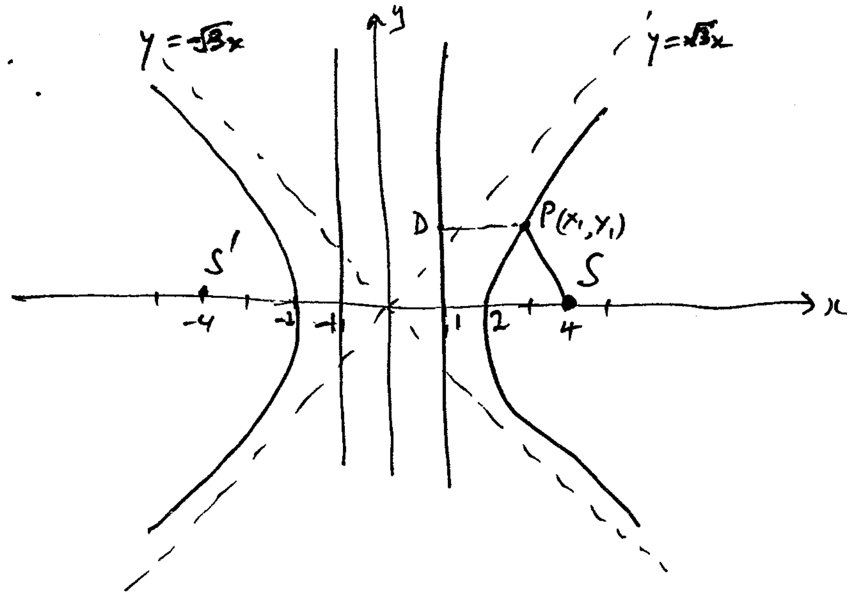
$$\therefore\ v = \frac{AV}{A+V}$$

Question 6

(a) (i) $a=2$, $b^2=12$ $\Rightarrow c^2 = 4+12=16$, $c=4$; $e=\frac{4}{2}=2$

$\therefore$ Foci $(\pm 4, 0)$, directrices $x = \pm\frac{2}{2} = \pm 1$, asymptotes $y = \pm\frac{\sqrt{12}}{2}x = \pm\sqrt{3}x$

$\therefore$



(ii) $PS = e\,PD = 2(x_1 - 1)$

(b) (i) $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$

$$= -\frac{b^2 a\cos\theta}{a^2 b\sin\theta} \quad \text{at } P$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ Tangent at P is $y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\therefore\ ay\sin\theta - ab\sin^2\theta = -b\cos\theta\, x + ab\cos^2\theta$

$\Rightarrow\ bx\cos\theta + ay\sin\theta = ab(\cos^2\theta + \sin^2\theta) = ab$

ie. $bx\cos\theta + ay\sin\theta - ab = 0$

(ii) $pq = \dfrac{(abe\cos\theta - ab)(-abe\cos\theta - ab)}{b^2\cos^2\theta + a^2\sin^2\theta}$

$$= \frac{a^2b^2(1-e\cos\theta)(1+e\cos\theta)}{b^2\cos^2\theta + a^2(1-\cos^2\theta)}$$

$$= \frac{a^2b^2(1-e^2\cos^2\theta)}{a^2 - (a^2-b^2)\cos^2\theta}$$

$$= \frac{a^2b^2(1-e^2\cos^2\theta)}{a^2 - a^2e^2\cos^2\theta} = b^2$$

(iii) For $P(a,0)$, $pq = (a-ae)(a+ae)$

$$= a^2 - a^2e^2$$

$$= b^2$$

(iv) $(p-q)^2 = p^2 + q^2 - 2pq$

$$= 6(a^2e^2) - 2(a^2 - a^2e^2) \quad \text{from (iii)}$$

$$= 2a^2(3e^2 - 1 + e^2) = 2a^2(4e^2-1)$$

But $(p-q)^2 \geq 0 \Rightarrow 4e^2 - 1 \geq 0$ or $e^2 \geq \frac{1}{4}$ ie $e \geq \frac{1}{2}$

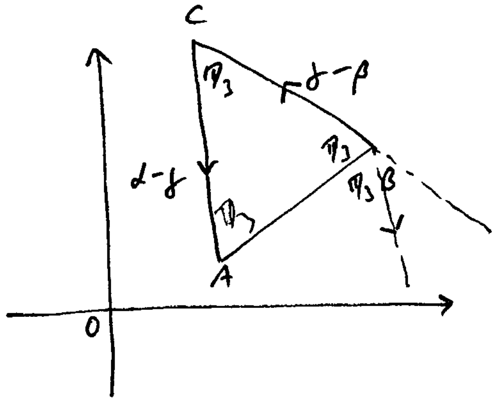
Question 7

(a) (i) $\omega^3 = \cos 2\pi + i \sin 2\pi = 1$

$$1 + \omega + \omega^2 = 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = 0$$ (ALTERNATIVES, OF COURSE)

(ii)



Now $\vec{CA} = \omega\,\vec{BC}$

$\Rightarrow \alpha - \gamma = \omega(\gamma - \beta)$

(iii) From (ii) $\quad \alpha - \gamma = \omega\gamma - \omega\beta$

$\therefore \alpha + \omega\beta - \gamma(1 + \omega) = 0$

$\Rightarrow \alpha + \omega\beta - \gamma(-\omega^2) = 0$ from (i)

i.e. $\alpha + \omega\beta + \omega^2\gamma = 0$

(iv) $\because \alpha + \omega\beta + \omega^2\gamma = 0$ and is the sum of the roots of $z^3 + pz + q = 0$

(v) $\alpha . \omega\beta . \omega^2\gamma = \alpha\beta\gamma\omega^3 = -q \Rightarrow \alpha\beta\gamma = -q$ from (i)

i.e. $q = -\alpha\beta\gamma$

(vi) Now $\sum \alpha^3 + p\sum\alpha + 3q = 0$

$\Rightarrow \alpha^3 + \omega^3\beta^3 + \omega^6\gamma^3 + p(0) - 3\alpha\beta\gamma = 0$

i.e. $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$ since $\omega^3 = 1$

(b) (i) $u_n - u_{n-1} = \int_0^{\pi/2} \frac{\sin 2n\theta - \sin 2(n-1)\theta}{\sin\theta}\, d\theta$

$$= \int_0^{\pi/2} \frac{2\cos(2n-1)\theta \sin\theta}{\sin\theta}\, d\theta$$

$$= \frac{2}{2n-1}\Big[\sin(2n-1)\theta\Big]_0^{\pi/2}$$

$$= \frac{2}{2n-1} \sin(2n-1)\frac{\pi}{2}$$

$$= (-1)^{n-1}\frac{2}{2n-1} \quad \text{since } \sin\frac{\pi}{2} = 1 \ \&\ \sin\frac{3\pi}{2} = -1$$

(ii) $\sum_{n=2}^{n}(u_n - u_{n-1}) = u_n - u_1 = \sum_{n=2}^{n}(-1)^{n-1}\frac{2}{2n-1}$

$$\therefore u_n = u_1 + \sum_{n=2}^{n}(-1)^{n-1}\frac{2}{2n-1}$$

$$= \int_0^{\pi/2}\frac{\sin 2\theta}{\sin\theta}\, d\theta + \sum_{n=2}^{n}(-1)^{n-1}\frac{2}{2n-1}$$

$$= \int_0^{\pi/2} 2\cos\theta\, d\theta + \sum_{n=2}^{n}(-1)^{n-1}\frac{2}{2n-1}$$

$$= 2\Big[\sin\theta\Big]_0^{\pi/2} + \sum_{n=2}^{n}(-1)^{n-1}\frac{2}{2n-1}$$

$$= 2 + 2\left(-\frac{1}{3} + \frac{1}{5} - \dots + (-1)^{n-1}\frac{1}{2n-1}\right)$$

$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \dots + (-1)^{n-1}\frac{1}{2n-1}\right)$$

Question 8

(a) (i) $u_2 = \frac{1}{2} + 1 = \frac{3}{2}$

$\therefore u_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$

(ii) $u_{n+1} = \frac{1}{\sqrt{2}}\left(\frac{1+A}{1-A}\right) + \frac{1}{\sqrt{2}}\left(\frac{1-A}{1+A}\right)$

$$= \frac{1}{\sqrt{2}} \cdot \frac{(1+A)^2 + (1-A)^2}{1-A^2}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{2(1+A^2)}{1-A^2} = \sqrt{2}\left(\frac{1+A^2}{1-A^2}\right)$$

(iii) $u_1 = \sqrt{2}\,\frac{(1-(\sqrt{2}-1)^2)}{1+(\sqrt{2}-1)^2} = \frac{\sqrt{2}(2\sqrt{2}-2)}{4-2\sqrt{2}} = \frac{4-2\sqrt{2}}{4-2\sqrt{2}} = 1$

$\therefore$ Assume $u_n = \sqrt{2}\left(\frac{1+A}{1-A}\right)$ where $A = (-1)^{2^{n-1}}(\sqrt{2}-1)^{2^n}$ for integers $n \geq 1$

Then $u_{n+1} = \sqrt{2}\left(\frac{1+A^2}{1-A^2}\right)$ from (ii) & using the assumption

$$= \sqrt{2}\,\frac{\left(1 + (-1)^{2^n}(\sqrt{2}-1)^{2^{n+1}}\right)}{1-(-1)^{2^n}(\sqrt{2}-1)^{2^{n+1}}}$$

since
$A^2 = \left((-1)^{2^{n-1}}(\sqrt{2}+1)^{2^n}\right)^2$
$= (-1)^{2^{n-1}\cdot 2}(\sqrt{2}+1)^{2^n \cdot 2}$
$= (-1)^{2^n}(\sqrt{2}-1)^{2^{n+1}}$

$\therefore$ by induction it's correct.

(iv) $0 < \sqrt{2}-1 < 1$ and $2^n \to \infty$ as $n \to \infty$

$\therefore \lim_{n\to\infty} (\sqrt{2}-1)^{2^n} = 0$

$\Rightarrow \lim_{n\to\infty} u_n = \sqrt{2}\left(\frac{1}{1}\right) = \sqrt{2}$

(b) (i) $9 - 8\sin\theta - 3\cos\theta = 9 - \sqrt{8^2+3^2}\,\sin(\theta+\alpha)$ for some $0 < \alpha < \frac{\pi}{2}$

$$\geqslant 9 - \sqrt{73} \quad \text{since } |\sin(\theta+\alpha)| \le 1$$

$$> 0 \quad \forall\, \theta$$

(ii) $f(\theta) = 1 + \dfrac{5 - 5\sin(\theta+\alpha)}{9 - 8\sin\theta - 3\cos\theta} \geqslant 1$ since $|\sin(\theta+\alpha)| \le 1$ & (i)

$f(\theta) = 2 - \dfrac{4 - 4\sin\theta}{9 - 8\sin\theta - 3\cos\theta} \le 2$ since $|\sin\theta| \le 1$ & (i)

ie. $1 \le f(\theta) \le 2 \quad \forall\, \theta$

(iii) If $\sin\theta = \frac{4}{5}$ then $\cos\theta = \frac{3}{5}$ since $0 < \theta < \frac{\pi}{2}$

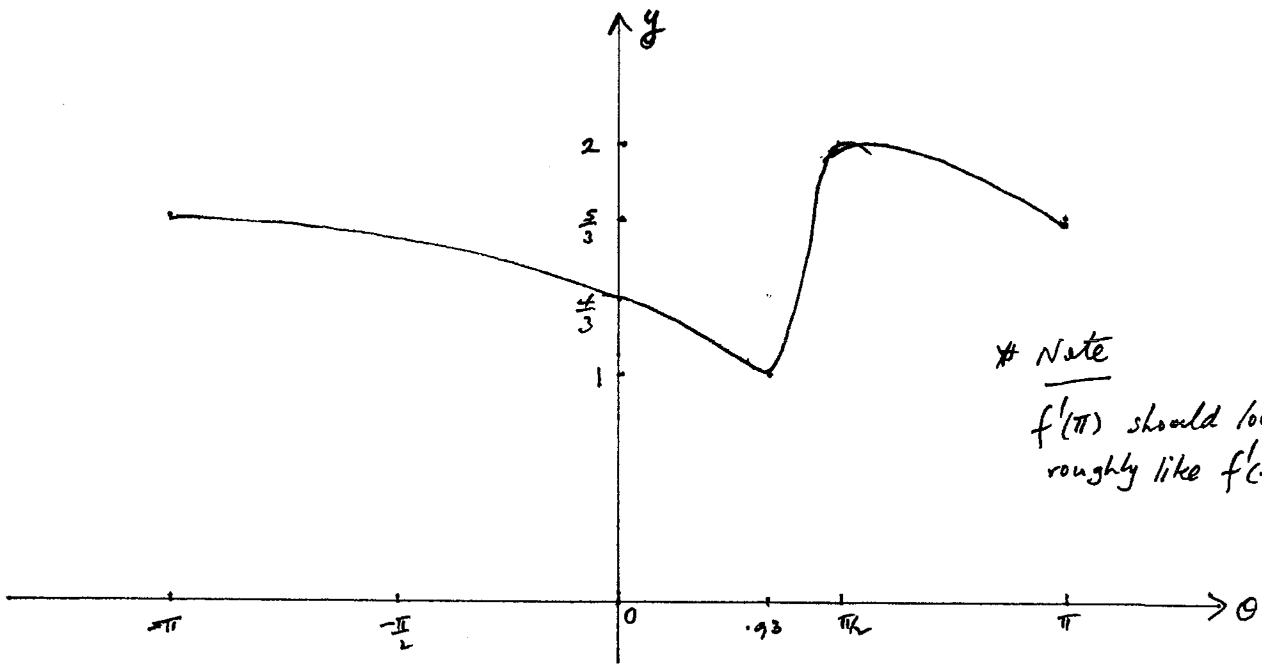
Then $f(\theta) = 1 + \dfrac{5 - \frac{16}{5} - \frac{9}{5}}{9 - 8\cdot\frac{4}{5} - \frac{9}{5}} = 1$

(iv) $f(\pi) = f(-\pi) = \dfrac{14+6}{9+3} = \dfrac{5}{3}$

$f(0) = \dfrac{14-6}{9-3} = \dfrac{4}{3}$

$\sin\theta = \frac{4}{5} \Rightarrow \theta \approx .93$

$f(\theta) = 2$ if $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$



\# Note

$f'(\pi)$ should look roughly like $f'(-\pi)$