THE KING'S SCHOOL

2010
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-8
- All questions are of equal value

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $\dfrac{x-1}{(x+1)(x^2+1)}$ in partial fractions. **3**

(ii) Find $\displaystyle\int \frac{x-1}{(x+1)(x^2+1)}\,dx$ **2**

(b) (i) Show that $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$ where $t = \tan\theta$ **2**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{4}{5 - 3\cos 2\theta}\,d\theta$ **3**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta}{\sqrt{1-\sin^4\theta}}\,d\theta$ **3**

(d) By using the result $a - \dfrac{a}{a+1} = \dfrac{a^2}{a+1}$, or otherwise, find $\displaystyle\int \frac{e^{-2x}}{e^{-x}+1}\,dx$ **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $z = 3 - 2i$ and $w = -1 + 3i$

Find

(i) $\overline{z + w}$ **1**

(ii) $|zw|$ **2**

(b) Let $z = 1 + ai$ and $w = a + i$, a real

Find $\arg(zw)$ **2**

(c) (i) Expand $(a + ib)^3$ **1**

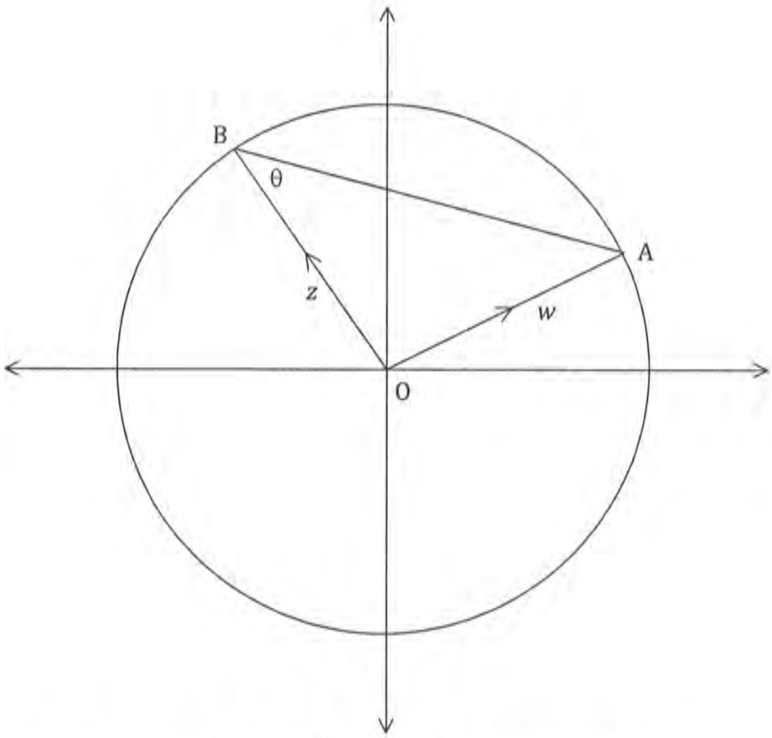
(ii) If $(a + ib)^3 = x + iy$, a, b, x, y *real* ,

show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$, $a, b \neq 0$ **2**

(d) Sketch the region in the complex plane where $|z - 2i| \geq |z + 2|$ **2**

Question 2 continues on the next page

(e)



The diagram shows two distinct points A and B that represent the complex numbers w and z respectively. These points lie on the circle with centre O and radius r.

Let $\angle OBA = \theta$

(i) Show that $z = w(-\cos 2\theta + i\sin 2\theta)$ **2**

(ii) Write down the complex number represented by $\overrightarrow{AB}$ and deduce that $|z + w| = \tan\theta\,|z - w|$ **3**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the gradient of the curve defined by $(x + 1)y + \cos \pi y + x^3 = 0$ at the point where $y = 1$ **4**

(b) Let $f(x) = \ln\left(\sqrt{x^2 + 1} - x\right)$

(i) Show that x can assume any real value. **1**

(ii) Show that the function is odd. **2**

(iii) Show that $f'(x) = -\dfrac{1}{\sqrt{x^2 + 1}}$ **2**

(iv) Given that $f''(x) = \dfrac{x}{(x^2 + 1)^{\frac{3}{2}}}$,

sketch the graph of $y = f(x)$ **2**

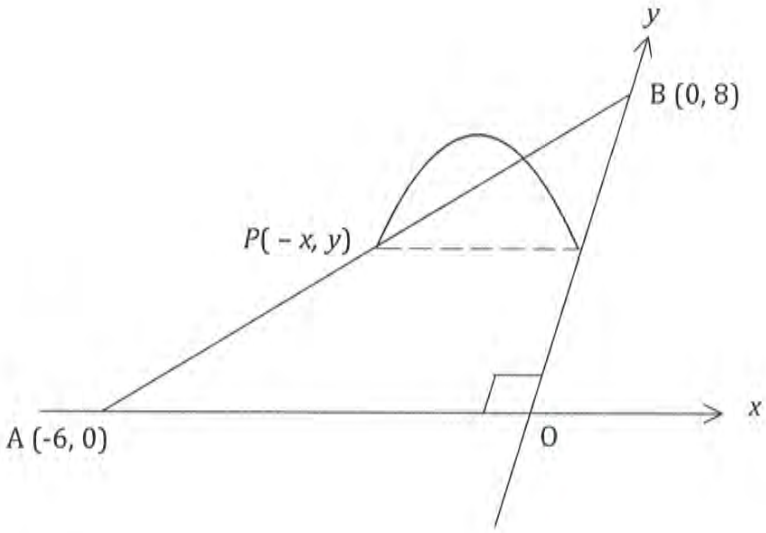
(v) Use integration by parts to evaluate $\displaystyle\int_0^1 \ln\left(\sqrt{x^2 + 1} - x\right) dx$ **4**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove that the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, is πab 3

(b)



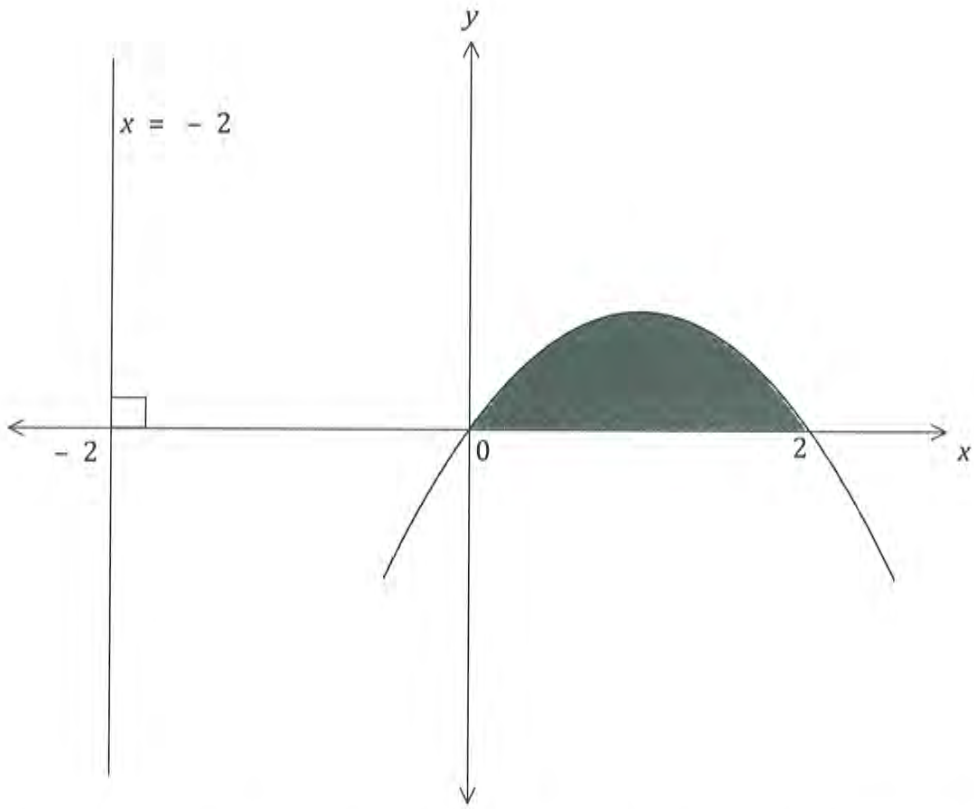
The base of a solid is the triangular region AOB where A = (-6, 0), O = (0, 0) and B (0, 8).

Vertical cross-sections of the solid parallel to AO are semi-ellipses where the major axis lies in the base of the solid and is twice the length of the minor axis.

Let P $(-x, y)$ be a point on AB. Find the volume of the solid. **5**

Question 4 continues on the next page

(c)



The diagram shows the region bounded by $y = x(2 - x)$ and the x axis, $0 \leq x \leq 2$

(i) Without using integration, show that the area of the region is $\frac{4}{3}$ **2**

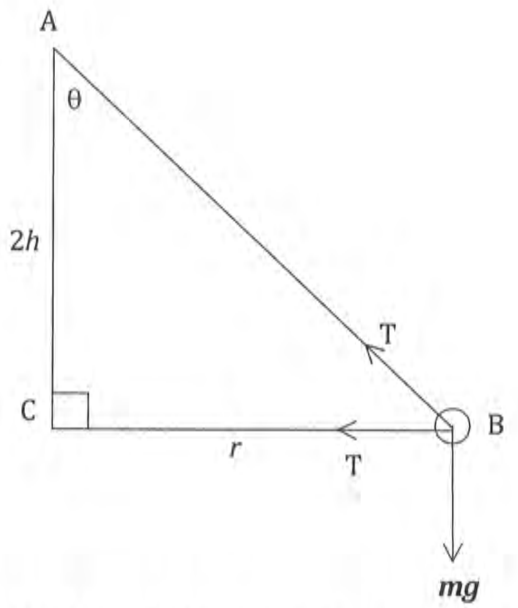
(ii) The region is revolved about the line $x = -2$.

By taking slices parallel to the line $x = -2$, find the volume of the solid generated. **5**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A string ABC of length $2l$ is attached at two points A and C. C is $2h$ vertically below A.

A smooth ring of mass m is free to move on the string and rotates about C in a horizontal plane with constant angular velocity w.

Let $\angle CAB = \theta$ and the radius of revolution CB be r.

(i) Show that $r < l$ **1**

(ii) Show that $T = mg \sec\theta$ **1**

(iii) Show that $w^2 r = g(\sec\theta + \tan\theta)$ **2**

(iv) Prove that $r = \dfrac{gl}{hw^2}$ **2**

(v) Deduce that $w > \sqrt{\dfrac{g}{h}}$ **1**

Question 5 continues on the next page

(b) (i) Prove by induction for positive integers n that

$$\frac{4.1}{1.3} - \frac{4.2}{3.5} + \frac{4.3}{5.7} - \dots + \frac{(-1)^n 4(n+1)}{(2n+1)(2n+3)} = 1 + (-1)^n \frac{1}{2n+3}$$ **4**

(ii) Find the sum $\frac{4.1}{1.3} - \frac{4.2}{3.5} + \frac{4.3}{5.7} - \dots$ **1**

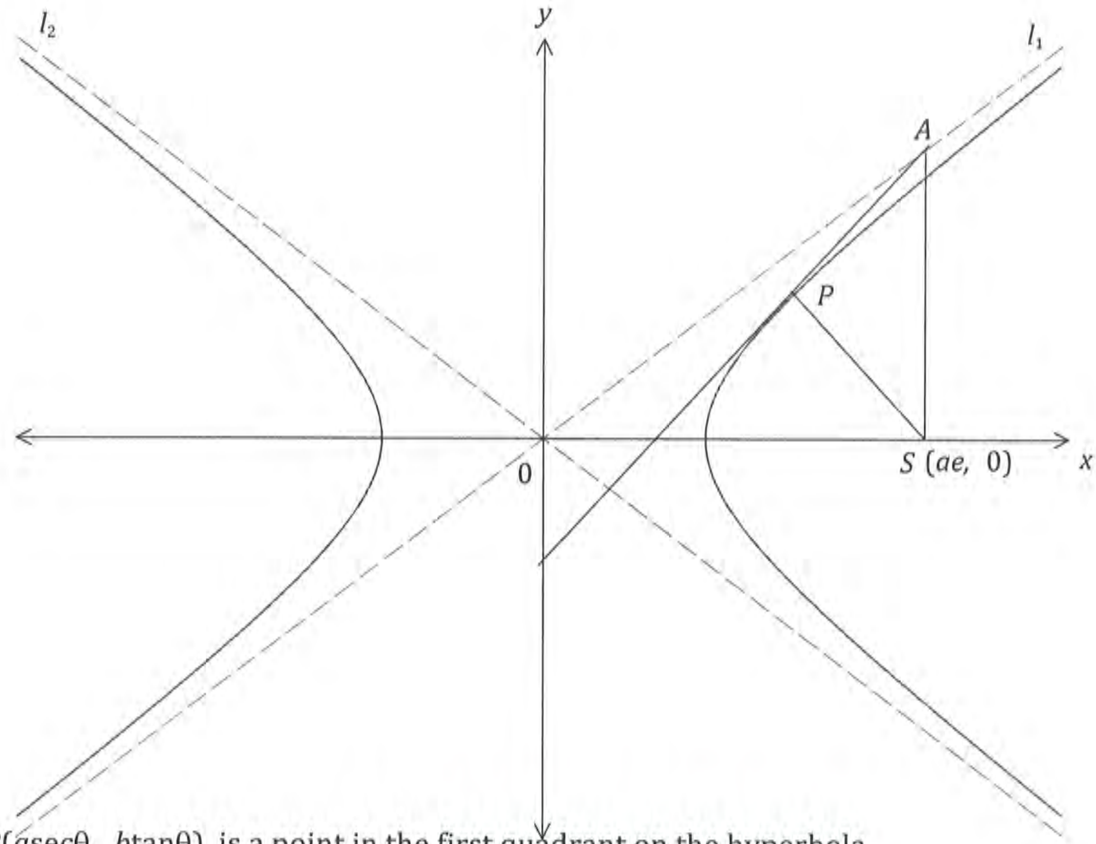
(c) Let $z = \cos\theta + i\sin\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Find the modulus and argument of the complex number $z^2 + 1$ **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



$P(a\sec\theta,\ b\tan\theta)$ is a point in the first quadrant on the hyperbola

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1\ ,\ \ a,\ b > 0$

$S(ae,\ 0)$ is a focus of the hyperbola, where e is the eccentricity.

$l_1\ ,\ \ l_2$ are the asymptotes of the hyperbola.

The line $x = ae$ and the tangent at P meet at A on l_1

(i) Show that $A = (ae\ ,\ be)$ **2**

(ii) Prove that the equation of the tangent at $P(a\sec\theta,\ b\tan\theta)$ is
$\dfrac{\sec\theta}{a}\,x - \dfrac{\tan\theta}{b}\,y = \sec^2\theta - \tan^2\theta$ **2**

(iii) Prove that $\sec\theta + \tan\theta = e$ **2**

(iv) Deduce that SP is parallel to l_2 **1**

Question 6 continues next page

(b) A particle moves on the x axis so that at anytime t its acceleration $\ddot{x}$ is given by $\ddot{x} = \frac{16}{v} + v$, where v is the velocity.

Initially, $v = V > 0$

Suppose the particle travels a distance X in time T to acquire a velocity of $4V$.

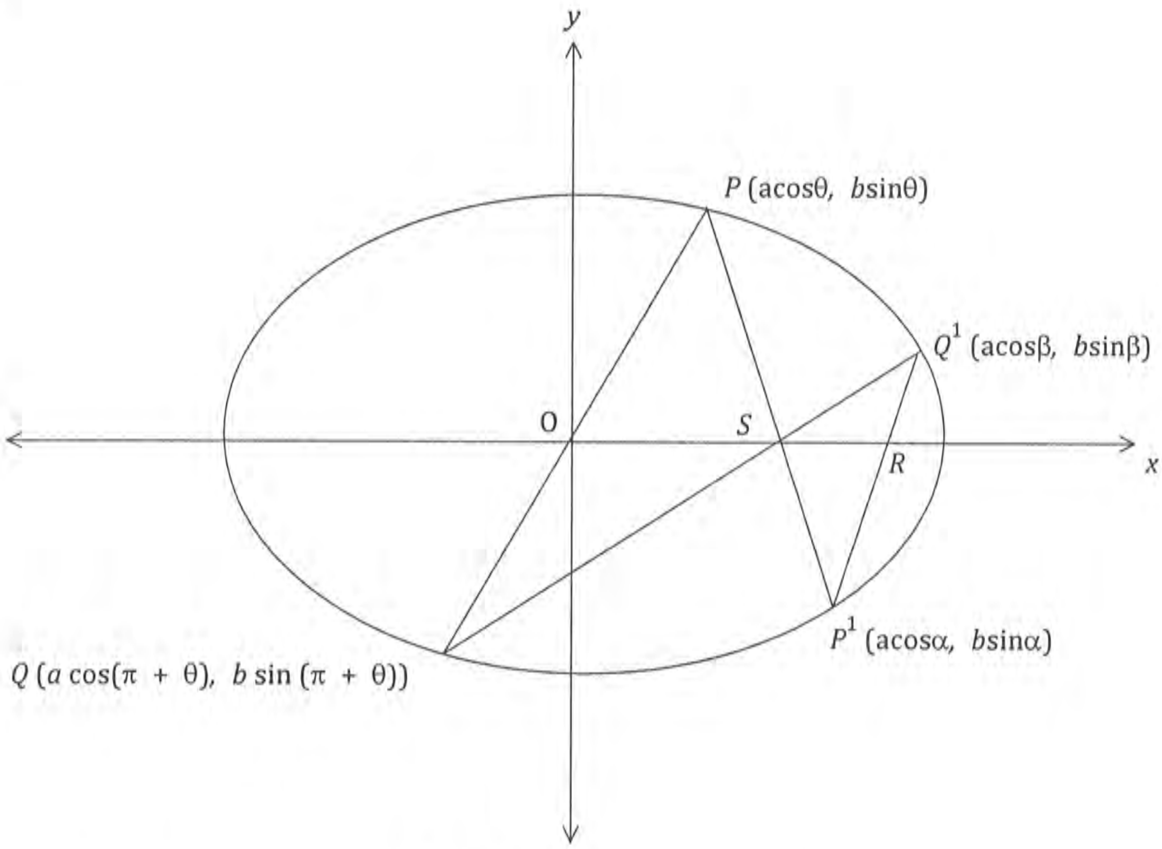
(i) Prove that $T < 2 \ln 2$ **4**

(ii) Prove that $X = 3V - 4\tan^{-1}\left(\frac{3V}{4 + V^2}\right)$ **4**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



PQ is a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$, passing through O (0, 0)

PP^1 and QQ^1 are focal chords passing through the focus $S(ae, 0)$, e the eccentricity.

The equation of the chord P^1Q^1 is

$$\frac{x}{a}\cos\left(\frac{\alpha + \beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\alpha - \beta}{2}\right)$$

[DO NOT PROVE THIS]

The chord P^1Q^1 meets the major axis at R.

Question 7 continues on the next page

(a) (i) Show that at R the x coordinate is given by

$$x = a\left(\frac{1 + \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}\right)$$ **2**

(ii) Use the focal chord PP^1 to show that

$$e\cos\left(\frac{\alpha + \theta}{2}\right) = \cos\left(\frac{\alpha - \theta}{2}\right)$$ **1**

(iii) Deduce that $\tan\frac{\alpha}{2}\tan\frac{\theta}{2} = \frac{e - 1}{e + 1}$ **2**

(iv) Hence show that $-\tan\frac{\beta}{2}\cot\frac{\theta}{2} = \frac{e - 1}{e + 1}$ **2**

(v) Deduce that R is a fixed point. **2**

(b) (i) Show that $(1 - x^b)^{n-1} - (1 - x^b)^n = x^b(1 - x^b)^{n-1}$ **1**

Let $u_n = \int_0^1 x^{a-1}(1 - x^b)^n\,dx$, $n = 0, 1, 2, 3, \ldots$ where a and b are positive integers.

(ii) Find u_0 **1**

(iii) Prove that $(a + bn)u_n = bn\,u_{n-1}$, $n \geq 1$ **4**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) $P(x) = 2x^3 - Ax - 2 = 0$ has the roots α, β, γ

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 = A$ **2**

(ii) Show that $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = A\alpha - \alpha^3$ **2**

(iii) Find a polynomial with the three roots

$$\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \ \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \ \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$ **4**

[YOU MAY LEAVE YOUR ANSWER IN UNEXPANDED FORM]

Question 8 continues on the next page

(b) (i) Find the sum S of the geometric series

$$1 + (1 + x) + (1 + x)^2 + \dots + (1 + x)^{n-1},\ x \neq 0$$ **1**

(ii) Deduce that $S = \binom{n}{1} + \binom{n}{2}x + \binom{n}{3}x^2 + \dots + \binom{n}{n}x^{n-1}$ **2**

(iii) Show that

$$\binom{n}{1} + \binom{n}{2}x + \binom{n}{3}x^2 + \dots + \binom{n}{n}x^{n-1}$$

$$= 1 + (1 + x) + (1 + x)^2 + \dots + (1 + x)^{n-1} \quad \text{if } x = 0$$ **1**

(iv) Prove that

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \dots + (-1)^{n+1}\frac{1}{n}\binom{n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$ **3**

End of Examination Paper

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

THE KING'S SCHOOL

2010
Higher School Certificate
Trial Examination

Mathematics Extension 2

Question (Marks)	Complex Numbers	Functions	Integration	Conics	Mechanics
1 (15)			(a)-(d) **15**		
2 (15)	(a)-(d) **15**				
3 (15)		(a),(b)(i)-(iv) **11**	(b)(v) **4**		
4 (15)			(a)-(c) **15**		
5 (15)	(c) **3**	(b) **5**			(a) **7**
6 (15)				(a) **7**	(b) **8**
7 (15)			(b) **6**	(a) **9**	
8 (15)		(a), (b) **15**			
Total (120)	**18**	**31**	**40**	**16**	**15**

TKS EXTENSION 2 TRIAL SOLUTIONS 2010

Question 1

(a) (i) Put $\dfrac{x-1}{(x+1)(x^2+1)} \equiv \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

$\therefore A(x^2+1) + (Bx+C)(x+1) \equiv x-1$

$x=-1, \quad 2A=-2, \quad A=-1$

$\therefore -1+B=0, \quad B=1$

$\& \quad -1+C=-1, \quad C=0$

$\Rightarrow \dfrac{-1}{x+1} + \dfrac{x}{x^2+1}$

(ii) $I = \displaystyle\int \frac{x}{x^2+1} - \frac{1}{x+1}\,dx = \frac{1}{2}\ln(x^2+1) - \ln(x+1) \quad (+C)$

(b) (i) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{\sec^2\theta} = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-t^2}{1+t^2}$

(ii) Put $t = \tan\theta$ $\qquad \theta=0,\ t=0$

$\dfrac{dt}{d\theta} = \sec^2\theta = 1+t^2 \qquad \theta=\frac{\pi}{4},\ t=1$

$\therefore I = \displaystyle\int_0^1 \frac{4}{5-3\cdot\frac{1-t^2}{1+t^2}}\,\frac{dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{4}{5+5t^2-3+3t^2}\,dt$

$= \displaystyle\int_0^1 \frac{2}{1+4t^2}\,dt$

$= 2\cdot\frac{1}{2}\left[\tan^{-1}2t\right]_0^1 = \tan^{-1}2$

(c) Put $u = \sin^2\theta$ $\qquad \theta = 0,\ u = 0$

$\frac{du}{d\theta} = 2\sin\theta\cos\theta = \sin 2\theta$ $\qquad \theta = \frac{\pi}{4},\ u = \frac{1}{2}$

$$\therefore\ I = \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}} = \left[\sin^{-1}u\right]_0^{\frac{1}{2}} = \frac{\pi}{6}$$

(d) $$I = \int e^{-x} - \frac{e^{-x}}{e^{-x}+1}\, dx$$

$$= -e^{-x} + \ln(e^{-x}+1) \quad (+c)$$

Question 2

(a) (i) $z + w = 2 + i \quad \therefore\ \overline{z+w} = 2 - i$

(ii) $|zw| = |z||w| = \sqrt{13}\,\sqrt{10} = \sqrt{130}$

(b) $zw = a + i + a^2 i - a = (a^2+1)i$

$\Rightarrow \arg zw = \frac{\pi}{2}$ since $a^2 + 1 > 0$

(c) (i) $a^3 + 3a^2 ib + 3a i^2 b^2 + i^3 b^3$ will do

$= a^3 - 3ab^2 + i(3a^2 b - b^3)$

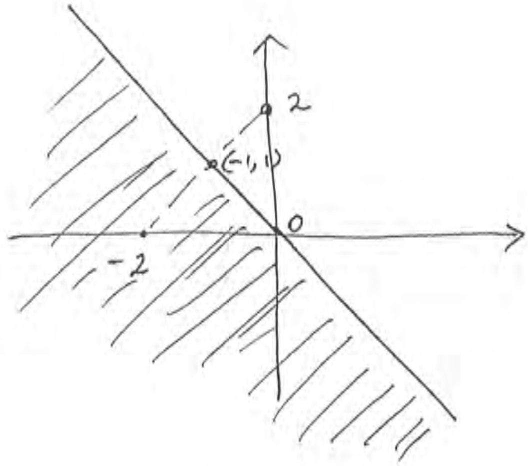
(ii) From (i) $\quad x = a^3 - 3ab^2$

$y = 3a^2 b - b^3$

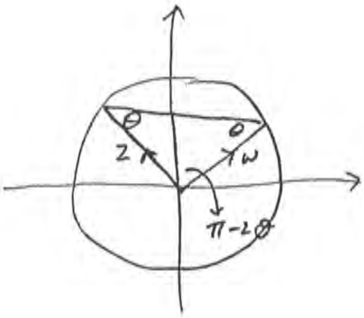
$\therefore\ \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2$

$= 4(a^2 - b^2)$

(d)



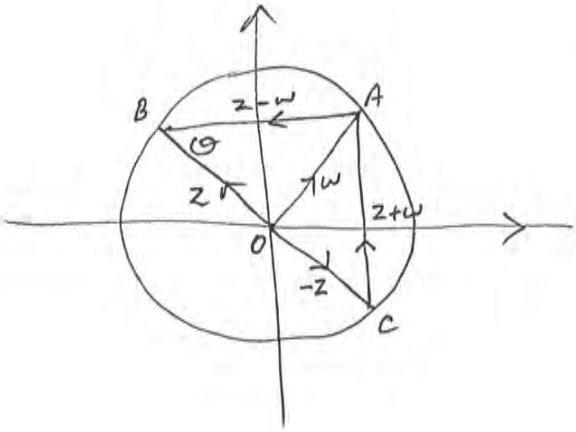
(e) (i)



$\therefore z = w\left(\cos(\pi - 2\theta) + i\sin(\pi - 2\theta)\right)$

i.e. $z = w\left(-\cos 2\theta + i\sin 2\theta\right)$

(ii) $\vec{AB} = z - w$



$\vec{OC} = -z$

$\vec{CA} = z + w$

$\angle BAC = \dfrac{\pi}{2}$ (∠ in semi-circle)

$\therefore \tan\theta = \dfrac{|z+w|}{|z-w|}$

i.e. $|z+w| = \tan\theta\,|z-w|$

Question 3

(a) $y = 1 \Rightarrow x + 1 - 1 + x^3 = 0$

i.e. $x(1 + x^2) = 0 \Rightarrow x = 0$

$$(x+1)\frac{dy}{dx} + y - \pi \sin \pi y \frac{dy}{dx} + 3x^2 = 0$$

$\therefore$ at $(0, 1)$, $\frac{dy}{dx} + 1 - 0 + 0 = 0$

$$\frac{dy}{dx} = -1$$

(b) (i) $\sqrt{x^2+1} > \sqrt{x^2} - x \geq 0 \ \forall x$ [ie. $|x| - x \geq 0$]

$\Rightarrow \sqrt{x^2+1} - x > 0 \ \forall x$

$\therefore$ domain is all real values for x

(ii) $f(x) + f(-x) = \ln(\sqrt{x^2+1} - x) + \ln(\sqrt{x^2+1} + x)$

$= \ln(x^2 + 1 - x^2)$

$= \ln 1 = 0$

$\Rightarrow f(x)$ is an odd function

(iii) $f'(x) = \dfrac{1}{\sqrt{x^2+1} - x} \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x - 1$

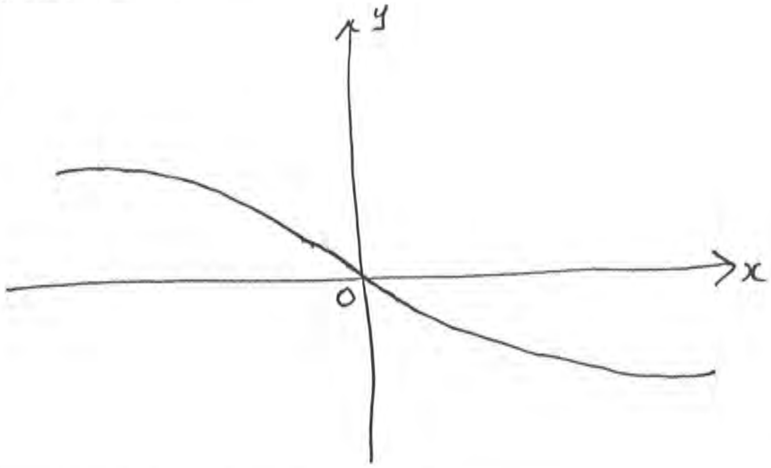
$= \dfrac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}\,(\sqrt{x^2+1} - x)} = \dfrac{-1}{\sqrt{x^2+1}}$

(iv) For $x > 0$, $f''(x) > 0$ ie concave upward

& for $x < 0$, $f''(x) < 0$ ie concave downward

$f'(0) = -1$

$\Rightarrow$



(v) Put $u = \ln(\sqrt{x^2+1} - x)$, $\frac{dv}{dx} = 1$

$$\Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{x^2+1}}, \quad v = x$$

$$\therefore I = \left[x \ln(\sqrt{x^2+1} - x)\right]_0^1 + \int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$$

For $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$, put $u = x^2+1$; $x = 0, u = 1$
$\frac{du}{dx} = 2x$; $x = 1, u = 2$

$$\therefore I = \ln(\sqrt{2} - 1) + \frac{1}{2}\int_1^2 \frac{1}{\sqrt{u}}\,du$$

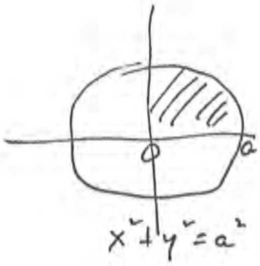
$$= \ln(\sqrt{2} - 1) + \frac{1}{2}\, 2\left[\sqrt{u}\right]_1^2$$

$$= \ln(\sqrt{2} - 1) + \sqrt{2} - 1$$

Question 4

(a) $A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\ dx$

$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\ dx \qquad \Rightarrow$



$= \frac{4b}{a}\cdot\frac{1}{4}\pi a^2$

$= \pi a b$

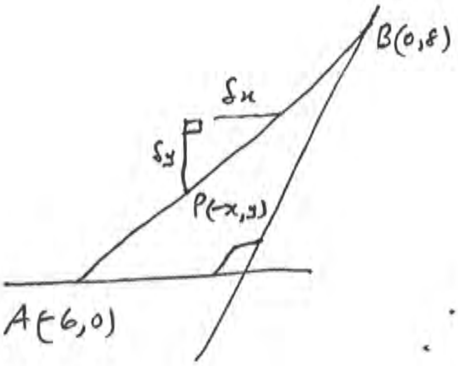
(b) Major axis length $= x$, minor axis length $= \frac{x}{2}$

$\Rightarrow 2a = x \ , \ 2b = \frac{x}{2}$

or $a = \frac{x}{2}\ , \ b = \frac{x}{4}$

$\therefore$ Area of slice $= \frac{1}{2}\cdot\pi\cdot\frac{x^2}{8}$ from (a)

$= \frac{\pi x^2}{16}$



$\Rightarrow \delta V \approx \frac{\pi x^2}{16}\delta y$ where $\frac{\delta y}{\delta x} = \frac{8}{6} = \frac{4}{3}$

or $\delta y = \frac{4}{3}\delta x$

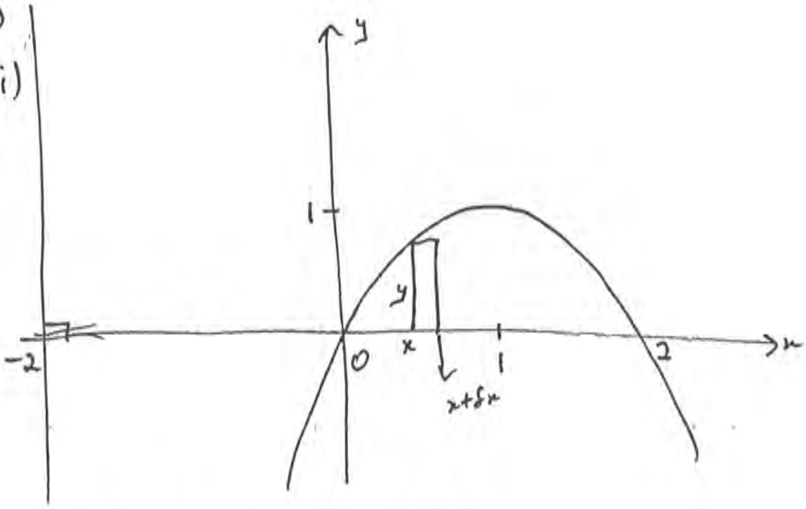
$\therefore V = \int_0^6 \frac{\pi x^2}{16}\,\frac{4}{3}\ dx$

$= \frac{\pi}{12}\left[\frac{x^3}{3}\right]_0^6$

$= 6\pi$

(c)

(i)



(i) at $x = 1$, $y = 1$

∴ using Simpson's rule (exact here, of course),

$$A = \frac{1}{6} \cdot 2 \left[0 + 0 + 4 \right] = \frac{4}{3}$$

(ii) $\delta V \approx \pi \left((x + \delta x + 2)^2 - (x+2)^2 \right) y$

$$\approx \pi \; 2 \, (x+2) \, \delta x \; y$$

$$\Rightarrow V = 2\pi \int_0^2 (x+2)\, x \,(2-x) \, dx$$

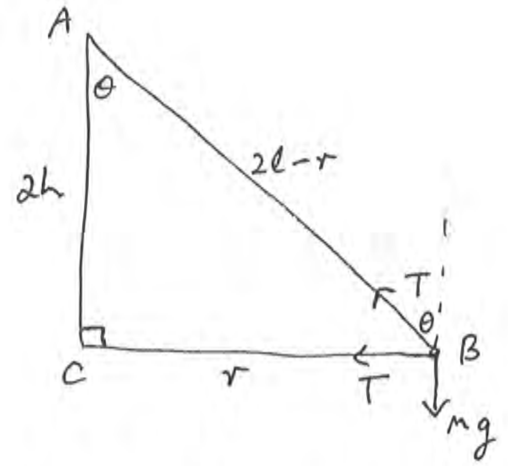
$$= 2\pi \int_0^2 4x - x^3 \, dx$$

$$= 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2$$

$$= 2\pi \,(8 - 4) = 8\pi$$

Question 5

(a)



(i) $AB > BC \Rightarrow 2l - r > r$ ie $r < l$

(ii) Resolving vertically at B, $mg = T\cos\theta$

ie $T = mg\sec\theta$

(iii) Resolving in direction $\overrightarrow{BC}$,

$$m\omega^2 r = T + T\sin\theta$$

$$= mg\sec\theta\,(1 + \sin\theta)$$

$$\Rightarrow \omega^2 r = g(\sec\theta + \tan\theta)$$

(iv) $\sec\theta + \tan\theta = \dfrac{2l - r}{2h} + \dfrac{r}{2h} = \dfrac{2l}{2h} = \dfrac{l}{h}$

$$\therefore \omega^2 r = g\frac{l}{h} \Rightarrow r = \frac{gl}{h\omega^2}$$

(v) Since $r < l$, $\dfrac{gl}{h\omega^2} < l$

or $\omega^2 > \dfrac{g}{h}$

ie $\omega > \sqrt{\dfrac{g}{h}}$

(b) (i) For $n=1$, $LS = \frac{4.1}{1.3} - \frac{4.2}{3.5} = \frac{4}{5}$

$$RS = 1 - \frac{1}{5} = \frac{4}{5}$$

$\therefore$ Assume $\frac{4.1}{1.3} - \dots + (-1)^n \frac{4(n+1)}{(2n+1)(2n+3)} = 1 + (-1)^n \frac{1}{2n+3}$ for some positive integer n

Then $\frac{4.1}{1.3} - \dots + \frac{(-1)^n 4(n+1)}{(2n+1)(2n+3)} + \frac{(-1)^{n+1} 4(n+2)}{(2n+3)(2n+5)}$

$= 1 + (-1)^n \frac{1}{2n+3} + \frac{(-1)^{n+1} 4(n+2)}{(2n+3)(2n+5)}$, using the assumption

$$= 1 + \frac{(-1)^{n+1}}{2n+3}\left(\frac{4(n+2)}{2n+5} - 1\right)$$

$$= 1 + \frac{(-1)^{n+1}}{2n+3}\left(\frac{4n+8-2n-5}{2n+5}\right)$$

$$= 1 + \frac{(-1)^{n+1}}{2n+3} \frac{(2n+3)}{2n+5}$$

$$= 1 + (-1)^{n+1} \cdot \frac{1}{2n+5}$$

$\therefore$ by induction it's true for $n \geq 1$

(ii) $S = \lim_{n \to \infty} \left(1 + \frac{(-1)^n}{2n+3}\right) = 1$

(c) $z^2 + 1 = \cos 2\theta + i \sin 2\theta + 1$

$= 2\cos^2\theta + 2i \sin\theta \cos\theta$

$= 2\cos\theta (\cos\theta + i \sin\theta)$

$\Rightarrow |z^2+1| = 2\cos\theta$, $\arg(z^2+1) = \theta$

Question 6

(a) (i) l_1 is $\frac{x}{a} - \frac{y}{b} = 0$

$\therefore$ at A, $\frac{ae}{a} - \frac{y}{b} = 0 \Rightarrow y = be$

i.e. $A = (ae, be)$

(ii) $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y} = \frac{b\sec\theta}{a\tan\theta}$ at P

$\therefore$ Tangent at P is $y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

or $\frac{\tan\theta}{b}y - \tan^2\theta = \frac{\sec\theta}{a}x - \sec^2\theta$

i.e. $\frac{\sec\theta}{a}x - \frac{\tan\theta}{b}y = \sec^2\theta - \tan^2\theta$

(iii) From (ii), at A, $x = ae$, $y = be$

$\Rightarrow e\sec\theta - e\tan\theta = \sec^2\theta - \tan^2\theta$

$\Rightarrow e = \frac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

(iv) l_2 is $\frac{x}{a} + \frac{y}{b} = 0$ with gradient $-\frac{b}{a}$

grad $SP = \frac{b\tan\theta}{a\sec\theta - ae} = -\frac{b}{a}\frac{\tan\theta}{\tan\theta}$ from (iii)

$= -\frac{b}{a}$

$\therefore l_2 \parallel SP$

(b) (i) $\frac{dv}{dt} = \frac{16+v^2}{v}$

$\Rightarrow \frac{dt}{dv} = \frac{v}{16+v^2}$

$\Rightarrow [t]_0^T = \frac{1}{2}\left[\ln(16+v^2)\right]_V^{4V}$

$\therefore T = \frac{1}{2}\left(\ln(16+16V^2) - \ln(16+V^2)\right)$

$= \frac{1}{2}\left(\ln 16 + \ln\left(\frac{1+V^2}{16+V^2}\right)\right)$

$= \frac{1}{2}\left(4\ln 2 + \ln\left(\frac{1+V^2}{16+V^2}\right)\right)$

$= 2\ln 2 + \frac{1}{2}\ln\left(\frac{1+V^2}{16+V^2}\right)$

$< 2\ln 2$ since $\frac{1+V^2}{16+V^2} < 1$

$\Rightarrow \ln\left(\frac{1+V^2}{16+V^2}\right) < 0$

(ii) $v\frac{dv}{dx} = \frac{16+v^2}{v}$

$\Rightarrow \frac{dx}{dv} = \frac{v^2}{16+v^2} = 1 - \frac{16}{16+v^2}$

$\therefore [x]_0^X = \left[v - \frac{16}{4}\tan^{-1}\frac{v}{4}\right]_V^{4V}$

$\Rightarrow X = 4V - 4\tan^{-1}V - \left(V - 4\tan^{-1}\frac{V}{4}\right)$

$= 3V - 4\left(\tan^{-1}V - \tan^{-1}\frac{V}{4}\right)$

$= 3V - 4\tan^{-1}\left(\frac{V - \frac{V}{4}}{1 + V\cdot\frac{V}{4}}\right) = 3V - 4\tan^{-1}\left(\frac{3V}{4+V^2}\right)$

Question 7

(a) (i) at R, $y=0$ $\therefore \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$

$$\therefore x = \frac{a\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

ie $$x = \frac{a\left(1 + \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$$

(ii) From (i), $\frac{ae}{a}\cos\left(\frac{\alpha+\theta}{2}\right) = \cos\left(\frac{\alpha-\theta}{2}\right)$

ie $e\cos\left(\frac{\alpha+\theta}{2}\right) = \cos\left(\frac{\alpha-\theta}{2}\right)$

(iii) Now $e\left(\cos\frac{\alpha}{2}\cos\frac{\theta}{2} - \sin\frac{\alpha}{2}\sin\frac{\theta}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\theta}{2} + \sin\frac{\alpha}{2}\sin\frac{\theta}{2}$

$\therefore \cos\frac{\alpha}{2}\cos\frac{\theta}{2}(e-1) = \sin\frac{\alpha}{2}\sin\frac{\theta}{2}(e+1)$

$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\theta}{2} = \frac{e-1}{e+1}$

(iv) From (iii), using chord QQ',

$\tan\frac{\beta}{2}\tan\frac{\pi+\theta}{2} = \frac{e-1}{e+1}$

or $\tan\frac{\beta}{2}\tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = \frac{e-1}{e+1}$

or $-\tan\frac{\beta}{2}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \frac{e-1}{e+1} = -\tan\frac{\beta}{2}\cot\frac{\theta}{2}$

(v) From (iii) and (iv),

$$-\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \left(\frac{e-1}{e+1}\right)^2 \text{ is constant}$$

But, at R, $x = \dfrac{a\left(1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$ from (i)

$\Rightarrow$ at R, x is constant

ie: R is a fixed point

(b) (i) $LS = (1-x^b)^{n-1}\left(1-(1-x^b)\right) = x^b(1-x^b)^{n-1}$

(ii) $u_0 = \int_0^1 x^{a-1}\,dx = \left[\dfrac{x^a}{a}\right]_0^1 = \dfrac{1}{a}$

(iii) Put $u = (1-x^b)^n$, $\dfrac{dv}{dx} = x^{a-1}$

$\therefore \dfrac{du}{dx} = n(1-x^b)^{n-1}(-bx^{b-1})$, $v = \dfrac{x^a}{a}$

$$\therefore u_n = \left[\frac{x^a}{a}(1-x^b)^n\right]_0^1 + \frac{nb}{a}\int_0^1 x^a . x^{b-1}(1-x^b)^{n-1}\,dx$$

$$\Rightarrow u_n = \frac{nb}{a}\int_0^1 x^{a-1}\, x^b(1-x^b)^{n-1}\,dx$$

$$= \frac{nb}{a}\int_0^1 x^{a-1}\left((1-x^b)^{n-1} - (1-x^b)^n\right)dx \text{ from (i)}$$

ie. $u_n = \dfrac{nb}{a}(u_{n-1} - u_n)$

or $au_n = nb\,u_{n-1} - nb\,u_n$

$\Rightarrow (a+bn)u_n = nb\,u_{n-1}$

Question 8

(a) (i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2\sum\alpha\beta$

$$= 0^2 - 2\left(-\frac{A}{2}\right) = A$$

(ii) $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{(A-\alpha^2)\alpha}{\alpha\beta\gamma}$

$$= A\alpha - \alpha^3 \quad \text{since } \alpha\beta\gamma = \tfrac{2}{2} = 1$$

(iii) From (ii), put $x = A\alpha - \alpha^3$

Now, $P(\alpha) = 2\alpha^3 - A\alpha - 2 = 0$

$\therefore x = \alpha^3 - 2 \quad$ or $\alpha^3 = x + 2$

$\therefore$ cubic is $2(x+2) - A(x+2)^{\frac{1}{3}} - 2 = 0$

or $A(x+2)^{\frac{1}{3}} = 2(x+1)$

$\Rightarrow A^3(x+2) = 8(x+1)^3$

i.e. $8(x+1)^3 - A^3(x+2) = 0$

(b) (i) $S = \frac{(1+x)^n - 1}{1+x-1} = \frac{(1+x)^n - 1}{x}, \; x \neq 0$

(ii) $S = \frac{1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n - 1}{x}$

$$= \binom{n}{1} + \binom{n}{2}x + \dots + \binom{n}{n}x^{n-1}$$

(iii) When $x = 0$

$\binom{n}{1} + \binom{n}{2}x + \dots + \binom{n}{n}x^{n-1} = \binom{n}{1} = n$

$1 + (1+x) + (1+x)^2 + \dots + (1+x)^{n-1} = n \times 1 = n$

$\therefore$ True for $n = 0$.

(iv) Now $S = \binom{n}{1} + \binom{n}{2}x + \cdots + \binom{n}{n}x^{n-1} = 1 + (1+x) + \cdots + (1+x)^{n-1}$

$$\therefore \int_{-1}^{0} \binom{n}{1} + \binom{n}{2}x + \cdots + \binom{n}{n}x^{n-1}\,dx = \int_{-1}^{0} 1 + (1+x) + \cdots + (1+x)^{n-1}\,dx$$

$$\Rightarrow \left[\binom{n}{1}x + \binom{n}{2}\frac{x^2}{2} + \cdots + \binom{n}{n}\frac{x^n}{n}\right]_{-1}^{0} = \left[x + \frac{(1+x)^2}{2} + \cdots + \frac{(1+x)^n}{n}\right]_{-1}^{0}$$

$$\therefore -\left(-\binom{n}{1} + \binom{n}{2}\frac{1}{2} - \cdots + \binom{n}{n}\frac{(-1)^n}{n}\right) = \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - (-1)$$

i.e. $\binom{n}{1} - \binom{n}{2}\frac{1}{2} + \cdots + (-1)^{n+1}\binom{n}{n}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$